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OSCILLATION THEOREMS FOR CERTAIN SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS†

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Abstract. In this paper, we consider the oscillation of the following certain second order nonlinear differential equations

$$\big(r(t)(x'(t))^{\alpha}\big)' + q(t)x^{\beta}(t) = 0,$$

where α and β are ratios of positive odd integers. New oscillation theorems are established, which are based on a class of new functions $\Phi = \Phi(t, s, l)$ defined in the sequel. Also, we establish some interval oscillation criteria for this equation.

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1. Introduction

In this paper, we are concerned with oscillation theorems for the following certain second order nonlinear differential equations

$$\big(r(t)(x'(t))^{\alpha}\big)' + q(t)x^{\beta}(t) = 0, \tag{1}$$

where α and β are ratios of positive odd integers, $q \in C([t_0, \infty), R)$, $q(t) \geq 0$. We assume that $r \in C^1([t_0, \infty), R)$, $r(t) > 0$ and $r'(t) \geq 0$. We shall consider the two cases

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(t)dt < \infty, \tag{2}$$

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(t)dt = \infty. \tag{3}$$

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By a solution of Eq. (1), we mean a function $x \in C^1([t_x,\infty),R)$, $t_x \geq t_0$, which has the property $r(t)(x'(t))^\alpha \in C^1([t_x,\infty),R)$ and satisfies Eq. (1) on $[t_x,\infty)$. A nontrivial solution of Eq. (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Eq. (1) is said to be oscillatory if all its solutions are oscillatory.

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions for different classes of second order differential equations, see [1–10]. Furthermore, there has been a great deal of works on the interval oscillation criteria for second order differential equations, and we refer the readers to the articles [11–16].

Elbert [1, 2], Kusano et al. [4–6], Mirzov [8, 9] have obtained some similar properties of second order differential equations

$$(r(t)x'(t))' + q_0(t)x(t) = 0, \tag{4}$$

under the assumption

$$\int_{t_0}^{\infty} \frac{1}{r(t)}dt = \infty.$$

Long and Wang [12] studied the oscillation of second-order nonlinear differential equations

$$(r(t)y'(t))' + Q(t, y(t), y'(t)) = 0, \tag{5}$$

where $1/r \in L_{loc}([t_0,\infty),R)$, the set of real-valued, locally integrable functions on $[t_0,\infty)$, and $r(t) > 0$ a.e. on $[t_0,\infty)$, and established new Kamenev-type criteria and interval criteria for oscillation of Eq. (5).

Clearly, (4) and (5) are different from (1). The problem of studying the oscillation and nonoscillation of all solutions of second order linear equations, e.g., Eq. (4), i.e., Eq. (1) with $\alpha = \beta = 1$, nonlinear equations, e.g., Eq. (1) with $\alpha = 1$ and $\beta \neq 1$, half-linear equations (4), e.g., Eq. (1) with $\alpha = \beta$ has been a very active area of research in the last few years.

In this paper, we consider the second order nonlinear equation (1) with $\alpha = \beta$ and $\alpha \neq \beta$. To the best of our knowledge nothing is known regarding the oscillation of Eq. (1) with $\alpha \neq \beta$.

The paper is organized as follows: In the next section, we present some definitions which will be used in the following results. In Section 3, by developing Riccati transformations technique and inequalities some sufficient conditions of new Kamenev-type criteria for oscillation of Eq. (1) are established, which are based on a class of new functions $\Phi = \Phi(t,s,l)$ defined in the sequel. In Section 4, we established interval criteria for oscillation of Eq. (1). In Section 5, we give two examples to illustrate Theorem 3.2 and Corollary 4.1, respectively.

2. Preliminaries

In this section, in order to prove our main results, we need the following definitions.

We say that a function $H = H(t,s)$ belongs to the function class Y, denoted by $H \in Y$, if $H \in C(D, R^+)$, where $D = \{(t,s) : t_0 \le s \le t < \infty\}$, which satisfies $H(t,t) = 0$, $H(t,s) > 0$ for $t > s$, and has partial derivative $\partial H/\partial s$ and $\partial H/\partial t$ on D such that

$$\frac{\partial H}{\partial t} = h_1(t,s)\sqrt{H(t,s)}, \quad \frac{\partial H}{\partial s} = -h_2(t,s)\sqrt{H(t,s)}, \tag{6}$$

where $h_1(t,s)$, $h_2(t,s)$ are locally integrable with respect to t and s, respectively, in D.

We will use the function classes X and Y to study the oscillation criteria for Eq. (1). We say that a function $\Phi = \Phi(t,s,l)$ belongs to the function class X, denoted by $\Phi \in X$, if $\Phi \in C(E, R)$, where $E = \{(t,s,l) : t_0 \le l \le s \le t < \infty\}$, which satisfies $\Phi(t,t,l) = 0$, $\Phi(t,l,l) = 0$, $\Phi(t,s,l) > 0$, $l < s < t$, and has the partial derivative $\partial\Phi/\partial s$ on E such that $\partial\Phi/\partial s$ is locally integrable with respect to s in E.

Definition 2.1. The operator $F[\cdot; l, t]$ is given by

$$F[g; l, t] = \int_l^t \Phi^n(t,s,l)g(s)ds, \quad t \ge s \ge l \ge t_0, \quad g \in C([t_0,\infty), R), \tag{7}$$

where n is a positive integer.

Definition 2.2. The function $\phi = \phi(t,s,l)$ is given by

$$\frac{\partial \Phi(t,s,l)}{\partial s} = \phi(t,s,l)\Phi(t,s,l). \tag{8}$$

It is easy to see that $F[\cdot; l, t]$ is a linear operator and satisfies

$$F[g'; l, t] = -nF[g\phi; l, t], \quad g \in C^1([t_0,\infty), R). \tag{9}$$

3. Kamenev-type oscillation criteria

In this section, we will establish some new Kamenev-type criteria for oscillation of Eq. (1).

It will be convenient to make the following notations:

$$d_+(t) = \max\{0, d(t)\}, \quad d_-(t) = \max\{0, -d(t)\},$$

$$\eta(t) = \left(\int_{t_1}^t r^{-\frac{1}{\alpha}}(s)ds\right)^{-1}, \quad \xi(t) = \int_t^\infty r^{-\frac{1}{\alpha}}(s)ds.$$

where t_1 is sufficiently large with $t_1 \ge t_0$.

Theorem 3.1. *Assume that* (3) *holds. If there exist functions* $\Phi \in X$ *and* $\psi \in C^1([t_0,\infty), R^+)$ *such that*

$$\limsup_{t\to\infty} F\left[\psi q\delta_1 - \frac{\psi r}{(\alpha+1)^{\alpha+1}}\left(n\phi + \frac{\psi'_+}{\psi}\right)^{\alpha+1}; l, t\right] > 0, \tag{10}$$

where the operator F is defined by (7)*, the function $\phi = \phi(t, s, l)$ is defined by* (8)*, and*

$$\delta_1(s) = \begin{cases} m_1, & m_1 \text{ is any positive constant}, \quad if\ \beta > \alpha, \\ 1, & if\ \beta = \alpha, \\ m_2\eta^{\alpha-\beta}(s), & m_2 \text{ is any positive constant}, \quad if\ \beta < \alpha, \end{cases}$$

then every solution of Eq. (1) *is oscillatory.*

Proof. Suppose to the contrary that x is a nonoscillatory solution of Eq. (1). We may assume without loss of generality that there exists a number $t_1 \geq t_0$, such that $x(t) > 0$, for all $t \geq t_1$. Then by Eq. (1) we have $(r(t)(x'(t))^\alpha)' = -q(t)x^\beta(t) \leq 0$, $t \geq t_1$, which implies that $r(t)(x'(t))^\alpha$ is decreasing, and it is eventually of one sign. So $x'(t)$ is eventually of one sign. We claim that $x'(t) > 0$, $t \geq t_1$. Otherwise, if there exists a $t_2 \geq t_1$ such that $r(t_2)(x'(t_2))^\alpha = a < 0$, then for all $t \geq t_2$, we have

$$r(t)(x'(t))^\alpha \leq r(t_2)(x'(t_2))^\alpha = a,$$

which implies that

$$x'(t) \leq -(-a)^{\frac{1}{\alpha}} \left(\frac{1}{r(t)}\right)^{\frac{1}{\alpha}}, \quad t \geq t_2.$$

Integrating the above inequality from t_2 to t, and letting $t \to \infty$, we get

$$x(t) \leq x(t_2) - (-a)^{\frac{1}{\alpha}} \int_{t_2}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} ds \to -\infty,$$

which gives a contradiction with $x(t) > 0$. Hence $x'(t) > 0$, $t \geq t_2$. Define the function ω by

$$\omega(s) = \psi(s)r(s)\left(\frac{x'(s)}{x(s)}\right)^\alpha, \quad s \geq t_1. \tag{11}$$

Then $\omega(s) > 0$ for $s \geq t_1$. Differentiating (11) and using (1), we obtain

$$\begin{aligned} \omega'(s) &= \frac{\psi'(s)}{\psi(s)}\omega(s) + \psi(s)\left(\frac{(r(s)(x'(s))^\alpha)'}{x^\alpha(s)} - \alpha r(s)\left(\frac{x'(s)}{x(s)}\right)^{\alpha+1}\right) \\ &= \frac{\psi'(s)}{\psi(s)}\omega(s) - \psi(s)q(s)x^{\beta-\alpha}(s) - \alpha\frac{\omega^{\frac{\alpha+1}{\alpha}}(s)}{(\psi(s)r(s))^{\frac{1}{\alpha}}} \\ &\leq \frac{\psi'_+(s)}{\psi(s)}\omega(s) - \psi(s)q(s)x^{\beta-\alpha}(s) - \alpha\frac{\omega^{\frac{\alpha+1}{\alpha}}(s)}{(\psi(s)r(s))^{\frac{1}{\alpha}}} \end{aligned}$$

for all $s \geq t_1$. It follows that

$$0 \leq -\omega'(s) + \frac{\psi'_+(s)}{\psi(s)}\omega(s) - \psi(s)q(s)x^{\beta-\alpha}(s) - \alpha\frac{\omega^{\frac{\alpha+1}{\alpha}}(s)}{(\psi(s)r(s))^{\frac{1}{\alpha}}}. \tag{12}$$

Applying $F[\cdot;t_1,t](t>t_1)$ to (12) and using (9), we have

$$\begin{aligned}0 &\le -F[\omega';t_1,t]+F[\frac{\psi'_+}{\psi}\omega;t_1,t]-F[\psi q x^{\beta-\alpha};t_1,t]-F\left[\alpha\frac{\omega^{\frac{\alpha+1}{\alpha}}}{(\psi r)^{\frac{1}{\alpha}}};t_1,t\right]\\ &=F\left[\left(n\phi+\frac{\psi'_+}{\psi}\right)\omega;t_1,t\right]-F[\psi q x^{\beta-\alpha};t_1,t]-F\left[\alpha\frac{\omega^{\frac{\alpha+1}{\alpha}}}{(\psi r)^{\frac{1}{\alpha}}};t_1,t\right]\\ &=F\left[\left(n\phi+\frac{\psi'_+}{\psi}\right)\omega-\alpha\frac{\omega^{\frac{\alpha+1}{\alpha}}}{(\psi r)^{\frac{1}{\alpha}}};t_1,t\right]-F[\psi q x^{\beta-\alpha};t_1,t]. \qquad (13)\end{aligned}$$

Set

$$G(v)=\left(n\phi+\frac{\psi'_+(s)}{\psi(s)}\right)v-\alpha\frac{v^{\frac{\alpha+1}{\alpha}}}{(\psi(s)r(s))^{\frac{1}{\alpha}}},\quad v>0.$$

By simple calculate, when $v=\psi(s)r(s)(n\phi+\psi'_+(s)/\psi(s))^{\alpha}/(\alpha+1)^{\alpha}$, we have

$$G(v)=G_{\max}=\frac{\psi(s)r(s)}{(\alpha+1)^{\alpha+1}}\left(n\phi+\frac{\psi'_+(s)}{\psi(s)}\right)^{\alpha+1}. \qquad (14)$$

From (13) and (14), we obtain

$$0\le F\left[\frac{\psi r}{(\alpha+1)^{\alpha+1}}\left(n\phi+\frac{\psi'_+}{\psi}\right)^{\alpha+1};t_1,t\right]-F[\psi q x^{\beta-\alpha};t_1,t]. \qquad (15)$$

Next, we consider the following three cases:
Case (i). Let $\beta>\alpha$. From $x'(t)>0$, there exists a constant m and a $t_2\ge t_1$, such that

$$x(s)\ge x(t_2)=m.$$

Thus

$$x^{\beta-\alpha}(s)\ge m_1,\quad s\ge t_2, \qquad (16)$$

where $m_1=m^{\beta-\alpha}$.
Case (ii). Let $\beta=\alpha$. Then

$$x^{\beta-\alpha}(s)=1,\quad s\ge t_1. \qquad (17)$$

Case (iii). Let $\beta<\alpha$. Then there exists a constant b such that

$$r(s)(x'(s))^{\alpha}\le r(t_1)(x'(t_1))^{\alpha}=b,\quad s\ge t_1,$$

or

$$x'(s)\le b^{\frac{1}{\alpha}}r^{-\frac{1}{\alpha}}(s),\quad s\ge t_1. \qquad (18)$$

Integrating (18) from t_1 to s, we get

$$x(s)\le x(t_1)+b^{\frac{1}{\alpha}}\int_{t_1}^{s}r^{-\frac{1}{\alpha}}(t)dt.$$

Thus there exist constants $b_1>0$ and $t_2\ge t_1$, such that

$$x(s)\le b_1\eta^{-1}(s),\quad s\ge t_2,$$

that is

$$x^{\beta-\alpha}(s) \geq m_2 \eta^{\alpha-\beta}(s), \quad s \geq t_2, \tag{19}$$

where $m_2 = b_1^{\beta-\alpha}$.

Combining (15) with (16), (17) and (19), we have

$$F\left[\psi q \delta_1 - \frac{\psi r}{(\alpha+1)^{\alpha+1}}\left(n\phi + \frac{\psi'_+}{\psi}\right)^{\alpha+1}; t_1, t\right] \leq 0,$$

which leads to a contradiction with (10). This completes the proof. □

Applying our approach used in [10], we establish the following result:

Theorem 3.2. *Assume that* (2) *holds. Furthermore, assume that there exists functions* $\Phi \in X$, *and* $\psi \in C^1([t_0, \infty), R^+)$. *If* (10) *holds, and*

$$\int^{\infty} \left[\delta_2 q(t) \xi^{\alpha}(t) - \frac{\alpha}{\xi(t) r^{\frac{1}{\alpha}}(t)}\right] dt = \infty, \tag{20}$$

where

$$\delta_2 = \begin{cases} 1, & if\ \beta = \alpha, \\ d_0, & d_0 \text{ is any positive constant}, \quad if\ \beta \neq \alpha, \end{cases}$$

then every solution of Eq. (1) *is oscillatory.*

Proof. Suppose to the contrary that x is a nonoscillatory solution of Eq. (1). We may assume without loss of generality that there exists a number $t_1 \geq t_0$, such that $x(t) > 0$, for all $t \geq t_1$. Then by Eq. (1) we have $(r(t)(x'(t))^{\alpha})' = -q(t)x^{\beta}(t) \leq 0$, $t \geq t_1$, which implies that $r(t)(x'(t))^{\alpha}$ is decreasing, and it is eventually of one sign. So $x'(t)$ is eventually of one sign. We shall distinguish the following two cases:

(I) $x'(t) > 0, \quad t \geq t_1$, and

(II) $x'(t) < 0, \quad t \geq t_1$.

Case (I). Proceeding as in the proof of Theorem 3.1, we get a contradiction with (10).

Case (II). Define the function υ by

$$\upsilon(t) = \frac{r(t)(x'(t))^{\alpha}}{x^{\alpha}(t)}, \quad t \geq t_1. \tag{21}$$

Then $\upsilon(t) < 0$ for $t \geq t_1$. Noting $(r(t)(x'(t))^{\alpha})' \leq 0$, we have

$$r(s)(x'(s))^{\alpha} \leq r(t)(x'(t))^{\alpha}, \quad s \geq t.$$

Dividing the above inequality by $r(s)$ and integrating it from t to l, we obtain

$$x(l) \leq x(t) + r^{\frac{1}{\alpha}}(t) x'(t) \int_t^l r^{-\frac{1}{\alpha}}(s) ds, \quad l \geq t. \tag{22}$$

Letting $l \to \infty$ in (22), we get

$$0 \leq x(t) + r^{\frac{1}{\alpha}}(t) x'(t) \xi(t), \quad t \geq t_1,$$

that is

$$r^{\frac{1}{\alpha}}(t)\xi(t)\frac{x'(t)}{x(t)} \geq -1, \quad t \geq t_1.$$

Thus

$$\frac{-r(t)(x'(t))^\alpha \xi^\alpha(t)}{x^\alpha(t)} \leq 1. \tag{23}$$

Applying (21) to (23), we have

$$-1 \leq \upsilon(t)\xi^\alpha(t) < 0. \tag{24}$$

Differentiating (21), using (1) and $x'(t) < 0$, we obtain

$$\upsilon'(t) = \frac{(r(t)(x'(t))^\alpha)' x^\alpha(t) - \alpha r(t)(x'(t))^\alpha x^{\alpha-1}(t)x'(t)}{x^{2\alpha}(t)},$$

which follows that

$$\upsilon'(t) \leq -q(t)x^{\beta-\alpha}(t). \tag{25}$$

Next, we consider the following three cases:
Case (i). Let $\beta > \alpha$. Then there exist a constant $d_1 \geq 0$ and $t_2 \geq t_1$, such that

$$\lim_{t\to\infty} x(t) = d_1.$$

Thus

$$x^{\beta-\alpha}(t) \geq d_2, \quad t \geq t_2, \tag{26}$$

where $d_2 = d_1^{\beta-\alpha}$.
Case (ii). Let $\beta = \alpha$. Then

$$x^{\beta-\alpha}(t) = 1, \quad t \geq t_1. \tag{27}$$

Case (iii). Let $\beta < \alpha$. Then there exists a constant $b < 0$, such that

$$r(t)(x'(t))^\alpha \leq r(t_1)(x'(t_1))^\alpha = b < 0, \quad t \geq t_1,$$

or

$$x'(t) \leq b^{\frac{1}{\alpha}} r^{-\frac{1}{\alpha}}(t), \quad t \geq t_1. \tag{28}$$

Integrating (28) from t_1 to t, we get

$$x(t) \leq x(t_1) + b^{\frac{1}{\alpha}} \int_{t_1}^{t} r^{-\frac{1}{\alpha}}(s)ds.$$

Thus there exist constants $m_0 > 0$ and $t_2 \geq t_1$, such that

$$x(t) \leq x(t_1) = m_0, \quad t \geq t_2,$$

that is

$$x^{\beta-\alpha}(t) \geq m_3, \quad t \geq t_2, \tag{29}$$

where $m_3 = m_0^{\beta-\alpha}$.

Combining (25) with (26), (27) and (29), we have

$$\upsilon'(t) \leq -\delta_2 q(t). \tag{30}$$

Multiplying (30) by $\xi^\alpha(t)$ and integrating it from t_1 to t, we obtain

$$\xi^\alpha(t)\upsilon(t)-\xi^\alpha(t_1)\upsilon(t_1)+\alpha\int_{t_1}^t \xi^{\alpha-1}(s)r^{-\frac{1}{\alpha}}(s)\upsilon(s)ds+\int_{t_1}^t \delta_2 q(s)\xi^\alpha(s)ds\le 0. \tag{31}$$

Therefore, it follows from (24) and (31) that

$$\xi^\alpha(t)\upsilon(t)\le \xi^\alpha(t_1)\upsilon(t_1)-\int_{t_1}^t\left(\delta_2 q(s)\xi^\alpha(s)-\frac{\alpha}{\xi(s)r^{\frac{1}{\alpha}}(s)}\right)ds.$$

Letting $t\to\infty$ in the above inequality, by (20), we get a contradiction with (24). This completes the proof. □

If we choose $\Phi(t,s,l)=z(s)(t-s)^a(s-l)^b$ for $a,b>1/2$ and $z\in C^1([t_0,\infty),R)$, then for $l<s<t$ we have

$$\phi(t,s,l)=\frac{z'(s)}{z(s)}+\frac{bt-(a+b)s+al}{(t-s)(s-l)}.$$

Thus taking $\psi(t)=1$ in Theorem 3.1, we have the following result.

Corollary 3.3. *Suppose that* (3) *holds. Eq.* (1) *is oscillatory provided that for each* $l\ge t_0$, *there exist a function* $z\in C^1([t_0,\infty),R)$ *and two constants* $a,b>1/2$, *such that*

$$\limsup_{t\to\infty}\int_l^t z^n(s)(t-s)^{na}(s-l)^{nb}\cdot$$
$$\left[q(s)\delta_1(s)-r(s)\left(\frac{n}{\alpha+1}\right)^{\alpha+1}\left(\frac{z'(s)}{z(s)}+\frac{bt-(a+b)s+al}{(t-s)(s-l)}\right)^{\alpha+1}\right]ds>0, \tag{32}$$

where δ_1 *is defined as in Theorem 3.1.*

Let $r(t)\equiv 1$. Taking $a=b=1$, $z(t)\equiv 1$ and $\alpha=n-1$ where n is even integer in Corollary 3.1, we have the following oscillation result.

Corollary 3.4. *Suppose that* (3) *holds. If for each* $l\ge t_0$,

$$\limsup_{t\to\infty}\frac{1}{t^{n+1}}\int_l^t(t-s)^n(s-l)^n q(s)\delta_1(s)ds>\frac{1}{n+1}, \tag{33}$$

where δ_1 *is defined as in Theorem 3.1, then Eq.* (1) *with* $r(t)\equiv 1$ *is oscillatory.*

Proof. Noting that

$$\int_l^t[t-2s+l]^n ds=\int_l^t[(t-s)-(s-l)]^n ds=\int_l^t(s-l)^n ds=\frac{1}{n+1}(t-l)^{n+1},$$

from (33), we get

$$\limsup_{t\to\infty}\frac{1}{t^{n+1}}\int_l^t(t-s)^n(s-l)^n\left[q(s)\delta_1(s)-\left(\frac{t-2s+l}{(t-s)(s-l)}\right)^n\right]ds$$
$$\ge\limsup_{t\to\infty}\frac{1}{t^{n+1}}\int_l^t(t-s)^n(s-l)^n q(s)\delta_1(s)ds-\limsup_{t\to\infty}\frac{1}{t^{n+1}}\int_l^t(t-2s+l)^n ds$$

$$= \limsup_{t\to\infty} \frac{1}{t^{n+1}} \int_l^t (t-s)^n (s-l)^n q(s)\delta_1(s)ds - \limsup_{t\to\infty} \frac{1}{n+1} \frac{(t-l)^{n+1}}{t^{n+1}}$$
$$= \limsup_{t\to\infty} \frac{1}{t^{n+1}} \int_l^t (t-s)^n (s-l)^n q(s)\delta_1(s)ds - \frac{1}{n+1} > 0,$$

which implies that (33) holds for $a = b = 1$, $z(t) \equiv 1$ and $\alpha = n-1$ where n is even integer. Then Eq. (1) with $r(t) \equiv 1$ is oscillatory by Corollary 3.1. This completes the proof. □

If we choose $\Phi(t, s, l) = \sqrt{H_1(s, l)H_2(t, s)}$, where $H_1, H_2 \in Y$, then from (8), we get

$$\phi(t, s, l) = \frac{1}{2}\left[\frac{h_1(s, l)}{\sqrt{H_1(s, l)}} - \frac{h_2(t, s)}{\sqrt{H_2(t, s)}}\right].$$

Thus taking $\psi(t) = 1$ in Theorem 3.1, we obtain the following result.

Corollary 3.5. *Suppose that* (3) *holds. If for each* $l \ge t_0$, *there exist* $H_1, H_2 \in Y$ *such that*

$$\limsup_{t\to\infty} \int_l^t (H_1(s, l)H_2(t, s))^{\frac{n}{2}} \cdot$$
$$\left[q(s)\delta_1(s) - r(s)\left(\frac{n}{\alpha+1}\right)^{\alpha+1} \left(\frac{h_1(s, l)}{\sqrt{H_1(s, l)}} - \frac{h_2(t, s)}{\sqrt{H_2(t, s)}}\right)^{\alpha+1}\right] ds > 0, \quad (34)$$

where δ_1 *is defined as in Theorem 3.1, then Eq.* (1) *is oscillatory.*

4. Interval oscillation criteria

In this section, we establish some interval oscillation criteria for Eq. (1). First, we give a lemma which will be used in the following results.

Lemma 4.1. *Suppose that* (3) *holds. If* x *is a solution of Eq.* (1) *and* $x(t) \neq 0$ *on* $[c, d] \subset [t_0, \infty)$, *then for any functions* $\Phi \in X$ *and* $\psi \in C^1([t_0, \infty), R^+)$,

$$F\left[\psi q\delta_1 - \frac{\psi r}{(\alpha+1)^{\alpha+1}}\left(n\phi + \frac{\psi'_+}{\psi}\right)^{\alpha+1}; c, d\right] \le 0, \quad (35)$$

where the operator F *is defined by* (7), *the function* $\phi = \phi(t, s, l)$ *is defined by* (8), *and* δ_1 *is defined as in Theorem 3.1.*

Proof. Define ω as in (11). We see that (12) holds. Applying $F[\cdot; c, d]$ to (12) and proceeding as in the proof of Theorem 3.1, we obtain

$$0 \le F\left[\frac{\psi r}{(\alpha+1)^{\alpha+1}}\left(n\phi + \frac{\psi'_+}{\psi}\right)^{\alpha+1}; c, d\right] - F[\psi q x^{\beta-\alpha}; c, d].$$

Combining the above inequality with (16), (17) and (19), we have

$$F\left[\psi q\delta_1 - \frac{\psi r}{(\alpha+1)^{\alpha+1}}\left(n\phi + \frac{\psi'_+}{\psi}\right)^{\alpha+1}; c, d\right] \le 0.$$

Hence, (35) holds. This completes the proof. □

Theorem 4.2. *Suppose that* (3) *holds. Furthermore, suppose that there exist functions* $\Phi \in X$ *and* $\psi \in C^1([t_0, \infty), R^+)$, *such that*

$$F\left[\psi q\delta_1 - \frac{\psi r}{(\alpha+1)^{\alpha+1}}\left(n\phi + \frac{\psi'_+}{\psi}\right)^{\alpha+1}; c, d\right] > 0, \tag{36}$$

where the operator F *is defined by* (7), *the function* $\phi = \phi(t, s, l)$ *is defined by* (8), *and* δ_1 *is defined as in Theorem 3.1. Then every solution of Eq.* (1) *has at least one zero in* $[c, d]$.

Proof. Suppose the contrary. We may assume without loss of generality that there exists a solution x of (1) such that $x(t) > 0$, for $t \in [c, d]$. By Lemma 4.1 we see that (35) holds for x, which contradicts the condition (36). This completes the proof. □

Corollary 4.3. *Suppose that* (3) *holds. If for each* $T_0 \geq t_0$, *there exist* $c, d \in R$, $\Phi \in X$ *and* $\psi \in C^1([t_0, \infty), R^+)$ *such that* $T_0 \leq c < d$ *and* (36) *holds, where the operator* F *is defined by* (7), *the function* $\phi = \phi(t, s, l)$ *is defined by* (8), *and* δ_1 *is defined as in Theorem 3.1, then every solution of Eq.* (1) *is oscillatory.*

Proof. Pick up a sequence $T_i \subset [t_0, \infty)$ such that $T_i \to \infty$ as $i \to \infty$. By the assumption, we find that for each $i \subset N$, there exist $c_i, d_i \subset R$ such that $T_i \leq c_i \leq d_i$ and (36) holds, where c, d are replaced by c_i, d_i, respectively. From Theorem 4.1, every solution x of (1) has at least one zero, $t_i \in [c_i, d_i]$. Since $d_i \geq t_i \geq c_i \geq T_i$, $i \in N$, it follows that every solution has arbitrary large zeros. Therefore, every solution of (1) is oscillatory. This completes the proof. □

Corollary 4.4. *Suppose that* (3) *holds. If for each* $\tau \geq t_0$, *there exist functions* $\Phi \in X$ *and* $\psi \in C^1([t_0, \infty), R^+)$, *such that*

$$F\left[\psi q\delta_1 - \frac{\psi r}{(\alpha+1)^{\alpha+1}}\left(n\phi + \frac{\psi'_+}{\psi}\right)^{\alpha+1}; \tau, t\right] > 0, \tag{37}$$

where the operator F *is defined by* (7), *the function* $\phi = \phi(t, s, l)$ *is defined by* (8), *and* δ_1 *is defined as in Theorem 3.1, then every solution of Eq.* (1) *is oscillatory.*

Proof. For any $T_0 \geq t_0$, we set $c = T_0$. In (37) we choose $\tau = c$. Then there exists $d > c$ such that (36) holds. Hence, the conclusion comes from Corollary 4.1. This completes the proof. □

Similar to the discussions in section 3, we have the following corollaries.

Corollary 4.5. *Suppose that* (3) *holds. If for each* $T_0 \geq t_0$, *there exist* $d > c \geq T_0$, $z \in C^1([m, l), R)$ *and two constants* $a, b > 1/2$, *such that*

$$\int_c^d z^n(s)(d-s)^{na}(s-c)^{nb}\cdot$$

$$\left[q(s)\delta_1(s) - r(s)\left(\frac{n}{\alpha+1}\right)^{\alpha+1}\left(\frac{z'(s)}{z(s)} + \frac{bd-(a+b)s+ac}{(d-s)(s-c)}\right)^{\alpha+1}\right] ds > 0,$$

then every solution of Eq. (1) *is oscillatory.*

Corollary 4.6. *Suppose that* (3) *holds. Eq.* (1) *is oscillatory provided that for each* $T_0 \geq t_0$*, there exist two constants* $d > c \geq T_0$ *and* $H_1, H_2 \in Y$*, such that*

$$\int_c^d (H_1(s,c)H_2(d,s))^{\frac{n}{2}} \cdot$$

$$\left[q(s)\delta_1(s) - r(s)\left(\frac{n}{\alpha+1}\right)^{\alpha+1}\left(\frac{h_1(s,c)}{\sqrt{H_1(s,c)}} - \frac{h_2(d,s)}{\sqrt{H_2(d,s)}}\right)^{\alpha+1}\right] ds > 0.$$

5. Examples

In this section, we will show the application of our oscillation criteria in two examples. Firstly we will give an example to illustrate Theorem 3.2.

Example 5.1. Consider the second order nonlinear differential equation

$$(t^4(x'(t))^3)' + \frac{t^\lambda}{27}x^5(t) = 0, \tag{38}$$

where $r(t) = t^4$, $\alpha = 3$, $\beta = 5$, $q(t) = t^\lambda/27$, and $\lambda > 0$ is a constant. Let $\psi(t) = 1$, $n = \alpha + 1 = 4$, and $\Phi(t,s,l) = (t-s)(s-l)$. Then we have

$$\int_{t_0}^\infty t^{-\frac{4}{3}} dt < \infty, \;\; \xi(t) = \int_t^\infty s^{-\frac{4}{3}} ds = 3t^{-\frac{1}{3}}.$$

Therefore, for all sufficiently large t_1, when $l \geq t_1$, we get

$$\limsup_{t\to\infty} F\left[\psi q \delta_1 - \frac{\psi r}{(\alpha+1)^{\alpha+1}}\left(n\phi + \frac{\psi'_+}{\psi}\right)^{\alpha+1}; l, t\right]$$

$$= \lim_{t\to\infty}\int_l^t (t-s)^4(s-l)^4\left[m_1\frac{s^\lambda}{27} - s^4\left(\frac{t-2s+l}{(t-s)(s-l)}\right)^4\right] ds$$

$$= \lim_{t\to\infty}\int_l^t \left[m_1(t-s)^4(s-l)^4\frac{s^\lambda}{27} - s^4(t-2s+l)^4\right] ds = \infty,$$

and we obtain

$$\int^\infty \left[q(t)\delta_2\xi^\alpha(t) - \frac{\alpha}{\xi(t)r^{\frac{1}{\alpha}}(t)}\right] dt = \int^\infty \left(d_0\frac{t^\lambda}{27}\frac{27}{t} - \frac{1}{t}\right) dt = \infty.$$

We can see that (10) and (21) hold. Hence, by Theorem 3.2, every solution of (38) is oscillatory.

The next example illustrates Corollary 4.1.

Example 5.2. Examine the second order nonlinear differential equation

$$((x'(t))^\alpha)' + (2n+1)!!|\text{sin}t|x^\alpha(t) = 0, \tag{39}$$

where $r(t) = 1$, $q(t) = (2n+1)!!|\sin t|$. For any $T_0 \geq t_0$, there exists a positive integer k such that $2k\pi \geq T_0$. Let $c = 2k\pi$, $d = (2k+1)\pi$. Moreover, choose $\Phi(t,s,l) = \sin(t-s)\sin(s-l)$, then we have $\Phi(d,s,c) = \sin^2 s$, $\Phi^n(d,s,c)\phi^n(d,s,c) = 2^n \sin^n s \cos^n s$. If we take $\psi(t) = 1$ and $\alpha = n-1$, where n is even integer, we have

$$\begin{aligned}
&F\left[\psi q\delta_1 - \frac{\psi r}{(\alpha+1)^{\alpha+1}}\left(n\phi + \frac{\psi'_+}{\psi}\right)^{\alpha+1}; c, d\right] \\
&\geq \int_c^d \Phi^n(d,s,c)(q(s) - \phi^n(d,s,c))ds \\
&= (2n+1)!! \int_{2k\pi}^{(2k+1)\pi} \sin^{2n+1} s ds - 2^n \int_{2k\pi}^{(2k+1)\pi} \sin^n s \cos^n s ds \\
&= (2n+1)!! \int_0^{\pi} \sin^{2n+1} s ds - 2^n \int_0^{\pi} \sin^n s \cos^n s ds \\
&\geq 2^{n+1} n! - 2^n \pi > 0.
\end{aligned}$$

From Corollary 4.1, we see that (39) is oscillatory.

One can easily see that the results obtained in [1, 2, 4–6, 8, 9, 12] cannot be applied in (38) and (39), so our results are new.

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