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DISTRIBUTIVE PROPERTIES OF ADDITION OVER MULTIPLICATION OF IDEMPOTENT MATRICES†

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Abstract. Let R be a ring with identity. If $a, b, c \in R$ such that $a+b+c = 1$, then the distributive laws from addition over multiplication hold in R, that is $a + (bc) = (a + b)(a + c)$ when $ab = ba$, and $(ab) + c = (a + c)(b + c)$ when $ac = ca$. An application to obtains, if $\mathbf{A}, \mathbf{B}$ are idempotent matrices and $\mathbf{AB} = \mathbf{BA} = \mathbf{0}$ then there exists an idempotent matrix $\mathbf{C}$ such that $\mathbf{A} + \mathbf{BC} = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{C})$, and also $\mathbf{A} + \mathbf{BC} = (\mathbf{I} - \mathbf{C})(\mathbf{I} - \mathbf{B})$. Some other cases and applications are also presented.

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1. Introduction

A *ring* (see [1, Definition 1.1. p. 115]) is a nonempty set R together with two binary operations denoted as addition (+) and multiplication such that:

(i) $(R, +)$ is an abelian group;
(ii) $(ab)c = a(bc)$ for all $a, b, c \in R$ (associative multiplication);
(iii) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ (left and right distributive laws).

If in addition:

(iv) $ab = ba$ for all $a, b \in R$,

then R is said to be a commutative ring. If R contains an element 1_R such that

(v) $1_R a = a 1_R = a$ for all $a \in R$,

then R is said to be a ring with identity.

Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. For a positive integer n, let M_n be the set of all $n \times n$ matrices over the field of complex numbers $\mathbb{C}$. A complex square matrix $\mathbf{A}$ is said to be idempotent, or a projector, whenever $\mathbf{A}^2 = \mathbf{A}$, we denote the set of all $n \times n$ idempotent matrices over complex

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numbers by $\mathcal{P}$. Moreover, we denote $\mathbf{I}$ is the identity matrix and $\mathbf{0}$ is the zero matrix.

Lemma 1.1 ([2, Theorem 10.1 (a)])**.** Let $\mathbf{A}$ be an $n \times n$ idempotent matrix. Then $\mathbf{I} - \mathbf{A}$ is also idempotent matrix.

Corollary 1.2. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix and* $\mathbf{I} - \mathbf{A}$ *be an idempotent matrix. Then* $\mathbf{A}$ *is an idempotent matrix.*

Proof. Suppose $\mathbf{I} - \mathbf{A}$ is an idempotent matrix. Then, Lemma 1.1 asserts that the matrix $\mathbf{I} - (\mathbf{I} - \mathbf{A})$ is also an idempotent. Therefore $\mathbf{A}$ is an idempotent matrix. So the proof is complete. ☐

Corollary 1.3. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix and* $\mathbf{I} + \mathbf{A}$ *be an idempotent matrix. Then* $-\mathbf{A}$ *is an idempotent matrix.*

Proof. Suppose $\mathbf{I} + \mathbf{A}$ is an idempotent matrix. Then, Lemma 1.1 asserts that the matrix $\mathbf{I} - (\mathbf{I} + \mathbf{A})$ is also an idempotent. Therefore $-\mathbf{A}$ is an idempotent matrix. So the proof is complete. ☐

Lemma 1.4 ([2, Theorem 10.3 (b)])**.** Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ idempotent matrices. Then $\mathbf{AB}$ is idempotent if $\mathbf{AB} = \mathbf{BA}$.

We already pointed out, the main theorem deal with the idempotency of linear combinations of idempotent matrices, given by J.K. Baksalary, O.M. Baksalary in [3, Theorem pp. 4-5], and given another approach to the first part of the theorem given by H. Özdemir, A.Y. Özban in [4, Theorem 3.1]. C. Bu and Y. Zhou (see[5]) were researched the problem of the linear combinations in the general case.

Theorem 1.5 ([3, Theorem pp. 4-5])**.** Given two different nonzero idempotent matrices $\mathbf{P}_1$ and $\mathbf{P}_2$, let $\mathbf{P}$ be their linear combination of the form

$$\mathbf{P} = c_1\mathbf{P}_1 + c_2\mathbf{P}_2$$

with nonzero scalars c_1 and c_2. Then there are exactly four situations, where $\mathbf{P}$ is an idempotent matrix:

(a) $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$ holds along with either one of the following sets of conditions:
  - (i) $c_1 = 1, c_2 = 1, \mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$;
  - (ii) $c_1 = 1, c_2 = -1, \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2$;
  - (iii) $c_1 = -1, c_2 = 1, \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1$;

(b) $\mathbf{P}_1\mathbf{P}_2 \neq \mathbf{P}_2\mathbf{P}_1$ holds along with the conditions $c_1 \in \mathbb{C}\backslash\{0, 1\}, c_2 = 1-c_1$, $(\mathbf{P_1} - \mathbf{P_2})^{\mathbf{2}} = \mathbf{0}$.

2. Main results

If R is a commutative ring with identity $1_R \neq 0$, and $a + b + c = 1_R$, then we can prove that the distributive laws from addition over multiplication hold in R.

Theorem 2.1. *If R is a ring with identity and $a+b+c=1_R$ where $a,b,c \in R$, then*

(i) $a+(bc)=(a+b)(a+c)$, *when* $ab=ba$;
(ii) $(ab)+c=(a+c)(b+c)$, *when* $ac=ca$.

Proof. We first prove the left distributive property from addition over multiplication. By the (usual) distributive laws, we have

$$\begin{aligned}(a+b)(a+c) &= a(a+c)+b(a+c)\\ &= (aa)+(ac)+(ba)+(bc)\\ &= [(aa)+(ac)+(ab)]+(bc)\\ &= a(a+b+c)+(bc)\\ &= a1_R+(bc)\\ &= a+(bc).\end{aligned}$$

Similarly, we have

$$\begin{aligned}(a+c)(b+c) &= a(b+c)+c(b+c)\\ &= (ab)+(ac)+(cb)+(cc)\\ &= (ab)+[(ca)+(cb)+(cc)]\\ &= (ab)+c(a+b+c)\\ &= (ab)+c.\end{aligned}$$

So the proof is complete. □

The left distributive property of addition over multiplication of idempotent matrices appears in the following theorem.

Theorem 2.2. *If* $\mathbf{A},\mathbf{B} \in \mathcal{P}$, *and* $\mathbf{AB}=\mathbf{BA}=\mathbf{0}$, *then there exists a matrix* $\mathbf{C} \in \mathcal{P}$ *such that* $\mathbf{A}+\mathbf{BC}=(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{C})$, *and also* $\mathbf{A}+\mathbf{BC}=(\mathbf{I}-\mathbf{C})(\mathbf{I}-\mathbf{B})$.

Proof. Let $\mathbf{A},\mathbf{B} \in \mathcal{P}$ and $\mathbf{AB}=\mathbf{BA}=\mathbf{0}$. By Theorem 1.5 (a)(i) asserts that $\mathbf{A}+\mathbf{B}$ is an idempotent matrix, let $\mathbf{A}+\mathbf{B}=\mathbf{P}$, we can write $\mathbf{P}=\mathbf{I}-\mathbf{C}$ for some square matrix $\mathbf{C}$. Since $\mathbf{P}$ is an idempotent, that is $\mathbf{I}-\mathbf{C}$ is an idempotent matrix, implies $\mathbf{C}$ is also an idempotent matrix, by Corollary 1.2. Then $\mathbf{A}+\mathbf{B}=\mathbf{I}-\mathbf{C}$, for some idempotent matrix $\mathbf{C} \in \mathcal{P}$. Therefore $\mathbf{A}+\mathbf{B}+\mathbf{C}=\mathbf{I}$.

In this case $\mathbf{A},\mathbf{B},\mathbf{C}$ are $n \times n$ idempotent matrices, $\mathbf{AB}=\mathbf{BA}$ and $\mathbf{A}+\mathbf{B}+\mathbf{C} = \mathbf{I}$, by Theorem 2.1 (i) asserts that

$$\mathbf{A}+\mathbf{BC}=(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{C}).$$

Replacing $\mathbf{A}+\mathbf{B}$ and $\mathbf{A}+\mathbf{C}$ by $\mathbf{I}-\mathbf{C}$ and $\mathbf{I}-\mathbf{B}$, respectively, then we have

$$\mathbf{A}+\mathbf{BC}=(\mathbf{I}-\mathbf{C})(\mathbf{I}-\mathbf{B}).$$

So the proof is complete. □

The proof of right distributive property of addition over multiplication of idempotent matrices is similar manner of Theorem 2.2, as the following.

Theorem 2.3. *If* $\mathbf{A},\mathbf{C} \in \mathcal{P}$, *and* $\mathbf{AC}=\mathbf{CA}=\mathbf{0}$ *then there exists a matrix* $\mathbf{B} \in \mathcal{P}$ *such that* $\mathbf{AB}+\mathbf{C}=(\mathbf{A}+\mathbf{C})(\mathbf{B}+\mathbf{C})$, *and also* $\mathbf{AB}+\mathbf{C}=(\mathbf{I}-\mathbf{B})(\mathbf{I}-\mathbf{A})$.

Proof. Since addition of matrices is commutative, it is obtained from Theorem 2.2 by replacing $\mathbf{A}$ with $\mathbf{C}$, $\mathbf{B}$ with $\mathbf{A}$, and $\mathbf{C}$ with $\mathbf{B}$. So the proof is complete. □

Corollary 2.4. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{P}$ *and* $\mathbf{A} + \mathbf{BC} = (\mathbf{I} - \mathbf{C})(\mathbf{I} - \mathbf{B})$. *If* $\mathbf{BC} = \mathbf{CB}$, *then* $\mathbf{A} + \mathbf{BC}$ *is an idempotent matrix.*

Proof. Since $(\mathbf{I} - \mathbf{C})$, and $(\mathbf{I} - \mathbf{B})$ are idempotent matrices, by Lemma 1.1; and

$$(\mathbf{I} - \mathbf{C})(\mathbf{I} - \mathbf{B}) = \mathbf{I} - \mathbf{C} - \mathbf{B} + \mathbf{CB} = \mathbf{I} - \mathbf{C} - \mathbf{B} + \mathbf{BC} = (\mathbf{I} - \mathbf{B})(\mathbf{I} - \mathbf{C}),$$

we obtain that $(\mathbf{I} - \mathbf{C})(\mathbf{I} - \mathbf{B})$ is an idempotent matrix, by using Lemma 1.4. Therefore $\mathbf{A} + \mathbf{BC}$ is an idempotent matrix. So the proof is complete. □

Corollary 2.5. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{P}$ *and* $\mathbf{AB} + \mathbf{C} = (\mathbf{I} - \mathbf{B})(\mathbf{I} - \mathbf{A})$. *If* $\mathbf{AB} = \mathbf{BA}$, *then* $\mathbf{AB} + \mathbf{C}$ *is an idempotent matrix.*

Proof. it is obtained from Corollary 2.4 by replacing $\mathbf{A}$ with $\mathbf{C}$, $\mathbf{B}$ with $\mathbf{A}$, and $\mathbf{C}$ with $\mathbf{B}$. So the proof is complete. □

Corollary 2.6. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in M_n$. *If* $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{I}$ *and* $\mathbf{AB} = \mathbf{BA}$ *then* $\mathbf{A} + \mathbf{BC} = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{C})$, *and also* $\mathbf{A} + \mathbf{BC} = (\mathbf{I} - \mathbf{C})(\mathbf{I} - \mathbf{B})$.

Proof. The first result is directly followed by Theorem 2.1 (i). Then, replacing $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} + \mathbf{C}$ by $\mathbf{I} - \mathbf{C}$ and $\mathbf{I} - \mathbf{B}$, respectively, then we have

$$\mathbf{A} + \mathbf{BC} = (\mathbf{I} - \mathbf{C})(\mathbf{I} - \mathbf{B}).$$

So the proof is complete. □

We also have the following corollary.

Corollary 2.7. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in M_n$. *If* $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{I}$ *and* $\mathbf{AC} = \mathbf{CA}$ *then* $\mathbf{AB} + \mathbf{C} = (\mathbf{A} + \mathbf{C})(\mathbf{B} + \mathbf{C})$, *and also* $\mathbf{AB} + \mathbf{C} = (\mathbf{I} - \mathbf{B})(\mathbf{I} - \mathbf{A})$.

For any square complex matrices of the same size $\mathbf{A}$ and $\mathbf{B}$ we know that

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}).$$

Theorem 2.8. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C} \in M_n$. *If* $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{I}$, *then*

(i) $\det(\mathbf{A} + \mathbf{BC}) = \det(\mathbf{I} - \mathbf{B})\det(\mathbf{I} - \mathbf{C})$, *when* $\mathbf{AB} = \mathbf{BA}$;
(ii) $\det(\mathbf{AB} + \mathbf{C}) = \det(\mathbf{I} - \mathbf{A})\det(\mathbf{I} - \mathbf{B})$, *when* $\mathbf{AC} = \mathbf{CA}$.

Proof. Suppose that $\mathbf{A}, \mathbf{B}, \mathbf{C} \in M_n$. If $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{I}$, and $\mathbf{AB} = \mathbf{BA}$, from Theorem 2.1 (i), then we have

$$\mathbf{A} + \mathbf{BC} = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{C}).$$

Therefore

$$\begin{aligned} \det(\mathbf{A} + \mathbf{BC}) &= \det[(\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{C})] \\ &= \det[(\mathbf{I} - \mathbf{C})(\mathbf{I} - \mathbf{B})] \\ &= \det(\mathbf{I} - \mathbf{B})\det(\mathbf{I} - \mathbf{C}). \end{aligned}$$

Similarly, if $\mathbf{AC} = \mathbf{CA}$, from Theorem 2.1 (ii), then we have

$$\det(\mathbf{AB} + \mathbf{C}) = \det(\mathbf{I} - \mathbf{A})\det(\mathbf{I} - \mathbf{B}).$$

So the proof is complete. □

We obtain the following corollaries.

Corollary 2.9. *If* $\mathbf{A}, \mathbf{B} \in \mathcal{P}$ *and* $\mathbf{AB} = \mathbf{BA} = \mathbf{0}$, *then there exists a matrix* $\mathbf{C} \in \mathcal{P}$ *such that* $\det(\mathbf{A} + \mathbf{BC}) = \det(\mathbf{A} + \mathbf{B}) \det(\mathbf{A} + \mathbf{C})$, *and also* $\det(\mathbf{A} + \mathbf{BC}) = \det(\mathbf{I} - \mathbf{B})) \det(\mathbf{I} - \mathbf{C})$.

Corollary 2.10. *If* $\mathbf{A}, \mathbf{C} \in \mathcal{P}$ *and* $\mathbf{AC} = \mathbf{CA} = \mathbf{0}$, *then there exists a matrix* $\mathbf{B} \in \mathcal{P}$ *such that* $\det(\mathbf{AB} + \mathbf{C}) = \det(\mathbf{A} + \mathbf{C}) \det(\mathbf{B} + \mathbf{C})$, *and also* $\det(\mathbf{AB} + \mathbf{C}) = \det(\mathbf{I} - \mathbf{A}) \det(\mathbf{I} - \mathbf{B}))$.

Similarly, consider the linear combinations $c_1\mathbf{A} + c_2\mathbf{BC}$ for the idempotent matrices, $\mathbf{A}, \mathbf{B} \in \mathcal{P}$, in case the scalars $c_1 = 1, c_2 = -1$, and $\mathbf{AB} = \mathbf{BA} = \mathbf{B}$. Then we have the following results.

Theorem 2.11. *If* $\mathbf{A}, \mathbf{B} \in \mathcal{P}$, *and* $\mathbf{AB} = \mathbf{BA} = \mathbf{B}$, *then there exists a matrix* $\mathbf{C} \in \mathcal{P}$ *such that* $\mathbf{A} - \mathbf{BC} = (\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{C})$, *and* $\mathbf{A} - \mathbf{BC} = (\mathbf{I} - \mathbf{C})(\mathbf{I} + \mathbf{B})$.

Theorem 2.12. *If* $\mathbf{A}, \mathbf{C} \in \mathcal{P}$, *and* $\mathbf{AC} = \mathbf{CA} = \mathbf{C}$, *then there exists a matrix* $\mathbf{B} \in \mathcal{P}$ *such that* $\mathbf{AB} - \mathbf{C} = (\mathbf{A} - \mathbf{C})(\mathbf{B} - \mathbf{C})$, *and* $\mathbf{AB} - \mathbf{C} = (\mathbf{I} - \mathbf{B})(\mathbf{I} - \mathbf{A})$.

Finally, consider the linear combinations $c_1\mathbf{A} + c_2\mathbf{BC}$ for $\mathbf{A}, \mathbf{B} \in \mathcal{P}$, the scalars $c_1 = -1, c_2 = 1$, and $\mathbf{AB} = \mathbf{BA} = \mathbf{A}$, Theorem 1.5 (a)(iii) asserts that the linear combinations $-\mathbf{A} + \mathbf{B}$ is an idempotent matrix. Then we have the following theorems.

Theorem 2.13. *If* $\mathbf{A}, \mathbf{B} \in \mathcal{P}$, *and* $\mathbf{AB} = \mathbf{BA} = \mathbf{A}$, *then there exists a matrix* $\mathbf{C} \in \mathcal{P}$ *such that* $-\mathbf{A} + \mathbf{BC} = (-\mathbf{A} + \mathbf{B})(-\mathbf{A} + \mathbf{C})$, *and also* $-\mathbf{A} + \mathbf{BC} = (\mathbf{I} - \mathbf{C})(\mathbf{I} - \mathbf{B})$.

Similarly, if $\mathbf{A}, \mathbf{C} \in \mathcal{P}$, and $\mathbf{AC} = \mathbf{CA} = \mathbf{A}$, Theorem 1.5 (a)(iii) asserts that the linear combinations $-\mathbf{A} + \mathbf{C}$ is an idempotent matrix. Then we have the following theorem.

Theorem 2.14. *If* $\mathbf{A}, \mathbf{C} \in \mathcal{P}$, *and* $\mathbf{AC} = \mathbf{CA} = \mathbf{A}$, *then there exists a matrix* $\mathbf{B} \in \mathcal{P}$ *such that* $-\mathbf{AB} + \mathbf{C} = (-\mathbf{A} + \mathbf{C})(\mathbf{B} + \mathbf{C})$, *and also* $-\mathbf{AB} + \mathbf{C} = (\mathbf{I} - \mathbf{B})(\mathbf{I} + \mathbf{A})$.

3. Conclusion

The unusual distributive properties of addition over multiplication hold in a commutative ring with identity under the condition that the sum of the three elements are identity. There are only three situations for applied this properties to idempotent matrices to agree with the first part of Theorem 1.5.

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