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AN EXACT PENALTY FUNCTION METHOD FOR SOLVING A CLASS OF NONLINEAR BILEVEL PROGRAMS†

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Abstract. In this paper, a class of nonlinear bilevel programs, i.e. the lower level problem is linear programs, is considered. Aiming at this special structure, we append the duality gap of the lower level problem to the upper level objective with a penalty and obtain a penalized problem. Using the penalty method, we give an existence theorem of solution and propose an algorithm. Then, a numerical example is given to illustrate the algorithm.

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1. Introduction

Bilevel programming (BLP) has increasingly been addressed in literature, both from the theoretical and computational points of view. It is characterized by the existence of two optimization problems in which the constraint region of the first-level problem is implicitly determined by another optimization problem. Let $F, f : R^{m+n} \to R$, $G : R^m \to R^p, g : R^{m+n} \to R^q$ be continuously differentiable functions and $x \in R^m, y \in R^n$, then the BLP problem can be formulated as [2, 5]:

$$\begin{aligned} &\min_x F(x,y) \\ &s.t. \;\; G(x) \le 0 \\ &\min_y f(x,y) \\ &s.t. \;\; g(x,y) \le 0 \end{aligned} \tag{1}$$

Due to its nested structure a BLP problem, even in the linear case, i.e, both the upper and lower level problems are linear, is a non-convex optimization

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and NP-hard problem[1, 3]. However, BLP problems have been used so wildly in management(network facility location, environment regulation), economic planning(electric power price, oil production), engineering design and optimal control[2] that many researchers have devoted to this promising field, and some feasible approaches have been developed for the BLP problem, which includes branch and bound approach[6], decent approach[10], trust region approach[4], et al. In this paper, we pay attention to a class of nonlinear BLP problem, which can be written as:

$$\begin{aligned}
&\max_{x\geq 0} F(x,y)\\
&s.t.\quad \max_{y\geq 0} f(x,y)=cx+dy\\
&s.t.\quad Ax+By\leq r
\end{aligned}\tag{2}$$

where $c\in R^m$, $d\in R^n$, $r\in R^q$, $A\in R^{q\times m}$, $B\in R^{q\times n}$.

Noted that for the above problem (2), Wang[11] proposed a global optimization approach based on genetic algorithm. However, in [11] the objective function $F(x,y)$ is limited to quadratic, and to execute the algorithm all the vertices of the feasible region of the lower level's duality must be founded in advance, then the practicability of the algorithm proposed in [11] is weakened greatly. Here, we aim to propose a practical algorithm for the nonlinear BLP problem (2). Our strategy can be outlined as follows. By appending the duality gap of the lower level problem to the upper level objective with a penalty, we obtain a penalized problem and give an existence theorem of solution using the penalty method. Then, inspired from our previous work[8], we propose an efficient algorithm for the nonlinear BLP problem (2).

Towards these ends, the rest of the paper is organized as follows. In Section 2, we present the main theoretical results. In section 3, we give the algorithm and an example to illustrate the algorithm proposed. Finally in section 4, we conclude the paper.

2. Main results

Throughout the paper, we assume that the following assumption is satisfied:

(H) The constraint region of the nonlinear BLP problem $S=\{(x,y):Ax+By\leq r\}$is nonempty and compact.

In fact, if the assumption (H) is satisfied, then similar to Theorem 1 in [9], the inducible region of problem (2) is nonempty.

For a given x, cx in problem (2) is just a constant, thus the follower's can become the following:

$$\begin{aligned}
&\max_{y\geq 0} dy\\
&s.t.\quad By\leq r-Ax
\end{aligned}\tag{3}$$

The duality of problem (3) is the following:

$$\min_{w\geq 0} w^T(r-Ax) \qquad (4)$$
$$s.t. \quad w^T B \geq d$$

Following the theory of duality, it is obvious that $w^T(r-Ax)-dy \geq 0$, and if the terms y and w solve problem (3) and (4) respectively, $w^T(r-Ax)-dy=0$. Then we can construct the following penalized problem for problem (2):

$$\begin{aligned} P(K) = \max_{(x,y,w)} & F^{'}(x,y,w,k) = F(x,y) - K[w^T(r-Ax)-dy] \\ s.t. \quad & Ax+By \leq r \\ & w^T B \geq d \\ & x,y,w \geq 0 \end{aligned} \qquad (5)$$

where $K>0$ is penalty value.

Let $W=\{w : w^T B \geq d, w \geq 0\}$ and W_v denote the extreme points of W, then we have the following result.

Theorem 1. *Suppose that assumption* (H) *is satisfied and for a given value of* w *and fixed* K*, suppose we define:*

$$Q(w,K) = \max_{(x,y)\in Z} \; F^{'}(x,y,w,K)$$

Then a solution to the problem

$$\max_{w\in W} \; Q(w,K)$$

will occur at some $w^* \in W_v$.

Proof. Firstly, we will prove that $Q(w,K)$ is convex. Let $w_1, w_2 \in W$ and $\lambda \in (0,1)$. Then

$$\begin{aligned} & Q(\lambda w_1 + (1-\lambda)w_2, K) \\ & = \max_{(x,y)\in Z} \; \{F(x,y) - K[(\lambda w_1 + (1-\lambda)w_2)^T(r-Ax)-dy]\} \\ & \leq \lambda \max_{(x,y)\in Z} \; \{F(x,y) - K[w_1^T(r-Ax)-dy]\} \\ & +(1-\lambda) \max_{(x,y)\in Z} \; \{F(x,y) - K[w_2^T(r-Ax)-dy]\} \\ & \leq \lambda Q(w_1,K) + (1-\lambda)Q(w_2,K) \end{aligned}$$

Thus, $Q(w,K)$ is convex. As W is polyhedron, then maximizing a convex function $Q(w,k)$ over W will yield an optimal solution at a vertex of W. The proof is completed. □

Theorem 1 is based on a fixed value of K. We now show that finite value of K would yield an exact solution to the overall problem (5), where the duality gap $w^T(r-Ax)-dy$ becomes zero.

Theorem 2. *Let assumption (H) hold. Let $\{(x_K, y_K, w_K)\}$ be a sequence of solutions of the problem $P(K)$, then there exists $K_1 \in R_+$, such that for all $K \geq K_1$, $w_K^T(r - Ax_K) - dy_K = 0$.*

Proof. Supposing (x^*, y^*, w^*) is the optimal solution of problem (2), then $(w^*)^T(r - Ax^*) - dy^* = 0$.

For $(x_K, y_K, w_K) \in \arg\max\ F^{'}(x, y, w, K)$, then

$$F(x_K, y_K) - K[w_K^T(r - Ax_K) - dy_K] \geq F(x^*, y^*)$$

It means that:

$$\begin{aligned} w_K^T(r - Ax_K) - dy_K &\leq \frac{F(x_K, y_K) - F(x^*, y^*)}{K} \\ &\leq \frac{\max\limits_{(x,y)\in Z}[F(x, y) - F(x^*, y^*)]}{K} \leq \frac{m}{K} \end{aligned}$$

where m is some constant. Note that $w_K^T(r - Ax_K) - dy_K \geq 0$ for all $(x_K, y_K, w_K) \in Z \times W$. Thus, as $K \to \infty$. However, since W_v is finite, $w_K^T(r - Ax_K) - dy_K = 0$ for some large finite value of K, say K_1. □

We now show that, by increasing K monotonically, we can achieve the local solution of problem (2). For that, we need the following theorem that is also the essence of penalty function methods, then we omit the proof.

Theorem 3. *In problem (5), the objective function $F(x, y)$ and the duality gap $w^T(r - Ax) - dy$ are both monotonically non-increasing in the penalty value K .*

Now, we are able to establish the following theorem which shows that the penalty is exact.

Theorem 4. *Assume that assumption (H) is satisfied. Let (x_K, y_K, w_K) be a sequence of solutions of problem $P(K)$, $K \in R_+$. Then there exists $K^* \in R_+$, such that for all $K \geq K^*$, (x_K, y_K) solves problem (2).*

Proof. Let $K^* = K_1$, then the theorem is obvious following theorem 2. □

The following theorem and remark will be used for a test of optimality in the algorithm proposed in the following section.

Theorem 5. *Assume that assumption (H) is satisfied. Let $K \in R_+$, and $u, w \in W$. Let $(x(w), y(w))$ be a solution to the problem $Q(w, K)$. Then,*

$$Q(u, K) \geq Q(w, K) - K(u - w)^T(r - Ax(w)) \quad (6)$$

Proof. As $(x(w), y(w))$ solves $Q(w, K)$, we have

$$Q(w, K) = F(x(w), y(w)) - K[w^T(r - Ax(w)) - dy(w)] \quad (7)$$

and

$$Q(u, K) \geq F(x(w), y(w)) - K[u^T(r - Ax(w)) - dy(w)] \quad (8)$$

Following (7) and (8), we can deduce that $Q(u, K) \geq Q(w, K) - K(u-w)^T(r - Ax(w))$. □

Remark. Let

$$\alpha_K(w) = \min_{u \in W} (u-w)^T(r - Ax(w)).$$

Following theorem 4, if $\alpha_K(w) < 0$, then

$$w \neq w^* \in \arg\max\{Q(w, K) : w \in W\} \tag{9}$$

By (9), if $\alpha_K(w^*) < 0$, and is reached at $u = u^*$, then we can select u^* as the next vertex to go to. If $\alpha_K(w^*) \geq 0$, thus the solution $[x(w^*), y(w^*)]$ is the best solution for the current value of K. At this value of K, if the duality gap is zero, then by Theorem 3, we are at a local optimal solution for the problem. If not, we increase K, and go through another iteration.

3. The algorithm

From Theorem 5, we can propose an algorithm, which needs only to solve a serials of nonlinear and linear programs to obtain the optimal solutions of problem (2).

Algorithm

Step 0: Choose $K > 0$(K large), $w^0 \in W_v$ and $\lambda > 0$, $i = 0$.

Step 1: Solve $\max\limits_{(x,y) \in Z} F^{'}(x, y, w^i, K)$, get a solution $[x(w^i), y(w^i)]$.

Step 2: Solve $\min\limits_{w \in W} (w - w^i)^T(r - Ax(w^i))$, obtain a solution $w^*(w^i)$ and optimal value $\alpha_K(w^i)$.

Step 3: 1) If $\alpha_K(w^i) < 0$, let $w^{i+1} = w^*(w^i)$, i=i+1, go to Step 1.

2) If $\alpha_K(w^i) \geq 0$ and $(w^i)^T(r - Ax(w^i)) - dy(w^i) > 0$, let $K = K + \lambda$, i=i, go to Step 1.

3) If $\alpha_K(w^i) \geq 0$ and $(w^i)^T(r - Ax(w^i)) - dy(w^i) = 0$, then the optimal solution of problem (2) is $[x(w^i), y(w^i)]$.

To illustrate the algorithm, we consider the following example[4].

$$\begin{aligned} &\max_x F(x, y) = -x^2 - y^2 + 16x + 5xy \\ &s.t.\ 0 \leq x \leq 20 \\ &\qquad \max_y f(x, y) = y \\ &\qquad s.t.\ x + y - 20 \leq 0 \\ &\qquad\qquad 0 \leq y \leq 10 \end{aligned} \tag{10}$$

By appending the duality gap of the lower level problem to the upper level objective with a penalty, we can obtain the following penalized problem:

$$\max_{(x,y,w)} \{-x^2 - y^2 + 16x + 5xy - K[w_1(20 - x) + 10w_2 - y]\}$$

$$\begin{aligned} s.t.\ & x - 20 \leq 0 \\ & x + y - 20 \leq 0 \\ & y - 10 \leq 0 \\ & w_1 + w_2 \geq 1 \\ & x, y, w \geq 0 \end{aligned}$$

Step 0: Choose $K = 100$ and $\lambda = 50$, $w^0 = (1, 0)$.

Step 1: Solve $\max_{(x,y)\in Z} [-x^2 - y^2 + 16x + 5xy - 100(20 - x - y)]$, obtain the solution $(x^0, y^0) = (11.84, 8.86)$.

Step 2: Solve $\min_{w\in W} [(w_1 - 1)(20 - x) + 10w_2]$, obtain the solution $w^* = (1, 0)$ and the optimal value is $\alpha_{100}(w^0) = 0$.

Step 3: As $\alpha_{100}(w^0) = 0$ and $w_1^0(20 - x^0) + 10w_2^0 - y^0 = 0$, then the optimal solution of the example is $(x^*, y^*) = (x^0, y^0) = (11.14, 8.86)$

The optimal solution from our method is the same as the result in the reference. It shows that the algorithm is feasible and efficient.

4. Conclusion

In this paper, a class of nonlinear bilevel programming problem is considered. By appending the duality gap of the lower level problem to the upper level objective with a penalty, we obtain the penalized problem of the nonlinear BLP problem. Through analyzing the characters of the penalized problem, we decompose the nonlinear BLP problem into a series of nonlinear and linear programming problems. Then we can obtain the optimal solution of the nonlinear BLP problem by traditional optimization approaches.

It deserves pointing out that the optimal solution obtained depends on the choice of the initial vector w^0. If the initial vector w^0 is chosen appropriately, the global optimal solution can be obtained. Otherwise, we can only get the local optimal solution. How to get the global optimal solution efficiently is our next objective.

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