J. Appl. Math. & Informatics Vol. 29(2011), No. 5 - 6, pp. 1511 - 1523
Website: http://www.kcam.biz

ANALYSIS OF SMOOTHING NEWTON-TYPE METHOD FOR NONLINEAR COMPLEMENTARITY PROBLEMS†

XIUYUN ZHENG* AND HONGWEI LIU

Abstract. In this paper, we consider the smoothing Newton method for the nonlinear complementarity problems with P_0-function. The proposed algorithm is based on a new smoothing function and it needs only to solve one linear system of equations and perform one line search per iteration. Under the condition that the solution set is nonempty and bounded, the proposed algorithm is proved to be convergent globally. Furthermore, the local superlinearly(quadratic) convergence is established under suitable conditions. Preliminary numerical results show that the proposed algorithm is very promising.

AMS Mathematics Subject Classification : 34G20, 34L30.
Key words and phrases : Nonlinear complementarity problem, Smoothing Newton method, P_0-function, Global and superlinear(quadratic) convergence.

1. Introduction

The nonlinear complementarity problem with P_0-function is to find a vector $x \in R^n$ such that

$$x \geq 0,\ F(x) \geq 0,\ x^T F(x) = 0. \tag{1}$$

where $F : R^n \to R^n$ is a continuously differentiable P_0-function.

The nonlinear complementarity problem has attracted much attention due to its various important applications. We refer the interested readers to the survey papers [5] and references therein.

Recently, increasing attention has been paid to smoothing Newton methods for the nonlinear complementarity problems [1, 15, 13, 7, 10, 17, 18]. Smoothing Newton methods employ a smoothing function to reformulate the problem concerned as a system of smooth equations and then to solve the smooth equations approximately by using Newton's method at each iteration. By making

Received September 12, 2010. Revised December 6, 2010. Accepted December 9, 2010.
*Corresponding author. †This work was supported by the NSF of China (No. 61072144).

the parameter to tend to zero, one can hope to obtain a solution of the original problem. Qi, Sun and Zhou [15] proposed a smoothing Newton method for the nonlinear complementarity problems and box constrained variational inequalities. It is proved that Qi-Sun-Zhou method converges to a solution of the problem under a nonsingularity assumption. Zhang, Han and Huang [6] proposed a one-step smoothing Newton method for the nonlinear complementarity problem with P_0-function based on the smoothing symmetric perturbed Fischer function. The algorithm solves only one linear system of equations, performs only one line search per iteration and has global and superlinear convergence under mild conditions. Lately, Zhang [17] proposed a different one-step smoothing Newton method based on a new smoothing function and proved that the algorithm is globally convergent. However, the convergence rate of the algorithm in [17] was not established.

Motivated by the above analysis, in this paper, we present a new smoothing Newton method for (1.1). The proposed algorithm is based on a new smoothing function which is proved to possess good properties and has a better performance than the method in [17]. It is testified that our algorithm has the following good properties: (a) We can obtain a solution of (1.1) from any accumulation point of the iteration sequence generated by our algorithm without requiring a priori the existence of an accumulation point. Moreover, if the solution set of (1.1) is nonempty and bounded, the iteration sequence is bounded. (b) Our algorithm needs only to solve one linear system of equations and perform one line search per iteration. (c) Under suitable conditions, we prove our algorithm has global and superlinear(quadratic) convergence. Moreover, our algorithm has better performance than the method in [17].

The rest of this paper is organized as follows. In the next section, we give some preliminary results and propose a new smoothing function. Based on the new smoothing function, we present a new smoothing Newton method. In Section 3, we establish the global and superlinear(quadratic) convergence of the proposed algorithm. In Section 4, we report some numerical experiments.

The following notations will be used. All vectors are column vectors, the subscript T denotes transpose, R^n (respectively, R) denotes the space of n-dimensional real column vectors (respectively, real numbers), R^n_+ (respectively, R^n_{++}) denotes the nonnegative (respectively, positive) orthants of R^n, R_+ (respectively, R_{++}) denotes the nonnegative (respectively, positive) orthants in R. Let $N = \{1, 2, \ldots, n\}$. For any $u \in R^n$, $diag\{u_i, i \in N\}$ denotes the diagonal matrix whose ith diagonal element is u_i and $vec\{u_i, i \in N\}$ the vector u. We use (u, v) for the column vector $(u^T, v^T)^T$. The symbol $\| \cdot \|$ stands for the 2-norm. S denotes the solution set of (1.1). For any $\alpha, \beta \in R_{++}$, $\alpha = O(\beta)$ (respectively, $\alpha = o(\beta)$ means α/β is uniformly bounded (respectively, tends to zero) as $\beta \to 0$.

2.Preliminaries and smoothing method

In this section, we review some useful preliminaries and propose a new smoothing function. Then we present a new smoothing Newton method for (1.1) based on the new smoothing function and show the method is well defined.

Definition 1. A matrix $M \in R^{n\times n}$ is said to be a P_0-matrix, if all its principal minors are non-negative.

Definition 2. A function $F : R^n \to R^n$ is said to be a P_0-function, if for all $x, y \in R^n$ with $x \neq y$, there exists an index $i_0 \in N$ such that $x_{i_0} \neq y_{i_0}$, $(x_{i_0} - y_{i_0})(F(x_{i_0}) - F(y_{i_0})) \geq 0$.

The Chen-Harker-Kanzow-Smale (CHKS) function[9] is defined by

$$\phi_{CHKS}(a, b) = a + b - \sqrt{(a-b)^2}. \tag{1}$$

Obviously, the CHKS function has the following property.

Lemma 1. *For any* $(a, b) \in R^2$*, we have*

$$\phi_{CHKS}(a, b) = 0 \iff a \geq 0, b \geq 0, ab = 0. \tag{2}$$

Our algorithm is based on the following smoothing function $\phi : R^3 \to R$ given by

$$\phi(\mu, a, b) = a + b - \sqrt{(a-b)^2 + \mu^2} + t(a + \sqrt{a^2 + \mu^2})(b + \sqrt{b^2 + \mu^2}), \tag{3}$$

where t is a non-negative constant.

Lemma 2. *For any* $(a, b) \in R^2$*, we have*

$$\phi(0, a, b) = 0 \iff a \geq 0,\ b \geq 0,\ ab = 0. \tag{4}$$

Proof. Suppose $\phi(0, a, b) = 0$. Then by (3), we get

$$a + b - \sqrt{(a-b)^2} + t(a + \sqrt{a^2})(b + \sqrt{b^2}) = 0. \tag{5}$$

If either $a < 0$ or $b < 0$, by (5), we have $a+b-\sqrt{(a-b)^2} = 0$, which contradicts Lemma 1. Hence, $a \geq 0, b \geq 0$. By a simple computation, we can easily get $ab = 0$ from (5). Conversely, suppose $a \geq 0, b \geq 0, ab = 0$. Then $\phi(0, a, b) = a + b - \sqrt{(a-b)^2}$. From Lemma 1, we obtain $\phi(0, a, b) = 0$. The proof is completed. □

Lemma 3. *For any* $(\mu, a, b) \in R_{++} \times R^2$*, we have*

$$\phi'_a(\mu, a, b) > 0, \phi'_b(\mu, a, b) > 0.$$

Proof. By a simple calculation, it follows that

$$\phi'_a(\mu, a, b) = 1 - \frac{a-b}{\sqrt{(a-b)^2 + \mu^2}} + t(1 + \frac{a}{\sqrt{a^2 + \mu^2}})(b + \sqrt{b^2 + \mu^2}),$$

$$\phi_b'(\mu,a,b) = 1 - \frac{a-b}{\sqrt{(a-b)^2+\mu^2}} + t(1+\frac{b}{\sqrt{b^2+\mu^2}})(a+\sqrt{a^2+\mu^2}).$$

Hence, $\phi_a'(\mu,a,b) > 0, \phi_b'(\mu,a,b) > 0$. The proof is completed. $\square$

Let $z := (\mu, x) \in R_{++} \times R^n$ and

$$H(z) := \begin{pmatrix} \sigma\mu \\ \Phi(\mu,x) \end{pmatrix}, \tag{6}$$

where

$$\Phi(\mu,x) := \begin{pmatrix} \phi(\mu,x_1,F_1(x)) \\ \vdots \\ \phi(\mu,x_n,F_n(x)) \end{pmatrix}, \tag{7}$$

and σ is a positive center parameter.

By (4), we can easily see that solving (1.1) is equivalent to solving the following equation:

$$H(z) = 0, \tag{8}$$

in the sense that their solution sets are coincident. $\square$

Lemma 4. *Let $H : R^{n+1} \to R^{n+1}$ and $\Phi : R^{n+1} \to R^n$ be defined by (6) and (7) , respectively. Then*
(a) Φ is continuously differentiable at any $z = (\mu, x) \in R_{++} \times R^n$.
(b) H is continuously differentiable at any $z = (\mu, x) \in R_{++} \times R^n$ with its Jacobian

$$H'(z) = \begin{pmatrix} \sigma & 0 \\ v(z) & w(z) \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned}
v(z) &:= vec\{\phi'(\mu,x_i,F_i(x)) : i \in N\}, \\
w(z) &:= D_1(x) + tD_2(x) + [D_3(x) + tD_4(x)]F'(x), \\
D_1(z) &:= diag\left\{1 - \frac{x_i - F_i(x)}{\sqrt{(x_i - F_i(x))^2+\mu^2}} : i \in N\right\}, \\
D_2(z) &:= diag\left\{\left[1 + \frac{x_i}{\sqrt{x_i^2+\mu^2}}\right]\left[F_i(x) + \sqrt{F_i^2(x)+\mu^2}\,\right] : i \in N\right\}, \\
D_3(z) &:= diag\left\{1 + \frac{x_i - F_i(x)}{\sqrt{(x_i - F_i(x))^2+\mu^2}} : i \in N\right\}, \\
D_4(z) &:= diag\left\{\left[1 + \frac{F_i(x)}{\sqrt{F_i^2(x)+\mu^2}}\right]\left[x_i + \sqrt{x_i^2+\mu^2}\,\right] : i \in N\right\}.
\end{aligned}$$

(c) If F is a P_0-function, then the matrix $H'(z)$ is nonsingular on $R_{++} \times R^n$.

Proof. (a). It is easy to see that Φ is continuously differentiable at any $z = (\mu, x) \in R_{++} \times R^n$.

(b). From (6) and (a), it follows that H is continuously differentiable on $R_{++} \times R^n$. For any $\mu > 0$, by a simple calculation from (6) , we can easily obtain (9) . Clearly, $(D_j(z))_{ii} > 0, j = 1, 2, 3, 4$ for all $i \in N$. Hence, $D_j(x), j = 1, 2, 3, 4$ are positive diagonal matrices. Since $t > 0$, the diagonal matrices $D_1(x)+tD_2(x)$ and $D_3(x)+tD_4(x)$ are positive. In order to show that $H'(z)$ is nonsingular, we need only to prove that the matrix $w(z)$ is. In fact, since F is a P_0-function, then $F'(x)$ is a P_0-matrix for all $x \in R^n$ by Theorem 5.8 in [12]. By a simple calculation, we can obtain that all principal minors of the matrix $(D_3(x) + tD_4(x))F'(x)$ are nonnegative. From Definition 1, the matrix $(D_3(x) + tD_4(x))F'(x)$ is a P_0-matrix. Hence, by Theorem 3.3 in [2], the matrix $D_1(x) + tD_2(x) + (D_3(x) + tD_4(x))F'(x)$ is nonsingular, which implies that the matrix $H'(z)$ is nonsingular. □

Now, we give our smoothing Newton algorithm for (1.1). Let $\gamma \in (0, 1)$ and define a function $\rho : R_{++} \times R^n \to R_+$ by

$$\rho(z) := \gamma\|H(z)\|^{\frac{1}{2}}\min\{\sigma, \|H(z)\|^{\frac{1}{2}}\}. \tag{10}$$

Obviously, we have $\rho(z) \leq \gamma\sigma\|H(z)\|^{\frac{1}{2}}$, $\rho(z) \leq \gamma\|H(z)\|$.

Algorithm 2.1.(A modified smoothing Newton method)

Step 0: Choose $\delta, \tau \in (0, 1), \sigma \in (0, 1]$ and $\mu_0 \in R_{++}$. Let $\bar{\mu} = (\sigma\mu_0, 0) \in R_{++} \times R^n$ and $x^0 \in R^n$ be an arbitrary point. Let $z^0 = (\mu_0, x^0)$. Choose $\gamma \in (0, 1)$ such that $\gamma\mu_0 < 1$ and $\gamma\|H(z^0)\| < 1$. Set $k := 0$.

Step 1: If $\|H(z^k)\| = 0$, stop. Otherwise, let $\rho_k = \rho(z^k)$.

Step 2: Compute $\Delta z^k = (\Delta\mu_k, \Delta x^k)$ by

$$H(z^k) + H'(z^k)\Delta z^k = \rho_k\bar{\mu}. \tag{11}$$

Step 3: Let m_k be the smallest nonnegative integer m such that

$$\|H(z^k + \delta^m\Delta z^k)\| \leq [1 - \tau(1 - \gamma\mu_0)\delta^m]\|H(z^k)\|, \tag{12}$$

and let $\lambda_k := \delta^{m_k}$.

Step 4: Set $z^{k+1} = z^k + \lambda_k\Delta z^k$ and $k := k + 1$. Go to Step 1.

Next, we give the following theorem which shows that Algorithm 2.1 is well-defined and generates an infinite sequence with nice properties.

Define the set Ω by

$$\Omega := \{z = (\mu, x) \in R^{n+1} : \rho(z)\mu_0 \leq \sigma\mu\}.$$

Theorem 1. *Let the sequence $\{z^k = (\mu_k, x^k)\}$ be generated by Algorithm 2.1. Then Algorithm 2.1 is well-defined.*

Proof. Suppose that $\mu_k > 0$. Since F is a continuously differentiable P_0-function, by Lemma 4 (c), it follows that the matrix $H'(z^k)$ is nonsingular. Hence, Step 2 is well-defined at the kth iteration. For any $\alpha \in (0, 1]$, let

$$R(\alpha) = H(z^k + \Delta z^k) - H(z^k) - \alpha H'(z^k)\Delta z^k. \tag{13}$$

From Lemma 4 (a), we can easily obtain that $\|R(\alpha)\| = o(\alpha)$.

By (10), (11) and (13) , we get

$$\begin{aligned}\|H(z^k + \alpha\Delta z^k)\| &= \|H(z^k) + \alpha H'(z^k)\Delta z^k + R(\alpha)\| \\ &= \|(1-\alpha)H(z^k) + \alpha\rho_k\bar{\mu} + R(\alpha)\| \\ &\leq (1-\alpha)\|H(z^k)\| + \alpha\gamma\mu_0\|H(z^k)\| + o(\alpha) \\ &= (1-\alpha(1-\gamma\mu_0))\|H(z^k)\| + o(\alpha), \end{aligned} \tag{14}$$

which implies that there exists constants $\bar{\alpha} \in (0,1]$ and $\tau \in (0,1)$ such that

$$\|H(z^k + \alpha\Delta z^k)\| \leq (1 - \tau(1-\gamma\mu_0)\alpha)\|H(z^k)\|$$

holds for any $\alpha \in (0, \bar{\alpha}]$. Hence, Step 3 is well-defined at the kth iteration. □

Theorem 2. *Let the sequence $\{z^k = (\mu_k, x^k)\}$ be generated by Algorithm 2.1. Then we have*
(a) $\mu_k \in R_{++}$ for all $k \geq 0$;
(b) the sequences $\{\rho_k\}$ and $\{\|H(z^k)\|\}$ are monotonically decreasing;
(c) $z^k \in \Omega$ for all $k \geq 0$;
(d) the sequence $\{\mu_k\}$ is monotonically decreasing.

Proof. (a) Obviously, $\mu_0 > 0$ by the choice of the starting point in Algorithm 2.1. Suppose $\mu_k > 0$. We obtain that $\mu_{k+1} > 0$ and thus the conclusion holds. In fact, for any $\alpha \in (0,1]$, by (11) and Step 4 in Algorithm 2.1, we have

$$\begin{aligned}\mu_{k+1} = \mu_k + \lambda_k\Delta\mu_k &= \mu_k + \lambda_k\frac{\rho_k\mu_0 - \sigma\mu_k}{\sigma} \\ &= (1-\lambda_k)\mu_k + \lambda_k\frac{\rho_k\mu_0}{\sigma} > 0. \end{aligned} \tag{15}$$

(b) From (12), it is obvious that $\|H(z^{k+1})\| \leq \|H(z^k)\|$, that is, the sequence $\{\|H(z^k)\|\}$ is monotonically decreasing. If $\|H(z^k)\|^{\frac{1}{2}} > \sigma$, then $\rho_k = \gamma\sigma\|H(z^k)\|^{\frac{1}{2}}$. By the definition of $\rho(\cdot)$, we have $\rho_{k+1} \leq \gamma\sigma\|H(z^{k+1})\|^{\frac{1}{2}}$. Hence, we have $\rho_k \geq \rho_{k+1}$. Otherwise, we have $\rho_k = \gamma\|H(z^k)\|$. Since $\|H(z^{k+1})\| \leq \|H(z^k)\|$ and $\rho_{k+1} \leq \gamma\|H(z^{k+1})\|$, it follows that $\rho_k \geq \rho_{k+1}$. Therefore, the sequence $\{\rho_k\}$ is monotonically decreasing.

(c) By the choice of the parameter γ and the definition of $\rho(\cdot)$, we have $\rho(z^0) \leq \sigma$. Then $\rho(z^0)\mu_0 \leq \sigma\mu_0$. Hence, $z^0 \in \Omega$. Suppose that $z^k \in \Omega$, that is, $\rho(z^k)\mu_0 \leq \sigma\mu_k$. By (11) and Step 4 in Algorithm 2.1, we have

$$\begin{aligned}\sigma\mu_{k+1} - \rho_{k+1}\mu_0 &= (1-\lambda_k)\sigma\mu_k + \lambda_k\rho_k\mu_0 - \rho_{k+1}\mu_0 \\ &\geq (\rho_k - \rho_{k+1})\mu_0 \\ &\geq 0,\end{aligned}$$

where the last inequality follows from (b). Hence, $z^{k+1} \in \Omega$.

(d) By (15), (a) and (c), we have that

$$\begin{aligned}0 < \mu_{k+1} &= (1-\lambda_k)\mu_k + \lambda_k\frac{\rho_k\mu_0}{\sigma} \leq (1-\lambda_k)\mu_k + \lambda_k\mu_k \\ &= \mu_k,\end{aligned}$$

which implies the sequence $\{\mu_k\}$ is monotonically decreasing. The whole proof of the theorem is completed. $\square$

3. Convergence analysis

In this section, we present the convergence analysis of Algorithm 2.1. By Theorem 2, Algorithm 2.1 generates an infinite sequence $\{z^k\}$. In the following, we will prove that any accumulation point of the iteration sequence $\{z^k\}$ is a solution of (8).

Assumption 1. The solution set $S = \{x \geq 0, F(x) \geq 0, x^T F(x) = 0\}$ of (1) is nonempty and bounded.

Define the level set

$$L(\mu, c) := \{x \in R^n : \|H(\mu, x)\| \leq c\}, \tag{1}$$

where $\mu > 0$ and $c > 0$.

Lemma 5. *For any* $0 < \mu_1 \leq \mu_2$*, the set*

$$L(c) := \bigcup_{\mu_1 \leq \mu \leq \mu_2} L(\mu, c),$$

is bounded.

Proof. Suppose $L(c)$ is unbounded. Then for some fixed $c > 0$, there exists a sequence $\{(\mu_k, x^k)\}$ such that

$$\mu_1 \leq \mu_k \leq \mu_2, \ \|H(\mu_k, x^k)\| \leq c, \ \|x^k\| \to +\infty.$$

Since $\{x^k\}$ is unbounded, the index set $J := \{i \in N : \{x_i^k\} \text{ is unbounded}\} \neq \emptyset$. Without loss of generality, we can assume that $|x_j^k| \to +\infty$ for all $j \in J$. Define the sequence $\{\hat{x}^k\}$ as follows:

$$\hat{x}_j^k = \begin{cases} x_j^k & \text{if } j \notin J, \\ 0 & \text{if } j \in J. \end{cases} \tag{2}$$

Then, $\{\hat{x}^k\}$ is bounded. Since F is a P_0-function, by Definition 2, there exists an index $j \in N$ such that

$$(x_j^k - \hat{x}_j^k)(F_j(x^k) - F_j(\hat{x}^k)) \geq 0. \tag{3}$$

Together with (3.2), we have

$$x_j^k(F_j(x^k) - F_j(\hat{x}^k)) \geq 0, j \in J. \tag{4}$$

Since $j \in J$, we get $|x_j^k| \to +\infty$.

In the following, we consider two cases:

Case 1: Suppose $\{x_j^k\} \to +\infty$. Since $F_j(\hat{x}^k)$ is bounded by the continuity of F_j, (3.4) implies that $F_j(x^k) \not\to -\infty$. Since $\mu_1 \leq \mu_k \leq \mu_2$, we have $x_j^k + F_j(x^k) - \sqrt{(x_j^k - F_j(x^k))^2 + \mu_k^2}$ is bounded below and $t(x_j^k + \sqrt{(x_j^k)^2 + \mu_k^2})(F_j(x^k) + \sqrt{(F_j(x^k))^2 + \mu_k^2}) \to +\infty$. Thus, $\phi(\mu_k, x_j^k, F_j(x^k)) \to +\infty$.

Case 2: Suppose $\{x_j^k\} \to -\infty$. Since $F_j(\hat{x}^k)$ is bounded by the continuity of F_j, (3.4) implies that $F_j(x^k) \not\to +\infty$. Since $\mu_1 \le \mu_k \le \mu_2$, we have $x_j^k + F_j(x^k) - \sqrt{(x_j^k - F_j(x^k))^2 + \mu_k^2} \to -\infty$ and $t(x_j^k + \sqrt{(x_j^k)^2 + \mu_k^2})\ (F_j(x^k) + \sqrt{(F_j(x^k))^2 + \mu_k^2})$ is bounded above. Thus, $\phi(\mu_k, x_j^k, F_j(x^k)) \to -\infty$.

In either case, we have that $\|\Phi(\mu_k, x^k)\| \to +\infty$, and thus $\|H(\mu_k, x^k)\| \to +\infty$, which contradicts $\|H(\mu_k, x^k)\| \le c$. The proof is completed. □

From Lemma 5, we can easily obtain the following result.

Corollary 1. *Suppose that F is a P_0-function and $\mu > 0$. Then we have*

$$\lim_{\|x\| \to +\infty} \|H(z)\| \to +\infty.$$

Lemma 6. *Let $H(\cdot)$ be defined by (6) and $\{z^k = (\mu_k, x^k)\}$ be the iteration sequence generated by Algorithm 2.1. Then, the sequence $\{H(z^k)\}$ is convergent. If it does not converge to zero, then the sequence $\{z^k\}$ is bounded.*

Proof. From Theorem 2 (b), we can easily obtain the sequences $\{H(z^k)\}$ and $\{\rho_k\}$ are convergent. Then there exists $\hat{\rho}$ such that $\rho_k \to \hat{\rho}$ as $k \to \infty$. If $\{H(z^k)\}$ does not converge to zero, then there exists $\hat{H} > 0$ such that $H(z^k) \to \hat{H}$ as $k \to \infty$. Then we have $\hat{\rho} = \gamma \hat{H}^{\frac{1}{2}} \min\{\sigma, \hat{H}^{\frac{1}{2}}\} > 0$.

By (15) and Theorem 2 (c), we can obtain that

$$\mu_{k+1} = \mu_k + \lambda_k \Delta\mu_k = (1 - \lambda_k)\mu_k + \frac{\lambda_k \rho_k \mu_0}{\sigma} \le \mu_k. \tag{5}$$

Thus, we have that

$$0 < \hat{\rho}\mu_0 \le \mu_k \le \mu_0, \text{for all } k \ge 0. \tag{6}$$

Let $c := \|H(z^0)\|$ where z^0 is the starting point in Algorithm 2.1 and $L(c) := \bigcup_{\hat{\rho}\mu_0 \le \mu_k \le \mu_0} L(\mu_k, c)$, where $L(\mu_k, c)$ is defined by (3.1). Since $x^k \in L(\mu_k, c)$, it is obvious that $x^k \in L(c)$. From Lemma 5, we have the sequence $\{x^k\}$ is bounded and thus the sequence $\{z^k\}$ is bounded. The proof is completed. □

Theorem 3. *Let $\{z^k = (\mu_k, x^k)\}$ be the iteration sequence generated by Algorithm 2.1. Suppose Assumption 1 holds. Then*
(a) the sequences $\{\|H(z^k)\|\}$ and $\{\mu_k\}$ tend to zero, and hence any accumulation point of $\{z^k\}$ is a solution of (8);
(b) $\{z^k\}$ is bounded and hence it has at least one accumulation point $\tilde{z} = (\tilde{\mu}, \tilde{x})$ with $H(\tilde{z}) = 0$ and thus $\tilde{x} \in S$.

Proof. From Lemma 6, we have that the limit point of $\{\|H(z^k)\|\}$ exists, denoted by $\hat{H}$. Suppose $\{\|H(z^k)\|\}$ does not converge to zero. Then we have $\hat{H} > 0$. By Lemma 6, it follows that $\{z^k\}$ is bounded. Let $\tilde{z} = (\tilde{\mu}, \tilde{x})$ be an accumulation point of $\{z^k\}$.

Without loss of generality, we assume that $\{z^k\}$ converges to $\tilde{z}$. Then, by the continuity of $H(\cdot)$, we have $\|H(\tilde{z})\| = \hat{H} > 0$. By the definition and continuity of $\rho(\cdot)$, we can get $\{\rho_k\}$ converges to $\tilde{\rho} > 0$. By (12), we have $\lim_{k \to +\infty} \lambda_k = 0$.

From Step 3 in Algorithm 2.1, it follows that

$$\|H(z^k + \delta^{m_k-1}\Delta z^k)\| > [1 - \tau(1-\gamma\mu_0)\delta^{m_k-1}]\|H(z^k)\|, \tag{7}$$

which implies

$$\frac{\|H(z^k + \delta^{m_k-1}\Delta z^k)\| - \|H(z^k)\|}{\delta^{m_k-1}} > -\tau(1-\gamma\mu_0)\|H(z^k)\|. \tag{8}$$

Let $k \to \infty$ in (8), we have

$$-\tau(1-\gamma\mu_0)\|H(\tilde{z})\| \leq \frac{H(\tilde{z})^T H'(\tilde{z})\Delta\tilde{z}}{\|H(\tilde{z})\|}. \tag{9}$$

On the other hand, let $k \to \infty$ in (11), we have

$$H(\tilde{z}) + H'(\tilde{z})\Delta\tilde{z} = \tilde{\rho}\bar{\mu}. \tag{10}$$

Combing (9) and (10), we get

$$\begin{aligned}-\tau(1-\gamma\mu_0)\|H(\tilde{z})\| &\leq \frac{H(\tilde{z})^T(\tilde{\rho}\bar{\mu} - H(\tilde{z}))}{\|H(\tilde{z})\|} = \tilde{\rho}\mu_0 - \|H(\tilde{z})\| \\ &\leq (\gamma\mu_0 - 1)\|H(\tilde{z})\|.\end{aligned}$$

Therefore, we deduce that $(1-\gamma\mu_0)(1-\tau) \leq 0$, which contradicts the fact $\gamma\mu_0 < 1$ and $\tau < 1$. Hence, $\hat{H} = 0$ and thus $\tilde{\mu} = 0$, $\tilde{z}$ is a solution of (1.1). (a) is proved.

We now give the proof of (b). From Theorem 2 (d), it follows that $\{\mu_k\}$ is bounded. It is sufficient to show that $\{x^k\}$ is bounded. From (a), we obtain $\|H(z^k)\| \to 0$ as $k \to \infty$. Therefore, by the famous mountain pass theorem [4] and by following the similar proof lines of [6], we get that $\{x^k\}$ is bounded and hence $\{z^k\}$ is. Hence, $\{z^k\}$ has at least one accumulation point $\tilde{z}$. By (a), we have $H(\tilde{z}) = 0$ and $\tilde{x} \in S$. □

Next we give the analysis of the convergence rate of Algorithm 2.1. To establish the convergence rate of Algorithm 2.1, we need the concept of semismoothness, which was originally introduced in [11] for functionals and extended in [16] for vector-valued functions. The composition of semismooth functions is still a semismooth function [11].

Lemma 7. *Suppose that $F : R^n \to R^n$ is a locally Lipschitzian function. Then (1) F has the generalized Jacobian $\partial F(x)$ as in Clarke* [3]. *And $F'(x;h)$, the directional derivative of F at x in the direction h, exists for any $h \in R^n$ if F is semismooth at x. Also, F is semismooth at $x \in R^n$ if and only if all its component functions are.*
(2) F is semismooth at x if and only if for any $V \in \partial F(x+h), h \to 0$,

$$\|Vh - F'(x;h)\| = o(\|h\|).$$

Also, $\|F(x+h) - F(x) - F'(x;h)\| = o(\|h\|)$.
(3) F is strongly semismooth at x if and only if for any $V \in \partial F(x+h), h \to 0$,

$$\|Vh - F'(x;h)\| = O(\|h\|^2).$$

Also, $\|F(x+h)-F(x)-F'(x;h)\| = O(\|h\|^2)$.

Similar to the proof of Theorem 8 in [15], we can present the following theorem.

Theorem 4. *Suppose that Assumption 1 is satisfied and* $\tilde{z}=(\tilde{\mu},\tilde{x})$ *is an accumulation point of the iteration sequence* $\{z^k\}$ *generated by Algorithm 2.1. If all* $V\in\partial H(\tilde{z})$ *are nonsingular. Then,*

(a) $\lambda_k\equiv 1$ *for all* z^k *sufficiently close to* $\tilde{z}$*;*

(b) $\{z^k\}$ *converges to* $\tilde{z}$ *superlinearly, i.e.,* $\|z^{k+1}-\tilde{z}\| = o(\|z^k-\tilde{z}\|)$ *and* $\mu_{k+1}=o(\mu_k)$*;*

(c) $\{z^k\}$ *converges to* $\tilde{z}$ *quadratically, if* F' *is Lipschitz continuous on* R^n*, i.e.,* $\|z^{k+1}-\tilde{z}\| = O(\|z^k-\tilde{z}\|^2)$ *and* $\mu_{k+1}=O(\mu_k^2)$. □

Proof. From Theorem 3, $H'(\tilde{z})=0$ and $\tilde{z}$ is a solution of (8). Since all $V\in\partial H(\tilde{z})$ are nonsingular, it follows from Proposition 3.1 in [16] that for all z^k sufficiently close to $\tilde{z}$, we have $\|H'(z^k)^{-1}\|\le C$ where $C>0$ is a constant. Since $H(\cdot)$ is semismooth (strongly semismooth if F' is Lipschitz continuous on R^n, respectively) at $\tilde{z}$, by Lemma 7, for all z^k sufficiently close to $\tilde{z}$, we have

$$\|H(z^k)-H(\tilde{z})-H'(z^k)(z^k-\tilde{z})\| = o(\|z^k-\tilde{z}\|)\ (O(\|z^k-\tilde{z}\|^2)). \tag{11}$$

Notice that $H(\cdot)$ is locally Lipschitz continuous near $\tilde{z}$. Hence, for all z^k sufficiently close to $\tilde{z}$, we get

$$\|H(z^k)\| = O(\|z^k-\tilde{z}\|). \tag{12}$$

From (11), (11) and (12) and the definition of $\rho(\cdot)$, it follows that

$$\begin{aligned}\|z^k+\triangle z^k-\tilde{z}\| &= \|z^k+H'(z^k)^{-1}[-H(z^k)+\rho_k\bar{\mu}]-\tilde{z}\|\\ &\le \|H'(z^k)^{-1}\|(\|H(\tilde{z})-H(z^k)+H'(z^k)(z^k-\tilde{z})\|+\rho_k\mu_0)\\ &\le C[\|H(\tilde{z})-H(z^k)+H'(z^k)(z^k-\tilde{z})\|+\gamma\sigma\mu_0\|H(z^k)\|]\\ &= o(\|z^k-\tilde{z}\|)\ (=O(\|z^k-\tilde{z}\|^2))\end{aligned} \tag{13}$$

Similar to the proof of Theorem 3.1 in [14], for all z^k sufficiently close to $\tilde{z}$, we get

$$\|z^k-\tilde{z}\| = O(\|H(z^k)-H(\tilde{z})\|). \tag{14}$$

Therefore, for all z^k sufficiently close to $\tilde{z}$, we have

$$\begin{aligned}\|H(z^k+\triangle z^k)\| &= O(\|z^k+\triangle z^k-\tilde{z}\|)\\ &= o(\|z^k-\tilde{z}\|)\ = O(\|z^k-\tilde{z}\|^2)\\ &= o(\|H(z^k)-H(\tilde{z})\|)\ (=O(\|H(z^k)-H(\tilde{z})\|^2))\\ &= o(\|H(z^k)\|)\ (=O(\|H(z^k)\|^2)).\end{aligned} \tag{15}$$

It follows from Theorem 3 that for all z^k sufficiently close to $\tilde{z}$, $\lambda_k=1$. Thus, $z^{k+1}=z^k+\triangle z^k$, which combining with (13) implies

$$\|z^{k+1}-\tilde{z}\| = o(\|z^k-\tilde{z}\|)\ (=O(\|z^k-\tilde{z}\|^2)).$$

By (11), for all sufficiently large k, we have

$$\mu_{k+1} = \mu_k + \triangle\mu_k = \rho_k\mu_0 = O(\|H(z^k)\|). \tag{16}$$

Therefore, for all z^k sufficiently close to $\tilde{z}$, $\mu_{k+1} = o(\mu_k)$ ($\mu_{k+1} = O(\mu_k^2)$). □

4. Numerical experiments

In this section, we report some numerical experiments. We tested our algorithm and Algorithm 2.4 in [17] respectively on PC with 1.60 GHz CPU professor and 1.5G memory and Windows XP operation system. The numerical results of all algorithms are reported on Table 1. The stopping rule is $\|H(z^k)\| \leq 10^{-6}$ and the starting point x^0 is randomly generated in the interval (0,1).

In Tables 1 and 2, the meanings of each column are listed below:

P: the tested problem;
Dim: the dimension of the problem;
NI: the total number of iterations;
Res: the value of $\|H(z^k)\|$ when the stopping rule is satisfied;
CPU: the CPU time for solving the underlying problem in seconds.

In our tests, for the proposed algorithm, we always set $\delta = 0.95$, $\tau = 10^{-2}$, $\mu_0 = 10^{-3}$ and t is randomly generated in the interval (0,5); for Algorithm 2.4 in [17], we choose the parameters as $\delta = 0.95$, $\gamma = 10^{-3}$, $\sigma = 0.8$, $\mu_0 = 10^{-3}$ and τ is randomly generated in (0,5). In the following, we give the test problems in detail, which are tested five times by using the proposed algorithm and Algorithm 2.4 in [17], respectively.

Problem 4.1. This is the fourth example tested in [8], where $F(x) = (f_1(x), \ldots, f_5(x))$ given by

$$f_i(x) = 2\exp(\sum_{i=1}^{5}(x_i - i + 2)^2)(x_i - i + 2), 1 \leq i \leq 5.$$

This problem is also tested by Qi-Sun-Zhou [15], etc, which has one solution: (0,0,1,2,3). The tested results are listed in Table1.

Problem 4.2. This is the fifth example tested in [8], where $F(x) = (f_1(x), \ldots, f_4(x))$ given by

$$\begin{aligned}
&f_1(x) = -x_2 + x_3 + x_4,\\
&f_2(x) = x_1 - (4.5x_3 + 2.7x_4)/(x_2 + 1),\\
&f_3(x) = 5 - x_1 - (0.5x_3 + 0.3x_4)/(x_3 + 1),\\
&f_4(x) = 3 - x_1.
\end{aligned}$$

This problem is also tested by Qi-Sun-Zhou [15], etc, which has infinitely many solutions: $(\lambda, 0, 0, 0)$, where $\lambda \in [0, 3]$. The tested results are listed in Table 1.

Problem 4.3. This problem is a nonlinear complementarity, which is taken from [17], defined by

$$F(x) = D(x) + Mx + q,$$

Table 1. Computational results for Problems 4.1 and 4.2

P	Dim	Algorithm 2.4			Proposed algorithm		
		NI	Res	CPU	NI	Res	CPU
4.1	5	18	1.1832e-007	4.1463e-003	14	8.0324e-012	3.2014e-003
	5	31	8.4084e-007	6.8096e-003	16	7.1019e-010	3.6527e-003
	5	26	4.8902e-007	5.7071e-003	15	3.4987e-009	3.4267e-003
	5	20	5.2526e-008	4.5642e-003	16	5.5450e-009	3.8983e-003
	5	36	9.2312e-007	7.8450e-003	16	9.5096e-010	3.8434e-003
4.2	4	11	6.6801e-007	2.2201e-003	3	2.7238e-009	8.3736e-004
	4	17	6.4827e-007	3.2927e-003	4	1.4028e-011	1.0018e-003
	4	12	6.8768e-007	2.3441e-003	4	4.6879e-012	9.8412e-004
	4	18	4.9124e-007	3.3944e-003	3	6.0615e-008	8.4231e-004
	4	20	6.9740e-007	3.6431e-003	4	4.4689e-013	1.0023e-003

Table 2. Computational results for Problem 4.3

Dim	Algorithm 2.4			Proposed algorithm		
	NI	Res	CPU	NI	Res	CPU
100	33	7.1485e-007	8.1854e-002	19	7.6769e-007	4.7559e-002
100	36	4.8804e-007	8.5149e-002	21	8.3991e-007	5.2393e-002
100	41	6.2296e-007	1.1817e-001	22	7.5553e-007	5.5149e-002
100	53	8.1238e-007	1.2391e-001	20	5.0326e-007	5.1009e-002
100	43	9.5949e-007	1.0575e-001	21	1.6449e-007	5.3153e-002
200	46	7.4997e-007	5.8056e-001	22	2.9439e-007	2.8395e-001
200	60	6.9471e-007	7.7789e-001	20	3.2470e-007	2.5379e-001
200	62	9.2396e-007	7.9508e-001	28	1.6676e-007	3.5690e-001
200	55	9.3677e-007	6.9999e-001	26	7.4338e-007	3.3682e-001
200	52	9.8390e-007	6.6783e-001	24	5.0895e-007	3.0456e-001
300	52	9.3808e-007	2.1754e+000	19	9.1306e-007	7.9048e-001
300	60	7.7297e-007	2.5111e+000	23	8.1698e-007	9.5973e-001
300	69	9.0646e-007	2.8845e+000	26	8.2650e-007	1.0831e+000
300	71	7.2812e-007	2.9716e+000	25	2.3413e-007	1.0424e+000
300	55	6.1184e-007	2.2751e+000	22	7.2638e-007	9.1250e-001

where $M = A^T A + B$, $A \in R^{n\times n}$ and its entries are randomly generated in the interval $(-5, 5)$ and a skew symmetric matrix B is generated in the same way; The components of q are generated from a uniform distribution in the interval $(-50, 50)$; The components of $D(x)$ are $D_j(x) = d_j \arctan(x_j)$, where $d_j \in (0, 5)$. In this test, we choose $n = 100, 200, 300$ as the dimension of the problem, respectively. The results are listed in Table 2.

From Tables 1 and 2, it is observed that the number of iterations and CPU time in our algorithm are better than that in Algorithm 2.4 [17]. In Table 2, for the Problem 4.3, we tested it by using different dimensions. We can see that the number of iterations in our algorithm do not change much. Therefore, we obtain that our algorithm is more effective than Algorithm 2.4 in [17].

References

1. B. Chen and X. Chen, *A global and local superlinear continuation smoothing method for* P_0 *and* R_0 NCP *or monotone* NCP, SIAM J. Optimization **9**, (1999) 624-645.
2. B. Chen and P. Harker, *A non-interior-point continuation method for linear complementarity problems*, SIAM J. Matrix Anal. Applications **14 (2)**, (1993), 1168-1190.
3. F. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.
4. F. Facchinei and C. Kanzow, *Beyond monotonicity in regularization methods for nonlinear complementarity problems*, SIAM J. Control and Optimization **37 (2)**, (1999), 1150-1161.
5. P. Harker and J.S. Pang, *Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications*, Math. Programming **48**, (1990), 161-220.
6. Z. Huang and J. Han and D. Xu and L. Zhang, *The non-interior continuation methods for solving the* P_0 *function nonlinear complementarity problem*,Science in China **44**, (2001), 1107-1114.
7. Z. Huang and L. Qi and D. Sun, *Sub-quadratic convergence of a smoothing newton algorithm for the* P_0*- and monotone* LCP, Math. Programming **99**, (2004), 423-441.
8. H. Jiang and L. Qi,*A new nonsmooth equations approach to nonlinear complementarity problems*, SIAM J. Control and Optimization **10**, (2000), 315-330.
9. C. Kanzow, *Some noninterior continuation methods for linear complementarity problems*, SIAM J. Matrix Anal. and Application **17**, (1996), 851-868.
10. C. Ma and X. Chen, *The convergence of a one-step smoothing newton method for* P_0-NCP *based on a new smoothing* NCP*-function* J. Comput. Appl. Mathematics **216**, (2008), 1-13.
11. R. Mifflin, *Semismooth and semiconvex functions in constrained optimization*, SIAM J. Control and Optimization **15**, (1977), 957-972.
12. J. Moré and W.C. Rheinboldt, *On* P*-and* S*-functions and related classes of n-dimensional non-linear mappings*, Linear Algebra and Applications **6**, (1973), 45-68.
13. H. Qi, *A regularized smoothing newton method for box constrained variational inequality problems with* P_0*-functions*, SIAM J. Optimization **10 (1)**, (2000), 315-330.
14. L. Qi, *Convergence analysis of some algorithms for solving nonsmooth equations*, Math. Oper. Research **18**, (1993), 227-244.
15. L. Qi and D. Sun and G. Zhou, *A new look at smoothing newton methods for nonlinear complementarity problems and box constrained variational inequality problems*, Math. Programming **87**, (2000), 1-35.
16. L. Qi and J. Sun,*A nonsmooth version of Newton's method*, Math. Programming **58**, (1993), 353-367.
17. J. Zhang and K. Zhang, *A variant smoothing newton method for* P_0-NCP *based on a new smoothing function*, J. Compu. Appl. Mathematics **225**, (2008), 1-8.
18. L.P. Zhang and J.Y. HAN and Z.H. Huang, *Superlinear/quadratic one-step smoothing newton method for* P_0-NCP, Acta Math. Sin. English Series **1**, (2005), 117-1128.

Xiuyun Zheng studied at Xidian University. Her research interests numerical optimization.

Department of Applied Mathematics, Xidian University, Xi'an 710071, China.
e-mail: xyzhengxd@gmail.com