

## A REVIEW ON THE MATHEMATICAL ASPECTS OF FLUID FLOW PROBLEMS IN AN INFINITE CHANNEL WITH ARBITRARY BOTTOM TOPOGRAPHY

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ABSTRACT. A special system of partial differential equations (PDEs) occur in a natural way while studying a class of irrotational inviscid fluid flow problems involving infinite channels. Certain aspects of solutions of such PDEs are analyzed in the context of flow problems involving multiple layers of fluids of different constant densities in a channel associated with arbitrary bottom topography. The whole analysis is divided into two parts-part A and part B. In part A the linearized theory is employed along with the standard Fourier analysis to understand such flow problems and physical quantities of interest are derived analytically. In part B, the same set of problems handled in part A are examined in the light of a weakly nonlinear theory involving perturbation in terms of a small parameter and it is shown that the original problems can be cast into KdV type of nonlinear PDEs involving the bottom topography occurring in one of the coefficients of these equations. Special cases of bottom topography are worked out in detail and expressions for quantities of physical importance are derived.

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### 1. Introduction

The problems of free surface fluid flow over submerged obstacles have created varieties of challenges to model the situations in engineering and in atmospheric and oceanographic sciences. The linearized solution can be found in Lamb [10] and others. Such problems were considered for their complete solution by Forbes and Schwartz [8], Vanden-Broeck [13], Forbes [7], Dias and Vanden-Broeck [3], Shen et.al. [12], Akylas [1], Dias and Vanden-Broeck [4] and

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many others. Problems for unsteady solutions were considered by Grimshaw and Smyth [9], Milewski and Vanden-Broeck [11]. The problem involving two layers of fluids where the fluid in each layer is inviscid and incompressible, were handled by Belward and Forbes [2], Dias and Vanden-Broeck [5, 6], assuming the upper fluid layer to be bounded by a rigid lid.

In this paper, we consider the flow problems involving two layers of fluids of different constant densities in a channel associated with arbitrary bottom topography where the upper fluid layer has two different constraints, as given by

- (i) The upper fluid layer is bounded by a rigid lid,
- (ii) The top surface of the upper fluid layer is free to the atmosphere.

The entire analysis is divided into two parts- part A and part B. In part A the linearized theory is employed along with the Fourier analysis and analytical expressions have been derived for the elevation of the interface as well as of the free surface. In part B, the same set of problems handled in part A are handled by using a specially designed weakly nonlinear theory and it is shown that the original problems can be solved approximately by the aid of KdV type of nonlinear PDEs, involving the bottom topography, arising in one of the coefficients of such KdV equation.

## 2. Description of the problems

We consider a system of two layers of fluids of different constant densities, one on the top of the other, flowing over an arbitrary topography in an infinite channel. The profile of the topography is given by  $y = \hat{h}(x)$  where the  $x$ -axis is chosen to be along the bottom of the channel and  $y$ -axis is chosen in the vertically upward direction. The fluid in each layer is assumed to be incompressible and inviscid and the flow is two-dimensional, irrotational with the far upstream velocity uniform. Quantities related to the upper layer of fluid will have the subscript 1, while those related to the lower layer will be indexed with 2. We denote the upstream depth of each layer by  $H_1$  and  $H_2$  and the upstream horizontal velocity in each layer by  $c_1$  and  $c_2$ . Densities, velocities and pressures in each layer are  $\rho_j$ ,  $\vec{q}_j$  and  $p_j$  at any point  $(x_j, y_j)$ ,  $j = 1, 2$ .

Let  $\phi_j$ , ( $j = 1, 2$ ) be the velocity potentials in layer  $j$ . So  $\vec{q}_j = (u_j, v_j) = (\phi_{j,x}, \phi_{j,y})$  where  $\phi_{j,x}$  denotes the partial derivative of  $\phi_j$  with respect to  $x$  and  $\phi_{j,y}$  denotes the partial derivative of  $\phi_j$  with respect to  $y$ . In the following sections  $\phi_{j,xx}$  denotes the second order partial derivative of  $\phi_j$  with respect to  $x$  and  $\phi_{j,yy}$  denotes the second order partial derivative of  $\phi_j$  with respect to  $y$ .

The above variables are non-dimensionalized using  $H_2$  as the length scale and  $c_2$  as the velocity scale. So the lower layer has an upstream uniform speed of 1 and upstream uniform height 1. The dimensionless quantities representing the ratio of depths of the layers  $\delta$ , the density ratio  $D$ , the ratio of upstream fluid speeds and also the Froude number in the lower layer are defined by the

relations:

$$\delta = \frac{H_1}{H_2}, \quad D = \frac{\rho_1}{\rho_2}, \quad \gamma = \frac{c_1}{c_2} \quad \text{and} \quad F_2 = \frac{c_2}{\sqrt{gH_2}}. \tag{1}$$

### 3. Part-A: Linear theory

We assume propagation of stationary waves with respect to the bottom profile, so that the partial derivatives with respect to time can be taken equal to zero. Then the following analysis can be employed.

**3.1. Problem of Two-layer fluid flow (upper fluid layer is bounded by a rigid lid).** The fluid flow problem involving two layers of fluids in an infinite channel associated with arbitrary bottom topography where the upper fluid layer is bounded by a rigid lid is considered. Hence,  $\phi_j$  satisfies the following equations:

$$\phi_{j,xx} + \phi_{j,yy} = 0, \quad j = 1, 2, \quad \text{within each fluid} \tag{2a}$$

$$\phi_{1,n} = 0, \quad \text{on } y = \delta + 1, \tag{2b}$$

$$\phi_{j,n} = 0, \quad j = 1, 2, \quad \text{on } y = S(x), \tag{2c}$$

$$\phi_{2,n} = 0, \quad \text{on } y = B(x), \tag{2d}$$

$$\frac{1}{2}F_2^2(q_2^2 - Dq_1^2) + (1 - D)S(x) = \frac{1}{2}F_2^2(1 - D\gamma^2) + (1 - D), \quad \text{on } y = S(x), \tag{2e}$$

where  $y = S(x)$  represents the interface and  $B(x) = \widehat{h}(x)/H_2$ .

The upstream conditions are

$$\vec{q}_1 \rightarrow \gamma \vec{i}, \quad \vec{q}_2 \rightarrow \vec{i}, \quad S(x) \rightarrow 1, \quad \text{as } x \rightarrow -\infty. \tag{3}$$

Here, we assume that the bottom profile is given by  $B(x) = hf(x)$  where  $h$  is the height of the bottom profile, a dimensionless small quantity. Now we use the asymptotic expansions of the form, for very small values of  $h$  ( $h \ll 1$ ):

$$\left. \begin{aligned} S(x) &= 1 + hS_1(x) + O(h^2) \\ \phi_1(x, y) &= \gamma x + h\phi_{11}(x, y) + O(h^2) \\ \phi_2(x, y) &= x + h\phi_{21}(x, y) + O(h^2) \end{aligned} \right\}, \tag{4}$$

in the above equations and get, to the order of  $h$ ,

$$\phi_{j1,xx} + \phi_{j1,yy} = 0, \quad \text{within each fluid} \tag{5a}$$

$$\phi_{11,y} = 0, \quad \text{on } y = 1 + \delta, \tag{5b}$$

$$\phi_{11,y} = \gamma S_1'(x), \quad \text{on } y = 1, \tag{5c}$$

$$\phi_{21,y} = S_1'(x), \quad \text{on } y = 1, \tag{5d}$$

$$F_2^2(\phi_{21,x} - D\gamma\phi_{11,x}) + (1 - D)S_1(x) = 0, \quad \text{on } y = 1, \tag{5e}$$

$$\phi_{21,y} = f'(x), \quad \text{on } y = 0, \tag{5f}$$

where  $S_1(x)$ ,  $\phi_{11}(x, y)$  and  $\phi_{21}(x, y)$  are to be determined.

The system, defined by the relations (5a)-(5f) gives rise to the linearized version of the original problem.

To solve the boundary value problem involving equations (5a) to (5f) we now assume that the first order potentials  $\phi_{j1}(x, y)$ ,  $j = 1, 2$  and the bottom profile  $f(x)$  are such that the Fourier transforms of  $\phi_{j1}$  and  $f(x)$  exist and are defined as follows:

$$\phi_{j1}(x, y) = \int_0^\infty \widehat{\phi}_{j1}(k, y) \sin(kx) dk, \quad (6a)$$

with the inverse transform

$$\widehat{\phi}_{j1}(k, y) = \frac{2}{\pi} \int_0^\infty \phi_{j1} \sin(kx) dx, \quad (6b)$$

and

$$f(x) = \int_0^\infty M(k) \cos(kx) dk, \quad (6c)$$

with the inverse transform

$$M(k) = \frac{2}{\pi} \int_0^\infty f(x) \cos(kx) dx, \quad (6d)$$

where  $M(k)$  is determined by the bottom profile. For this bottom profile, let's define  $S_1(x)$  as

$$S_1(x) = \int_0^\infty a(k) \cos(kx) dk. \quad (6e)$$

Applying Fourier sine transform to the above equations (5a)-(5f) and solving them, we obtain:

$$\phi_{11}(x, y) = \int_0^\infty \frac{\gamma a(k)}{\sinh k\delta} \cosh k(y-1-\delta) \sin kx dk, \quad (7a)$$

$$\phi_{21}(x, y) = \int_0^\infty \left[ \left\{ \frac{-a(k) \cosh k + M(k)}{\sinh k} \right\} \cosh k(y-1) - a(k) \sinh k(y-1) \right] \sin kx dk, \quad (7b)$$

$$\text{with } a(k) = \frac{F_2^2 k M(k) \sinh(k\delta)}{E(k)}, \quad (7c)$$

where

$$E(k) = \{F_2^2 k \cosh k - (1-D) \sinh k\} \sinh k\delta + \gamma^2 D F_2^2 k \sinh k \cosh k\delta. \quad (7d)$$

Here the dispersion relation is given by

$$E(k_0) = 0, \quad (8)$$

where  $k_0$  is the wave number of the downstream waves. Note that  $(-k_0)$  is also another real root of the equation (8).

*Example: Known bottom profile.* We consider the smooth bottom profile as given by

$$f(x) = \begin{cases} \frac{1}{2} \left(1 + \cos \frac{\pi x}{L}\right), & -L \leq x \leq L \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

and hence,  $a(k)$  can be determined as

$$a(k) = \frac{\pi F_2^2 \sin(kL) \sinh(k\delta)}{L^2 \{(\pi^2/L^2) - k^2\} E(k)}. \tag{10}$$

From the equations (6e) and (7d), we get

$$S_1(x) = \frac{\pi F_2^2}{4L^2} \left[ \int_{-\infty}^{\infty} \frac{\sin[k(x+L)] \sinh(k\delta)}{\{(\pi^2/L^2) - k^2\} E(k)} dk - \int_{-\infty}^{\infty} \frac{\sin[k(x-L)] \sinh(k\delta)}{\{(\pi^2/L^2) - k^2\} E(k)} dk \right]. \tag{11}$$

Now, if  $\frac{F_2^2(\gamma^2 D + \lambda)}{\lambda(1-D)} < 1$ ,  $E(k) = 0$  has a nonzero real solution. Each integral in the relation (11) is singular with poles on the real axis at  $k = \pm k_0$ , ( $k_0 > 0$ ), and then, the integrals in (11) have to be understood as CPV, with an indentation below the singularities  $k = \pm k_0$ .

Hence, we find that

$$S_1(x) = \begin{cases} \frac{-\pi F_2^2 \sinh k_0 \delta}{L^2 k_0 h(k_0)} \sin k_0 x \sin k_0 L, & \text{for } x > L \\ 0, & \text{for } x < -L \end{cases}, \tag{12}$$

where  $h(k) = \left(\frac{\pi^2}{L^2} - k^2\right) F(k)$  with  $E(k) = (k^2 - k_0^2)F(k)$ . The form (12) is oscillatory in nature, representing a wave, downstream and no wave upstream.

**3.2. Problem of Two-layer fluid flow (top surface of the upper layer is a free surface).** Considering the same problem of the previous case as described in the section 3.1, we proceed to handle the case where the top surface of the upper fluid layer is a free surface to the atmosphere. Hence,  $\phi_j$  satisfies the following equations:

$$\phi_{j,xx} + \phi_{j,yy} = 0, \quad j = 1, 2, \text{ within each fluid} \tag{13a}$$

$$\phi_{1,n} = 0, \text{ on } y = \widehat{S}(x), \tag{13b}$$

$$\phi_{j,n} = 0, \quad j = 1, 2, \text{ on } y = S(x), \tag{13c}$$

$$\phi_{2,n} = 0, \text{ on } y = B(x), \tag{13d}$$

$$\frac{1}{2} F_2^2 (q_1^2 - \gamma^2) + \widehat{S}(x) = 1 + \delta, \text{ on } y = \widehat{S}(x). \tag{13e}$$

$$\frac{1}{2} F_2^2 (q_2^2 - Dq_1^2) + (1 - D)S(x) = \frac{1}{2} F_2^2 (1 - D\gamma^2) + (1 - D), \text{ on } y = S(x) \tag{13f}$$

where  $y = S(x)$  represents the interface,  $y = \widehat{S}(x)$  represents the free surface and  $B(x)$  represents the same form as considered in section 3.1. Now we use the

regular perturbations of the forms (for  $h \ll 1$ ):

$$\left. \begin{aligned} \phi_1(x, y) &= \gamma x + h\phi_{11}(x, y) + O(h^2) \\ \phi_2(x, y) &= x + h\phi_{21}(x, y) + O(h^2) \\ S(x) &= 1 + hS_1(x) + O(h^2) \\ \widehat{S}(x) &= 1 + \delta + h\widehat{S}_1(x) + O(h^2) \end{aligned} \right\}. \tag{13g}$$

and obtain the linearized system:

$$\phi_{j1,xx} + \phi_{j1,yy} = 0, \quad \text{within each fluid} \tag{14a}$$

$$\phi_{11,y} = \gamma \widehat{S}'_1(x), \quad \text{on } y = 1 + \delta, \tag{14b}$$

$$\phi_{11,y} = \gamma S'_1(x), \quad \text{on } y = 1, \tag{14c}$$

$$\phi_{21,y} = S'_1(x), \quad \text{on } y = 1, \tag{14d}$$

$$\phi_{21,y} = f'(x), \quad \text{on } y = 0, \tag{14e}$$

$$F_2^2 \gamma \phi_{11,x} + \widehat{S}_1(x) = 0, \quad \text{on } y = 1 + \delta, \tag{14f}$$

$$F_2^2 (\phi_{21,x} - D\gamma \phi_{11,x}) + (1 - D)S_1(x) = 0, \quad \text{on } y = 1. \tag{14g}$$

where  $S_1(x), \widehat{S}_1(x), \phi_{11}(x, y)$  and  $\phi_{21}(x, y)$  are to be determined.

In order to solve the boundary value problem involving equations (14a) to (14g), we consider the same transforms of  $\phi_{j1}(x, y), j = 1, 2$  and  $f(x)$  as defined in the section 3.1 with the following:

$$\widehat{S}_1(x) = \int_0^\infty a_1(k) \cos(kx) dk, \tag{15a}$$

and

$$S_1(x) = \int_0^\infty a_2(k) \cos(kx) dk. \tag{15b}$$

Applying Fourier sine transform, we get:

$$\begin{aligned} \phi_{11}(x, y) &= \int_0^\infty \left[ \frac{\gamma \{a_2(k) - a_1(k) \cosh k\delta\}}{\sinh k\delta} \cosh k(y - 1 - \delta) \right. \\ &\quad \left. - \gamma a_1(k) \sinh k(y - 1 - \delta) \right] \sin kx dk, \end{aligned} \tag{16a}$$

$$\begin{aligned} \phi_{21}(x, y) &= \int_0^\infty \left[ \left\{ \frac{-a_2(k) \cosh k + M(k)}{\sinh k} \right\} \cosh k(y - 1) \right. \\ &\quad \left. - a_2(k) \sinh k(y - 1) \right] \sin kx dk, \end{aligned} \tag{16b}$$

with

$$a_1(k) = \frac{F_2^2 \gamma^2}{E_1(k)} a_2(k), \tag{16c}$$

$$a_2(k) = \frac{E_1(k)}{E_2(k)} F_2^2 k M(k) \sinh k\delta, \tag{16d}$$

where

$$E_1(k) = \frac{1}{k} \{F_2^2 \gamma^2 k \cosh k\delta - \sinh k\delta\} \tag{17}$$

$$E_2(k) = \left[ F_2^2 k \cosh k \sinh k \delta + \{ F_2^2 D \gamma^2 k \cosh k \delta - (1 - D) \sinh k \delta \} \sinh k \right] E_1(k) - \gamma^4 D F_2^4 k \sinh k. \tag{18}$$

Note that here  $E_1(-k) = E_1(k)$  and  $E_2(-k) = E_2(k)$ .

*Example: Known bottom profile.* For the known bottom profile given by the relation (9),  $a_2(k)$  and  $a_1(k)$  can be determined as

$$a_1(k) = \pi F_2^4 \gamma^2 \frac{\sin k L \sinh k \delta}{\pi^2 - L^2 k^2}, \tag{19a}$$

$$a_2(k) = \pi F_2^2 \frac{E_1(k)}{E_2(k)} \frac{\sin k L \sinh k \delta}{\pi^2 - L^2 k^2}, \tag{19b}$$

and hence, we get

$$\widehat{S}_1(x) = \frac{\pi F_2^4 \gamma^2}{4L^2} \int_{-\infty}^{\infty} \frac{\sinh(k\delta)}{\{(\pi^2/L^2) - k^2\} E_2(k)} [\sin k(x + L) - \sin k(x - L)] dk, \tag{20a}$$

$$S_1(x) = \frac{\pi F_2^2}{4L^2} \int_{-\infty}^{\infty} \frac{E_1(k) \sinh(k\delta)}{\{(\pi^2/L^2) - k^2\} E_2(k)} [\sin k(x + L) - \sin k(x - L)] dk. \tag{20b}$$

Here also, we observe that each integral in the relations (20a) and (20b) is singular with poles on the real axis at  $k = \pm k_0$ , ( $k_0 > 0$ ) which are zeros of  $E_2(k) = 0$  and then, the integrals in (20a) and (20b) have to be understood as CPV, with an indentation below the singularities  $k = \pm k_0$ . Hence, we find that

$$\widehat{S}_1(x) = \begin{cases} \frac{-\pi^2 F_2^4 \gamma^2}{L^2 k_0} \frac{\sinh k_0 \delta}{h_1(k_0)} \sin k_0 x \sin k_0 L, & \text{for } x > L \\ 0 & \text{for } x < -L \end{cases}, \tag{21a}$$

$$S_1(x) = \begin{cases} \frac{-\pi^2 F_2^2}{L^2 k_0} \frac{\sinh k_0 \delta}{h_2(k_0)} \sin k_0 x \sin k_0 L, & \text{for } x > L \\ 0 & \text{for } x < -L \end{cases}, \tag{21b}$$

where  $h_1(k) = \left(\frac{\pi^2}{L^2} - k^2\right) Q(k)$ ,  $h_2(k) = \left(\frac{\pi^2}{L^2} - k^2\right) \frac{Q(k)}{E_1(k)}$  with  $E_3(k) = (k^2 - k_0^2)Q(k)$ . We have thus shown the existence of wave downstream and no wave upstream in this case also.

#### 4. Part-B: Weakly nonlinear theory

##### 4.1. One-layer fluid flow Problem.

**4.1.1. Formulation of the problem.** Consider the fluid flow in an infinite channel where  $y = H + \eta(x, t)$  and  $y = 0$  represent, respectively, the free surface and the bottom. Hence the velocity potential  $\phi$  satisfies the following equations:

$$\nabla^2 \phi = 0, \quad -\infty < x < \infty, 0 \leq y \leq H + \eta(x, t). \tag{22}$$

with boundary conditions

$$\phi_y = \eta_t + \eta_x \phi_x, \quad \text{on } y = H + \eta(x, t), \tag{23}$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + g(H + \eta) = B(t), \quad \text{on } y = H + \eta(x, t), \tag{24}$$

$$\phi_y = 0, \text{ on } y = 0. \quad (25)$$

**4.1.2. Derivation of KdV equation.** Assuming  $\eta(x, t) = a\hat{\eta}(x, t)$  with  $a \ll 1$  and for making these equations dimensionless, we use the scaled variables:

$$\bar{x} = \frac{x}{\lambda}, \quad \bar{y} = \frac{y}{H}, \quad \bar{\phi} = \frac{H\phi}{\lambda a \sqrt{gH}}, \quad \bar{t} = \frac{t\sqrt{gH}}{\lambda}. \quad (26)$$

Hence, the dimensionless system is

$$\varepsilon^2 \bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{y}\bar{y}} = 0, \quad -\infty < \bar{x} < \infty, \quad 0 \leq \bar{y} \leq 1 + \alpha\hat{\eta}(\bar{x}, \bar{t}), \quad (27)$$

$$\bar{\phi}_{\bar{y}} = \varepsilon^2 (\hat{\eta}_{\bar{t}} + \alpha\bar{\phi}_{\bar{x}}\hat{\eta}_{\bar{x}}), \quad \text{on } \bar{y} = 1 + \alpha\hat{\eta}(\bar{x}, \bar{t}) \quad (28)$$

$$\bar{\phi}_{\bar{t}} + \frac{1}{2}\alpha(\bar{\phi}_{\bar{x}}^2 + \varepsilon^{-2}\bar{\phi}_{\bar{y}}^2) + \hat{\eta} = (B(\bar{t}) - gH)/ag, \quad \text{on } \bar{y} = 1 + \alpha\hat{\eta}(\bar{x}, \bar{t}) \quad (29)$$

$$\bar{\phi}_{\bar{y}} = 0, \quad \text{on } \bar{y} = 0. \quad (30)$$

where  $\varepsilon = \frac{H}{\lambda} \ll 1$  and  $\alpha = \frac{a}{H} \ll 1$  are two small parameters.

To remove  $\varepsilon$  from the equations, introducing the transformation

$$z = \frac{\alpha^{1/2}}{\varepsilon}(\bar{x} - \bar{t}), \quad \tau = \frac{\alpha^{3/2}}{\varepsilon}\bar{t}, \quad \psi = \frac{\alpha^{1/2}}{\varepsilon} \left[ \bar{\phi} - \int_0^{\bar{t}} \frac{B(s) - gH}{ag} ds \right], \quad (31)$$

we get,

$$\alpha\psi_{zz} + \psi_{\bar{y}\bar{y}} = 0, \quad -\infty < z < \infty, \quad 0 \leq \bar{y} \leq 1 + \alpha\hat{\eta}(z, \tau), \quad (32)$$

$$\psi_{\bar{y}} = \alpha(-\hat{\eta}_z + \alpha\hat{\eta}_\tau + \alpha\psi_z\hat{\eta}_z), \quad \text{on } \bar{y} = 1 + \alpha\hat{\eta}(z, \tau), \quad (33)$$

$$\hat{\eta} - \psi_z + \alpha\psi_\tau + \frac{1}{2}(\alpha\psi_z^2 + \psi_{\bar{y}}^2) = 0, \quad \text{on } \bar{y} = 1 + \alpha\hat{\eta}(z, \tau), \quad (34)$$

$$\psi_{\bar{y}} = 0, \quad \text{on } \bar{y} = 0. \quad (35)$$

Now, we use the asymptotic expansion of the form:

$$\psi = \psi_0 + \alpha\psi_1 + \alpha^2\psi_2 + O(\alpha^2), \quad (36)$$

$$\hat{\eta} = \hat{\eta}_0 + \alpha\hat{\eta}_1 + O(\alpha). \quad (37)$$

Substituting relation (36) in the equations (32) and (35), we get

$$O(\alpha^0) : \quad \psi_{0,\bar{y}\bar{y}} = 0 \Rightarrow \psi_0 = B_0(z, \tau), \quad (38)$$

$$O(\alpha) : \quad \psi_{1,\bar{y}\bar{y}} = -\psi_{0,zz} \Rightarrow \psi_1 = -\frac{1}{2}\bar{y}^2 B_{0,zz} + B_1(z, \tau), \quad (39)$$

$$O(\alpha^2) : \quad \psi_{2,\bar{y}\bar{y}} = -\psi_{1,zz} \Rightarrow \psi_2 = \frac{1}{24}\bar{y}^4 B_{0,zzzz} - \frac{1}{2}\bar{y}^2 B_{1,zz} + B_2(z, \tau). \quad (40)$$

Using the expansion forms (36) and (37), the Bernoulli equation (34), at leading order, gives:

$$O(\alpha^0) : \quad \hat{\eta}_0(z, \tau) = \psi_{0,z} = B_{0,z}, \quad (41)$$

$$O(\alpha) : \quad \hat{\eta}_1 - B_{1,z} + \frac{1}{2}B_{0,zzz} + B_{0,\tau} + \frac{1}{2}B_{0,z}^2 = 0, \quad (42)$$

and the Kinematic boundary condition (33) gives:

$$O(\alpha^2) : -\widehat{\eta}_0 B_{0,zz} + \frac{1}{6} B_{0,zzzz} - B_{1,zz} + \widehat{\eta}_{1,z} - \widehat{\eta}_{0,\tau} - B_{0,z} \widehat{\eta}_{0,z} = 0. \quad (43)$$

By the help of the equation (42), we can eliminate  $\widehat{\eta}_1$  and  $B_1$  from the equation (43) and hence, we can write:

$$-\widehat{\eta}_0 B_{0,zz} - \frac{1}{3} B_{0,zzzz} - B_{0,z\tau} - B_{0,z} B_{0,zz} = \widehat{\eta}_{0,\tau} + B_{0,z} \widehat{\eta}_{0,z}. \quad (44)$$

Finally, from the relations (41) and (44), we get

$$2\widehat{\eta}_{0,\tau} + 3\widehat{\eta}_0 \widehat{\eta}_{0,z} + \frac{1}{3} \widehat{\eta}_{0,zzz} = 0. \quad (45)$$

which is named as Korteweg-de Vries (KdV) Equation.

In the equation (45), we can write  $\widehat{\eta}_{0,\tau}$  as

$$\widehat{\eta}_{0,\tau} = \frac{\varepsilon}{\alpha^{3/2}} \widehat{\eta}_{0,\bar{t}} - \frac{1}{\alpha} \widehat{\eta}_{0,z}. \quad (46)$$

So, in the case of time independent problem, the KdV equation (45) reduces to

$$-\frac{1}{\alpha} \widehat{\eta}_{0,z} + \frac{3}{2} \widehat{\eta}_0 \widehat{\eta}_{0,z} + \frac{1}{6} \widehat{\eta}_{0,zzz} = 0. \quad (47)$$

#### 4.2. Problem of one-layer fluid flow over an arbitrary topography.

**4.2.1. Formulation of the problem.** Consider the fluid flow over an arbitrary topography. The profile of the topography is given by  $y = \widehat{h}(x) = ah_0(x)$ . Here we consider the BVP involving the Laplace's equation (22) and boundary conditions (23) and (24) with

$$\phi_y - \widehat{h}_x(x) \phi_x = 0, \quad \text{on } y = \widehat{h}(x). \quad (48)$$

**4.2.2. Derivation of KdV equation.** Assuming the same form of  $\eta(x, t)$  as above and using the same dimensionless variables as given in the relation (26), we get a system involving the equations (27)-(29) with

$$\varepsilon^{-2} \bar{\phi}_{\bar{y}} - \alpha h_{0,x}(x) \bar{\phi}_{\bar{x}} = 0, \quad \text{on } \bar{y} = \alpha h_0(\bar{x}). \quad (49)$$

Considering the transformation as defined by the relation (31), we get BVP involving the equations (32)-(34) with

$$\psi_{\bar{y}} - \alpha^2 h_{0,z} \psi_z = 0, \quad \text{on } \bar{y} = \alpha h_0(z). \quad (50)$$

Now, using the same asymptotic expansion of  $\psi$  given by the relation (36) in the relations (32) and (50) and then solving, we get

$$O(\alpha^0) : \psi_{0,\bar{y}\bar{y}} = 0 \Rightarrow \psi_0 = B_0(z, \tau), \quad (51)$$

$$O(\alpha) : \psi_{1,\bar{y}\bar{y}} = -\psi_{0,zz} \Rightarrow \psi_1 = -\frac{1}{2} \bar{y}^2 B_{0,zz} + B_1(z, \tau), \quad (52)$$

$$O(\alpha^2) : \psi_{2,\bar{y}\bar{y}} = -\psi_{1,zz} \Rightarrow \psi_2 = \frac{\bar{y}^4}{24} B_{0,zzzz} - \frac{\bar{y}^2}{2} B_{1,zz} + \bar{y} h_{0,z} B_{0,z} + B_2(z, \tau). \quad (53)$$

Using the asymptotic expansion of  $\eta$  given by the relations (37), the Bernoulli equation (34) gives (at leading order) :

$$O(\alpha^0) : \hat{\eta}_0(z, \tau) = \psi_{0,z} = B_{0,z} , \quad (54)$$

$$O(\alpha) : \hat{\eta}_1 - B_{1,z} + \frac{1}{2}B_{0,zzz} + B_{0,\tau} + \frac{1}{2}B_{0,z}^2 = 0, \quad (55)$$

and the Kinematic boundary condition (33) gives:

$$O(\alpha^2) : -\hat{\eta}_0 B_{0,zz} + \frac{1}{6}B_{0,zzzz} - B_{1,zz} + h_{0,z}B_{0,z} + \hat{\eta}_{1,z} - \hat{\eta}_{0,\tau} - B_{0,z}\hat{\eta}_{0,z} = 0. \quad (56)$$

Using equation (55) in the equation (56), we can write:

$$-\hat{\eta}_0 B_{0,zz} - \frac{1}{3}B_{0,zzzz} - B_{0,z\tau} - B_{0,z}B_{0,zz} + h_{0,z}B_{0,z} - \hat{\eta}_{0,\tau} - B_{0,z}\hat{\eta}_{0,z} = 0. \quad (57)$$

Now, from the relations (54) and (57), we get

$$2\hat{\eta}_{0,\tau} + 3\hat{\eta}_0\hat{\eta}_{0,z} + \frac{1}{3}\hat{\eta}_0\hat{\eta}_{0,zz} - h_{0,z}\hat{\eta}_0 = 0. \quad (58)$$

which is called a Generalized Korteweg-de Vries Equation.

It is observed that, in the particular case, when there is no hump i.e.  $\hat{h}(x) = 0$ , the generalized KdV equation (58) gives rise to the KdV equation (45) for the flow in a channel without hump.

In the time-independent case, the generalized KdV equation (58) reduces to [see relation (46)]:

$$-\frac{1}{\alpha}\hat{\eta}_{0,z} + \frac{3}{2}\hat{\eta}_0\hat{\eta}_{0,z} + \frac{1}{6}\hat{\eta}_0\hat{\eta}_{0,zz} - \frac{1}{2}\hat{\eta}_0 h_{0,z} = 0. \quad (59)$$

### 4.3. Problem of Two-layer fluid flow (upper fluid layer is bounded by a rigid lid).

**4.3.1. Formulation of the problem.** The irrotational flow of two layers of incompressible inviscid fluids of different densities over a hump is considered. The upper fluid layer is bounded by a rigid lid. The interface between the two fluids is given by  $y = H_2 + \eta(x, t) = H_2 + a\hat{\eta}(x, t)$  and the profile of the hump is given by  $y = \hat{h}(x) = ah_0(x)$ . The governing equations are the following:

$$\nabla^2\phi_1 = 0, \quad -\infty < x < \infty, \quad H_2 + \eta(x, t) \leq y \leq H_1 + H_2, \quad (60)$$

$$\nabla^2\phi_2 = 0, \quad -\infty < x < \infty, \quad \hat{h}(x) \leq y \leq H_2 + \eta(x, t), \quad (61)$$

$$\phi_{1,y} = 0, \quad \text{on } y = H_1 + H_2, \quad (62)$$

$$\phi_{2,y} - \hat{h}_x(x)\phi_{2,x} = 0, \quad \text{on } y = \hat{h}(x). \quad (63)$$

$$\phi_{1,y} = \eta_t + \eta_x\phi_{1,x}, \quad \text{on } y = H_2 + \eta(x, t), \quad (64)$$

$$\phi_{2,y} = \eta_t + \eta_x\phi_{2,x}, \quad \text{on } y = H_2 + \eta(x, t), \quad (65)$$

$$\begin{aligned} &(\phi_{2,t} - D\phi_{1,t}) + \frac{1}{2}[\phi_{2,x}^2 + \phi_{2,y}^2 - D(\phi_{1,x}^2 + \phi_{1,y}^2)] + g\eta(1 - D) \\ &= B_2(t) - DB_1(t) - gH_2(1 - D), \quad \text{on } y = H_2 + \eta(x, t), \end{aligned} \quad (66)$$

where  $D = \rho_1/\rho_2$ .

**4.3.2. Derivation of KdV equation.** Using the same dimensionless variables  $\bar{x}, \bar{y}, \bar{t}$  as given by the relation (26) with

$$\bar{\phi}_j = \frac{H_2 \phi_j}{\lambda a \sqrt{g H_2}}, (j = 1, 2), \delta = \frac{H_1}{H_2}, \tag{67}$$

we get the system as:

$$\varepsilon^2 \bar{\phi}_{1,\bar{x}\bar{x}} + \bar{\phi}_{1,\bar{y}\bar{y}} = 0, \quad -\infty < \bar{x} < \infty, \quad 1 + \alpha \hat{\eta}(\bar{x}, \bar{t}) \leq \bar{y} \leq 1 + \delta, \tag{68}$$

$$\varepsilon^2 \bar{\phi}_{2,\bar{x}\bar{x}} + \bar{\phi}_{2,\bar{y}\bar{y}} = 0, \quad -\infty < \bar{x} < \infty, \quad \alpha h_0(\bar{x}) \leq \bar{y} \leq 1 + \alpha \hat{\eta}(\bar{x}, \bar{t}), \tag{69}$$

$$\bar{\phi}_{1,\bar{y}} = 0, \quad \text{on } \bar{y} = 1 + \delta, \tag{70}$$

$$\varepsilon^{-2} \bar{\phi}_{2,\bar{y}} - \alpha h_{0,x}(x) \bar{\phi}_{2,\bar{x}} = 0, \quad \text{on } \bar{y} = \alpha h_0(\bar{x}). \tag{71}$$

$$\bar{\phi}_{1,\bar{y}} = \varepsilon^2 (\hat{\eta}_{\bar{t}} + \alpha \bar{\phi}_{1,\bar{x}} \hat{\eta}_{\bar{x}}), \quad \text{on } \bar{y} = 1 + \alpha \hat{\eta}(\bar{x}, \bar{t}) \tag{72}$$

$$\bar{\phi}_{2,\bar{y}} = \varepsilon^2 (\hat{\eta}_{\bar{t}} + \alpha \bar{\phi}_{2,\bar{x}} \hat{\eta}_{\bar{x}}), \quad \text{on } \bar{y} = 1 + \alpha \hat{\eta}(\bar{x}, \bar{t}) \tag{73}$$

$$\begin{aligned} & (\bar{\phi}_{2,\bar{t}} - D \bar{\phi}_{1,\bar{t}}) + \frac{\alpha}{2} [\bar{\phi}_{2,\bar{x}}^2 + \varepsilon^{-2} \bar{\phi}_{2,\bar{y}}^2 - D \{ \bar{\phi}_{1,\bar{x}}^2 + \varepsilon^{-2} \bar{\phi}_{1,\bar{y}}^2 \}] + \hat{\eta}(1 - D) \\ & = C_2(t) - D C_1 t, \quad \text{on } \bar{y} = 1 + \alpha \hat{\eta}(\bar{x}, \bar{t}) \end{aligned} \tag{74}$$

where

$$C_2(t) = \frac{B_2(t) - g H_2(1 - D)}{a g}, \quad C_1(t) = \frac{B_1(t)}{a g}, \quad \varepsilon = \frac{H_2}{\lambda}, \quad \alpha = \frac{a}{H_2}.$$

Considering the transformations as defined by the relation (31) with  $\psi_j = \frac{\alpha^{1/2}}{\varepsilon} \left[ \bar{\phi}_j - \int_0^{\bar{t}} C_j(s) ds \right]$ , we get

$$\alpha \psi_{1,zz} + \psi_{1,\bar{y}\bar{y}} = 0, \quad -\infty < z < \infty, \quad 1 + \alpha \hat{\eta}(z, \tau) \leq \bar{y} \leq 1 + \delta, \tag{75}$$

$$\alpha \psi_{2,zz} + \psi_{2,\bar{y}\bar{y}} = 0, \quad -\infty < z < \infty, \quad \alpha h_0(z) \leq \bar{y} \leq 1 + \alpha \hat{\eta}(z, \tau), \tag{76}$$

$$\psi_{1,\bar{y}} = 0, \quad \text{on } \bar{y} = 1 + \delta, \tag{77}$$

$$\psi_{2,\bar{y}} - \alpha^2 h_{0,z} \bar{\psi}_{2,z} = 0, \quad \text{on } \bar{y} = \alpha h_0(z), \tag{78}$$

$$\psi_{1,\bar{y}} = \alpha (-\hat{\eta}_z + \alpha \hat{\eta}_\tau + \alpha \psi_{1,z} \hat{\eta}_z), \quad \text{on } \bar{y} = 1 + \alpha \hat{\eta}(z, \tau), \tag{79}$$

$$\psi_{2,\bar{y}} = \alpha (-\hat{\eta}_z + \alpha \hat{\eta}_\tau + \alpha \psi_{2,z} \hat{\eta}_z), \quad \text{on } \bar{y} = 1 + \alpha \hat{\eta}(z, \tau), \tag{80}$$

$$\begin{aligned} & \alpha (\psi_{2,\tau} - D \psi_{1,\tau}) - (\psi_{2,z} - D \psi_{1,z}) + \frac{1}{2} [\alpha \psi_{2,z}^2 + \psi_{2,\bar{y}}^2 - D \{ \alpha \psi_{1,z}^2 + \psi_{1,\bar{y}}^2 \}] \\ & + \hat{\eta}(1 - D) = 0, \quad \text{on } \bar{y} = 1 + \alpha \hat{\eta}(z, \tau). \end{aligned} \tag{81}$$

Now we use the same asymptotic expansion of  $\hat{\eta}$  as given by the relation (37) and

$$\psi_i = \psi_{i0} + \alpha \psi_{i1} + \alpha^2 \psi_{i2} + O(\alpha^2), \quad i = 1, 2 \tag{82}$$

in the relations (75) - (81), we get the following BVPs:

$O(\alpha^0)$  :

$$\psi_{10,\bar{y}\bar{y}} = 0, \quad -\infty < z < \infty, \quad 1 \leq \bar{y} \leq 1 + \delta, \tag{83a}$$

$$\psi_{10,\bar{y}} = 0, \quad \text{on } \bar{y} = 1 + \delta, \tag{83b}$$

$$\psi_{10,\bar{y}} = 0, \quad \text{on } \bar{y} = 1, \tag{83c}$$

and

$$\psi_{20,\bar{y}\bar{y}} = 0, \quad -\infty < z < \infty, \quad 0 \leq \bar{y} \leq 1, \quad (84a)$$

$$\psi_{20,\bar{y}} = 0, \quad \text{on } \bar{y} = 1, \quad (84b)$$

$$\psi_{20,\bar{y}} = 0, \quad \text{on } \bar{y} = 0, \quad (84c)$$

$O(\alpha)$  :

$$\psi_{11,\bar{y}\bar{y}} + \psi_{10,zz} = 0, \quad -\infty < z < \infty, \quad 1 \leq \bar{y} \leq 1 + \delta, \quad (85a)$$

$$\psi_{11,\bar{y}} = 0, \quad \text{on } \bar{y} = 1 + \delta, \quad (85b)$$

$$\psi_{11,\bar{y}} + \hat{\eta}_0 \psi_{10,\bar{y}\bar{y}} = -\hat{\eta}_{0,z}, \quad \text{on } \bar{y} = 1, \quad (85c)$$

and

$$\psi_{21,\bar{y}\bar{y}} + \psi_{20,zz} = 0, \quad -\infty < z < \infty, \quad 0 \leq \bar{y} \leq 1, \quad (86a)$$

$$\psi_{21,\bar{y}} + \hat{\eta}_0 \psi_{20,\bar{y}\bar{y}} = -\hat{\eta}_{0,z}, \quad \text{on } \bar{y} = 1, \quad (86b)$$

$$\psi_{21,\bar{y}} = 0, \quad \text{on } \bar{y} = 0, \quad (86c)$$

$O(\alpha^2)$  :

$$\psi_{12,\bar{y}\bar{y}} + \psi_{11,zz} = 0, \quad -\infty < z < \infty, \quad 1 \leq \bar{y} \leq 1 + \delta, \quad (87a)$$

$$\psi_{12,\bar{y}} = 0, \quad \text{on } \bar{y} = 1 + \delta, \quad (87b)$$

$$\psi_{12,\bar{y}} + \hat{\eta}_0 \psi_{11,\bar{y}\bar{y}} + \hat{\eta}_1 \psi_{10,\bar{y}\bar{y}} + \frac{1}{2} \hat{\eta}_0 \psi_{10,\bar{y}\bar{y}\bar{y}} = \hat{\eta}_{0,\tau} - \hat{\eta}_{1,z} + \hat{\eta}_{0,z} \psi_{10,z}, \quad \text{on } \bar{y} = 1, \quad (87c)$$

and

$$\psi_{22,\bar{y}\bar{y}} + \psi_{21,zz} = 0, \quad -\infty < z < \infty, \quad 0 \leq \bar{y} \leq 1, \quad (88a)$$

$$\psi_{22,\bar{y}} = h_{0,z} \psi_{20,z}, \quad \text{on } \bar{y} = 0, \quad (88b)$$

$$\psi_{22,\bar{y}} + \hat{\eta}_0 \psi_{21,\bar{y}\bar{y}} + \hat{\eta}_1 \psi_{20,\bar{y}\bar{y}} + \frac{1}{2} \hat{\eta}_0 \psi_{20,\bar{y}\bar{y}\bar{y}} = \hat{\eta}_{0,\tau} - \hat{\eta}_{1,z} + \hat{\eta}_{0,z} \psi_{20,z}, \quad \text{on } \bar{y} = 1. \quad (88c)$$

From the equations (83a)-(83c),  $\psi_{10}$  can be found as

$$\psi_{10} = B_0(z, \tau). \quad (89)$$

We can determine  $\psi_{11}$  and  $B_{0,zz}$  from the equations (85a)-(85c):

$$\psi_{11} = -B_{0,zz} \left[ \frac{\bar{y}^2}{2} - \bar{y}(1 + \delta) \right] + B_1(z, \tau), \quad (90)$$

$$\text{with } B_{0,zz} = \frac{-1}{\delta} \hat{\eta}_{0,z}. \quad (91)$$

From the equations (87a)-(87c), we obtain

$$\begin{aligned} \psi_{12} = & \frac{-1}{\delta} \hat{\eta}_{0,zzz} \left\{ \frac{\bar{y}^4}{24} - (1 + \delta) \frac{\bar{y}^3}{6} + \frac{(1 + \delta)^3}{3} \bar{y} \right\} \\ & - B_{1,zz} \left\{ \frac{\bar{y}^2}{2} - (1 + \delta) \bar{y} \right\} + B_2(z, \tau) \end{aligned} \quad (92a)$$

$$B_{1,zz} = \frac{1}{\delta} \left[ B_{0,z} \hat{\eta}_{0,z} + \hat{\eta}_{0,\tau} - \hat{\eta}_{1,z} - \frac{\hat{\eta}_0}{\delta} \hat{\eta}_{0,z} \right]$$

$$+ \frac{1}{\delta} \left\{ \frac{1}{6} - \frac{(1 + \delta)}{2} + \frac{(1 + \delta)^3}{3} \right\} \widehat{\eta}_{0,zzz} \Big]. \tag{92b}$$

Similarly, from equations (84a)-(84c) and (86a)-(86c), we obtain, respectively,

$$\psi_{20} = A_0(z, \tau), \tag{93a}$$

$$\psi_{21} = -A_{0,zz} \frac{\bar{y}^2}{2} + A_1(z, \tau), \tag{93b}$$

$$\text{with } A_{0,zz} = \widehat{\eta}_{0,z}. \tag{93c}$$

From the equations (88a)-(88c), we obtain

$$\psi_{22} = \widehat{\eta}_{0,zzz} \frac{\bar{y}^4}{24} - A_{1,zz} \frac{\bar{y}^2}{2} + A_{0,z} h_{0,z} \bar{y} + A_2(z, \tau) \tag{94a}$$

$$\text{with } A_{1,zz} = A_{0,z} (h_{0,z} - \widehat{\eta}_{0,z}) + \frac{1}{6} \widehat{\eta}_{0,zzz} - \widehat{\eta}_0 \widehat{\eta}_{0,z} - \widehat{\eta}_{0,\tau} + \widehat{\eta}_{1,z}. \tag{94b}$$

From the interface condition (81), we get

$$O(\alpha^0) : A_{0,z} - DB_{0,z} = (1 - D)\widehat{\eta}_0. \tag{95a}$$

$$O(\alpha) : A_{0,\tau} - DB_{0,\tau} - \{-\widehat{\eta}_{0,zz} \frac{y^2}{2} + A_{1,z}\} + D \left[ \frac{1}{\sigma} \widehat{\eta}_{0,zz} \left\{ \frac{y^2}{2} - (1 + \delta)\bar{y} \right\} + B_{1,z} \right] + \frac{1}{2} \{A_{0,z}^2 - DB_{0,z}^2\} + (1 - D)\widehat{\eta}_1 = 0, \text{ on } \bar{y} = 1. \tag{95b}$$

From the relations (91), (93c) and (95b), we obtain (neglecting  $\widehat{\eta}_{1,z}$ ):

$$2 \left( 1 + \frac{D}{\sigma} \right) \widehat{\eta}_{0,\tau} + \left[ \frac{1}{3} - \frac{D(1 + 2\sigma)}{2\sigma} + \frac{D}{\sigma^2} \left\{ \frac{1}{6} - \frac{(1 + \sigma)}{2} + \frac{(1 + \sigma)^3}{3} \right\} \right] \widehat{\eta}_{0,zzz} + 3 \left( 1 - \frac{D}{\sigma^2} \right) \widehat{\eta}_0 \widehat{\eta}_{0,z} = \widehat{\eta}_0 h_{0,z}. \tag{96}$$

which is **the generalized Korteweg-de Vries equation**, for this problem.

From the equations (91), (93c) and (95a), we conclude that these equations will be satisfied only when  $\widehat{\eta} = 0$ .

In particular, when there is no upper layer i.e., when  $D = 0$ , the generalized KdV equation (96) exactly matches with the KdV equation (58) for the flow over an arbitrary topography in a channel.

**4.4. Problem of Two-layer fluid flow ( the top surface of the upper fluid layer is a free surface).**

**4.4.1. Formulation of the problem.** The irrotational flow of two layers of incompressible inviscid fluids of different densities over a hump is considered. The top surface of the upper fluid layer is free to the atmosphere, given by,  $y = H_1 + H_2 + \eta_1(x, t) = H_1 + H_2 + a\widehat{\eta}_1(x, t)$ . The interface between the two fluid is given by  $y = H_2 + \eta_2(x, t) = H_2 + a\widehat{\eta}_2(x, t)$  and the profile of the hump is given by  $y = \widehat{h}(x) = ah_0(x)$ . The governing equations are the following:

$$\nabla^2 \phi_1 = 0, \quad -\infty < x < \infty, \quad H_2 + \eta_2(x, t) \leq y \leq H_1 + H_2 + \eta_1(x, t), \tag{97a}$$

$$\nabla^2 \phi_2 = 0, \quad -\infty < x < \infty, \quad \widehat{h}(x) \leq y \leq H_2 + \eta_2(x, t), \tag{97b}$$

$$\phi_{1,y} = \eta_{1,t} + \eta_{1,x}\phi_{1,x}, \quad \text{on } y = H_1 + H_2 + \eta_1(x, t), \tag{97c}$$

$$\begin{aligned} \phi_{1,t} + \frac{1}{2} [\phi_{1,x}^2 + \phi_{1,y}^2] + g(H_1 + H_2 + \eta_1) &= B_3(t), \\ \text{on } y = H_1 + H_2 + \eta_1(x, t), \end{aligned} \tag{97d}$$

with the equations (63) on the bottom  $y = \hat{h}(x)$ , and (64), (65) and (66) on  $y = H_2 + \eta_2(x, t)$ .

**4.4.2. Derivation of KdV equation.** Using the same dimensionless variables  $\bar{x}, \bar{y}, \bar{t}$  as given by the relation (26) with  $\bar{\phi}_j, (j = 1, 2)$ , given by the relation (67), we get the system as:

$$\varepsilon^2 \bar{\phi}_{1,\bar{x}\bar{x}} + \bar{\phi}_{1,\bar{y}\bar{y}} = 0, \quad -\infty < \bar{x} < \infty, \quad 1 + \alpha\hat{\eta}_2(\bar{x}, \bar{t}) \leq \bar{y} \leq 1 + \delta + \alpha\hat{\eta}_1(\bar{x}, \bar{t}), \tag{98a}$$

$$\varepsilon^2 \bar{\phi}_{2,\bar{x}\bar{x}} + \bar{\phi}_{2,\bar{y}\bar{y}} = 0, \quad -\infty < \bar{x} < \infty, \quad \alpha h_0(\bar{x}) \leq \bar{y} \leq 1 + \alpha\hat{\eta}_2(\bar{x}, \bar{t}), \tag{98b}$$

$$\bar{\phi}_{1,\bar{y}} = \varepsilon^2 (\hat{\eta}_{1,\bar{t}} + \alpha\bar{\phi}_{1,\bar{x}}\hat{\eta}_{1,\bar{x}}), \quad \text{on } \bar{y} = 1 + \delta + \alpha\hat{\eta}_1(\bar{x}, \bar{t}) \tag{98c}$$

$$\bar{\phi}_{1,\bar{t}} + \frac{\alpha}{2} [\bar{\phi}_{1,\bar{x}}^2 + \varepsilon^{-2}\bar{\phi}_{1,\bar{y}}^2] + \hat{\eta}_1 = C_3(t), \quad \text{on } \bar{y} = 1 + \delta + \alpha\hat{\eta}_1(\bar{x}, \bar{t}), \tag{98d}$$

with the equations (71) on  $\bar{y} = \alpha h_0(\bar{x})$ , (72), (73) and (74) on  $\bar{y} = 1 + \alpha\hat{\eta}_2(x, t)$ , where

$$C_3(t) = \frac{B_3(t) - g(H_1 + H_2)}{ag}.$$

Considering the transformation as defined by the relation (31) with

$$\psi_1 = \begin{cases} \frac{\alpha^{1/2}}{\varepsilon} \left[ \bar{\phi}_1 - \int_0^{\bar{t}} C_3(s) ds \right], & \text{when } \bar{y} = 1 + \delta + \alpha\hat{\eta}_1(\bar{x}, \bar{t}) \\ \frac{\alpha^{1/2}}{\varepsilon} \left[ \bar{\phi}_1 - \int_0^{\bar{t}} C_1(s) ds \right], & \text{when } \bar{y} = 1 + \alpha\hat{\eta}_2(\bar{x}, \bar{t}) \end{cases} \tag{99a}$$

$$\psi_2 = \frac{\alpha^{1/2}}{\varepsilon} \left[ \bar{\phi}_2 - \int_0^{\bar{t}} C_2(s) ds \right], \tag{99b}$$

we get

$$\alpha\psi_{1,zz} + \psi_{1,\bar{y}\bar{y}} = 0, \quad -\infty < z < \infty, \quad 1 + \alpha\hat{\eta}_2(z, \tau) \leq \bar{y} \leq 1 + \delta + \alpha\hat{\eta}_1(z, \tau), \tag{100a}$$

$$\alpha\psi_{2,zz} + \psi_{2,\bar{y}\bar{y}} = 0, \quad -\infty < z < \infty, \quad \alpha h_0(z) \leq \bar{y} \leq 1 + \alpha\hat{\eta}_2(z, \tau), \tag{100b}$$

$$\psi_{1,\bar{y}} = \alpha(-\hat{\eta}_{1,z} + \alpha\hat{\eta}_{1,\tau} + \alpha\psi_{1,z}\hat{\eta}_{1,z}), \quad \text{on } \bar{y} = 1 + \delta + \alpha\hat{\eta}_1(z, \tau), \tag{100c}$$

$$\alpha\psi_{1,\tau} - \psi_{1,z} + \frac{1}{2} [\alpha\psi_{1,z}^2 + \psi_{1,\bar{y}}^2] + \hat{\eta}_1 = 0, \quad \text{on } \bar{y} = 1 + \delta + \alpha\hat{\eta}_1(z, \tau), \tag{100d}$$

with equations (78) on  $\bar{y} = \alpha h_0(z)$ , and equations (79), (80) and (81) on  $\bar{y} = 1 + \alpha\hat{\eta}_2(z, \tau)$ .

Now using the same asymptotic expansion of  $\hat{\eta}$  as given by the relation (37) for  $\hat{\eta}_1$  and  $\hat{\eta}_2$  and  $\psi_i (i = 1, 2)$ , given by the above relation (82), in the above relations, we get the following BVPs:

$O(\alpha^0)$  :

$$\psi_{10,\bar{y}\bar{y}} = 0, \quad -\infty < z < \infty, \quad 1 \leq \bar{y} \leq 1 + \delta, \tag{101a}$$

$$\psi_{10,\bar{y}} = 0, \quad \text{on } \bar{y} = 1 + \delta, \tag{101b}$$

$$\hat{\eta}_{10} - \psi_{10,z} + \frac{1}{2}\psi_{10,\bar{y}}^2 = 0, \quad \text{on } \bar{y} = 1 + \delta, \tag{101c}$$

$$\psi_{10,\bar{y}} = 0, \quad \text{on } \bar{y} = 1, \tag{101d}$$

and

$$\psi_{20,\bar{y}\bar{y}} = 0, \quad -\infty < z < \infty, \quad 0 \leq \bar{y} \leq 1, \tag{102a}$$

$$\psi_{20,\bar{y}} = 0, \quad \text{on } \bar{y} = 1, \tag{102b}$$

$$\psi_{20,\bar{y}} = 0, \quad \text{on } \bar{y} = 0, \tag{102c}$$

$O(\alpha)$  :

$$\psi_{11,\bar{y}\bar{y}} + \psi_{10,zz} = 0, \quad -\infty < z < \infty, \quad 1 \leq \bar{y} \leq 1 + \delta, \tag{103a}$$

$$\psi_{11,\bar{y}} + \hat{\eta}_{10}\psi_{10,\bar{y}\bar{y}} = -\hat{\eta}_{10,z}, \quad \text{on } \bar{y} = 1 + \delta, \tag{103b}$$

$$\hat{\eta}_{11} - \psi_{11,z} - \hat{\eta}_{10}\psi_{10,z\bar{y}} + \psi_{10,\tau} + \psi_{10,\bar{y}}\{\psi_{11,\bar{y}} + \hat{\eta}_{10}\psi_{10,\bar{y}\bar{y}}\} + \frac{1}{2}\psi_{10,z}^2 = 0, \quad \text{on } \bar{y} = 1 + \delta, \tag{103c}$$

$$\psi_{11,\bar{y}} + \hat{\eta}_{20}\psi_{10,\bar{y}\bar{y}} = -\hat{\eta}_{20,z}, \quad \text{on } \bar{y} = 1, \tag{103d}$$

and

$$\psi_{21,\bar{y}\bar{y}} + \psi_{20,zz} = 0, \quad -\infty < z < \infty, \quad 0 \leq \bar{y} \leq 1, \tag{104a}$$

$$\psi_{21,\bar{y}} + \hat{\eta}_{20}\psi_{20,\bar{y}\bar{y}} = -\hat{\eta}_{20,z}, \quad \text{on } \bar{y} = 1, \tag{104b}$$

$$\psi_{21,\bar{y}} = 0, \quad \text{on } \bar{y} = 0, \tag{104c}$$

$O(\alpha^2)$  :

$$\psi_{12,\bar{y}\bar{y}} + \psi_{11,zz} = 0, \quad -\infty < z < \infty, \quad 1 \leq \bar{y} \leq 1 + \delta, \tag{105a}$$

$$\psi_{12,\bar{y}} + \hat{\eta}_{10}\psi_{11,\bar{y}\bar{y}} + \hat{\eta}_{11}\psi_{10,\bar{y}\bar{y}} + \frac{1}{2}\hat{\eta}_{10}\psi_{10,\bar{y}\bar{y}\bar{y}} = \hat{\eta}_{10,\tau} - \hat{\eta}_{11,z} + \hat{\eta}_{10,z}\psi_{10,z}, \quad \text{on } \bar{y} = 1 + \delta, \tag{105b}$$

$$\psi_{11,\tau} - \psi_{12,z} - \hat{\eta}_{10}\psi_{11,z\bar{y}} + \hat{\eta}_{10}\psi_{11,z} + \frac{1}{2}\psi_{11,\bar{y}}^2 = 0, \quad \text{on } \bar{y} = 1 + \delta, \tag{105c}$$

$$\psi_{12,\bar{y}} + \hat{\eta}_{20}\psi_{11,\bar{y}\bar{y}} + \hat{\eta}_{21}\psi_{10,\bar{y}\bar{y}} + \frac{1}{2}\hat{\eta}_{20}\psi_{10,\bar{y}\bar{y}\bar{y}} = \hat{\eta}_{20,\tau} - \hat{\eta}_{21,z} + \hat{\eta}_{20,z}\psi_{10,z}, \quad \text{on } \bar{y} = 1, \tag{105d}$$

and

$$\psi_{22,\bar{y}\bar{y}} + \psi_{21,zz} = 0, \quad -\infty < z < \infty, \quad 0 \leq \bar{y} \leq 1, \tag{106a}$$

$$\psi_{22,\bar{y}} = h_{0,z}\psi_{20,z}, \quad \text{on } \bar{y} = 0, \tag{106b}$$

$$\begin{aligned} \psi_{22,\bar{y}} + \hat{\eta}_{20}\psi_{21,\bar{y}\bar{y}} + \hat{\eta}_{21}\psi_{20,\bar{y}\bar{y}} + \frac{1}{2}\hat{\eta}_{20}\psi_{20,\bar{y}\bar{y}\bar{y}} \\ = \hat{\eta}_{20,\tau} - \hat{\eta}_{21,z} + \hat{\eta}_{20,z}\psi_{20,z}, \quad \text{on } \bar{y} = 1. \end{aligned} \tag{106c}$$

From the equations (101a)-(101d),  $\psi_{10}$  can be found as

$$\psi_{10} = B_0(z, \tau), \tag{107}$$

$$\text{with } B_{0,z} = \hat{\eta}_{10}. \tag{108}$$

We can determine  $\psi_{11}$  from the equations (103a)-(103d):

$$\psi_{11} = -\hat{\eta}_{10,z} \left[ \frac{\bar{y}^2}{2} - \delta\bar{y} \right] + B_1(z, \tau), \tag{109a}$$

$$\text{with } \hat{\eta}_{10,z} = \frac{1}{1-\delta}\hat{\eta}_{20,z}, \tag{109b}$$

$$\text{and } B_{1,zz} = \hat{\eta}_{11,z} + \hat{\eta}_{10,\tau} + \hat{\eta}_{10}\hat{\eta}_{10,z} + \frac{(1-\delta^2)}{2}\hat{\eta}_{10,zzz}. \tag{109c}$$

From the equations (105a)-(105d), we obtain

$$\begin{aligned}
 & (\delta + 1)\widehat{\eta}_{10,\tau} - \widehat{\eta}_{20,\tau} - \widehat{\eta}_{20}\widehat{\eta}_{10,z} + (\delta + 2)\widehat{\eta}_{10}\widehat{\eta}_{10,z} - \widehat{\eta}_{10}\widehat{\eta}_{20,z} + \left\{ \frac{1}{6} + \frac{\delta}{2} \right. \\
 & \left. \times (1 + \delta)^2 - \frac{\delta}{2} - \frac{(1 + \delta)^3}{6} + \frac{\delta(1 - \delta^2)}{2} \right\} \widehat{\eta}_{10,zzz} + (\delta - 1)\widehat{\eta}_{11,z} + \widehat{\eta}_{21,z} = 0. \quad (110)
 \end{aligned}$$

Similarly, from equations (102a)-(102c) and (104a)-(104c), we obtain, respectively,

$$\psi_{20} = A_0(z, \tau), \quad (111a)$$

$$\psi_{21} = -A_{0,zz} \frac{\bar{y}^2}{2} + A_1(z, \tau), \quad (111b)$$

$$\text{with } A_{0,zz} = \widehat{\eta}_{20,z}. \quad (111c)$$

Similarly, from the equations (106a)-(106c), we obtain

$$\psi_{22} = \widehat{\eta}_{20,zzz} \frac{\bar{y}^4}{24} - A_{1,zz} \frac{\bar{y}^2}{2} + A_{0,z} h_{0,z} \bar{y} + A_2(z, \tau), \quad (112a)$$

$$\text{with } A_{1,zz} = A_{0,z} (h_{0,z} - \widehat{\eta}_{20,z}) + \frac{1}{6} \widehat{\eta}_{20,zzz} - \widehat{\eta}_{20}\widehat{\eta}_{20,z} - \widehat{\eta}_{20,\tau} + \widehat{\eta}_{21,z}. \quad (112b)$$

From the interface condition (pressure continuity condition: relation (81)) on  $\bar{y} = 1 + \alpha \widehat{\eta}_2(z, \tau)$ , we get

$$O(\alpha^0) : A_{0,z} - DB_{0,z} = (1 - D)\widehat{\eta}_{20}, \quad (113a)$$

$$\begin{aligned}
 O(\alpha) : & A_{0,\tau} - DB_{0,\tau} - \left( \frac{-1}{2} \widehat{\eta}_{20,zz} + A_{1,z} \right) + D \left[ - \left( \frac{1}{2} - \sigma \right) \widehat{\eta}_{10,zz} + B_{1,z} \right] \\
 & + \frac{1}{2} (A_{0,z}^2 - DB_{0,z}^2) + (1 - D)\widehat{\eta}_{21} = 0. \quad (113b)
 \end{aligned}$$

Hence, from the equations (108), (111c) and (113b), we obtain (neglecting  $\widehat{\eta}_{11,z}$  and  $\widehat{\eta}_{21,z}$ ):

$$\begin{aligned}
 2\widehat{\eta}_{20,\tau} + D(\widehat{\eta}_{10,\tau} - \widehat{\eta}_{20,\tau}) + \frac{1}{3}\widehat{\eta}_{20,zzz} + D \left\{ \frac{1 - \delta^2}{2} - \frac{1}{2} + \delta \right\} \widehat{\eta}_{10,zzz} \\
 + 3\widehat{\eta}_{20}\widehat{\eta}_{20,z} = \widehat{\eta}_{20}h_{0,z}, \quad (114)
 \end{aligned}$$

which is **the generalized Korteweg-de Vries equation**, for this problem.

In the absence of the upper layer of fluid i.e.,  $D = 0$ , the generalized KdV equation (114) exactly matches with the KdV equation (58) for the flow over an arbitrary topography in a channel.

*Notes.*

- (1) With the help of the relation (109b), the equation (110) gives rise to an equation for  $\widehat{\eta}_{10}$  and the equation (114) gives rise to an equation for  $\widehat{\eta}_{20}$ .
- (2) From the relations (108), (109b), (111c) and (113a), we conclude that these equations will be valid only when  $\delta$  is very very small.

- (3) When  $\delta$  is very very small, the equation (110) is satisfied and the equation (114) gives rise to the KdV equation (58) for the flow over a hump in a channel.

**4.5. Solution of the KdV equations.**

**4.5.1. KdV equation of type-1 (constant coefficients).** Let the KdV equation be of the form

$$A_0 \widehat{\eta}_{0,\tau}(z, \tau) + A_1 \widehat{\eta}_0 \widehat{\eta}_{0,z} + A_2 \widehat{\eta}_{0,zzz} = 0. \tag{115a}$$

For the solution, one can assume that

$$\widehat{\eta}_0(z, \tau) = u(z - \beta\tau) = u(\xi), \tag{115b}$$

where

$$\xi = z - \beta\tau. \tag{115c}$$

Substituting the relation (115b) in the KdV equation (115a), we get

$$-A_0 \beta \frac{\partial u}{\partial \xi} + A_1 u \frac{\partial u}{\partial \xi} + A_2 \frac{\partial^3 u}{\partial \xi^3} = 0, \tag{116a}$$

whose solution can be determined as (see Whitham [14])

$$u = \frac{3A_0\beta}{A_1} \operatorname{sech}^2 \left( \frac{\xi}{2} \sqrt{\frac{\beta A_0}{A_2}} \right), \tag{116b}$$

$$= \frac{3A_0\beta}{A_1} \left[ 1 - \frac{A_0\beta}{4A_2} \xi^2 + \frac{1}{24} \left( \frac{A_0\beta}{A_2} \right)^2 \xi^4 - \dots \right]. \tag{116c}$$

We finally get from the relation (116b):

$$\widehat{\eta}_0(z, \tau) = \frac{3A_0\beta}{A_1} \operatorname{sech}^2 \left( \frac{1}{2} \sqrt{\frac{\beta A_0}{A_2}} (z - \beta\tau) \right). \tag{116d}$$

Alternatively, let

$$u(\xi) = \sum_{n=0}^{\infty} a_n \xi^n, \tag{116e}$$

be a solution of the equation (116a), where  $a_n$ 's are constants to be determined.

Substituting the relation (116e) in the equation (116a) and equating the coefficients of  $\xi^0, \xi, \xi^2, \dots$ , we get

$$\begin{aligned} a_3 &= \frac{1}{6A_2} [A_0\beta a_1 - A_1 a_0 a_1] \\ a_4 &= \frac{1}{24A_2} [2A_0\beta a_2 - A_1(a_1^2 + 2a_0 a_2)] \\ a_5 &= \frac{1}{60A_2} [3A_0\beta a_3 - A_1(3a_1 a_2 + a_0 a_3)] \\ &\vdots \end{aligned}$$

Choosing  $a_0 = \frac{3A_0\beta}{A_1}$ ,  $a_1 = 0$  and  $a_2 = \frac{-3}{4} \frac{A_0^2\beta^2}{A_1A_2}$ , we obtain

$$a_3 = 0, \quad \text{and} \quad a_4 = \frac{1}{8} \frac{A_0^3\beta^3}{A_1A_2^2},$$

which exactly match with the coefficients of  $\xi^3$  and  $\xi^4$  respectively of the relation (116c).

*Particular case.* Substituting  $A_0 = 2$ ,  $A_1 = 3$  and  $A_2 = \frac{1}{3}$  in the relation (116b) or in the relation (116c), we get the solution of the KdV equation (45).

**4.5.2. KdV Equation of type-2 (variable coefficients).** Let the KdV equation be of the form

$$A_0\widehat{\eta}_{0,\tau}(z, \tau) + A_1\widehat{\eta}_0\widehat{\eta}_{0,z} + A_2\widehat{\eta}_0\widehat{\eta}_{0,zzz} - \widehat{\eta}_0h_{0,z}(z) = 0. \quad (117a)$$

whose solution has to be determined for a known bottom profile  $h_0(z)$ , which is a variable quantity.

For the solution, we assume that there is a series solution of the form given by the relation (116e), where  $a_n$ 's are constants to be determined.

Now for a known bottom profile as given by

$$h_0(z) = b \sin lz, \quad (117b)$$

substituting the relation (116e) in the equation (117a) and equating the coefficients of  $\xi^3$ ,  $\xi^4$ ,  $\xi^5$ ,  $\dots$ , we get:

$$a_3 = \frac{1}{6A_2} [A_0\beta a_1 - A_1a_0a_1 + bl(a_0 - \frac{l}{2}a_0^3)], \quad (117c)$$

$$a_4 = \frac{1}{24A_2} [2A_0\beta a_2 - A_1(a_1^2 + 2a_0a_2) + bl(a_1 - \frac{3l}{2}a_0^2a_1)], \quad (117d)$$

$$a_5 = \frac{1}{60A_2} [3A_0\beta a_3 - A_1(3a_1a_2 + a_0a_3) + bl(a_2 - \frac{3l}{2}(a_0^2a_2 + a_0a_1^2))], \quad (117e)$$

$\vdots$

Setting

$$a_0 = 1, a_1 = 0 \text{ and } a_2 = \frac{-3}{4} \frac{A_0^2\beta^2}{A_1A_2}, \quad (117f)$$

we get the values of  $a_3, a_4, a_5, \dots$  and hence a special series solution of the equation (117a) can be determined.

*Particular case.* Substituting  $A_0 = 2$ ,  $A_1 = 3$  and  $A_2 = \frac{1}{3}$  in the above relation (117c)-(117f), we get the series solution (116e) of the KdV equation (58).

By utilizing similar ideas we can obtain the series solutions of the type (116e) of the two generalized KdV equations (96) and (114) derived above. Work in this direction is in progress.

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### REFERENCES

1. T.R. Akylas, *On the excitation of long nonlinear water waves by a moving pressure distribution*, J. Fluid Mech. **141** (1984), 455-466.
2. S. R. Belward and L. K. Forbes, *Fully non-linear two-layer flow over arbitrary topography*, J. Engng. Math. **27** (1993), 419-432.
3. F. Dias and J.-M. Vanden-Broeck, *Open channel flows with submerged obstructions*, J. Fluid Mech. **206** (1989), 155 -170.
4. F. Dias and J.-M. Vanden-Broeck, *Generalised critical free-surface flows*, J. Engng. Math. **42** (2002), 291-301.
5. F. Dias and J.-M. Vanden-Broeck, *Steady two-layer flows over an obstacle*, Phil. Trans. R. Soc. Lond. A. **360** (2002), 2137-2154.
6. F. Dias and J.-M. Vanden-Broeck, *Two-layer hydraulic falls over an obstacle*, European J. Mech. B/Fluids. **23** (2004), 879-898.
7. L.K. Forbes, *Critical free-surface flow over a semi-circular obstruction*, J. Eng. Math. **22** (1988), 3 -13.
8. L.K. Forbes and L.W. Schwartz, *Free-surface flow over a semicircular obstruction*, J. Fluid Mech. **114** (1982), 299-314.
9. R. H. J. Grimshaw and N. Smyth, *Resonant flow of a stratified fluid over topography*, J. Fluid Mech. **16** (1986), 429-464.
10. H. Lamb, *Hydrodynamics*, 6th ed. Cambridge University Press. (1932), 738 pp.
11. P. Milewski and J.-M. Vanden-Broeck, *Time dependent gravitycapillary flows past an obstacle*, Wave Motion. **29** (1999), 63-79.
12. S. P. Shen, M. C. Shen and S. M. Sun, *A model equation for steady surface waves over a bump*, J. Engng. Math. **23** (1989), 315-323.
13. J.-M. Vanden-Broeck, *Free-surface flow over a semi-circular obstruction in a channel*, Phys. Fluids. **30** (1987), 2315-2317.
14. G. B. Whitham, *Linear and Nonlinear waves*, Wiley-Interscience. 1974.

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