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IDEAL THEORY OF d-ALGEBRAS BASED ON $\mathcal{N}$-STRUCTURES

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ABSTRACT. The notions of $\mathcal{N}$-subalgebra, (positive implicative) $\mathcal{N}$-ideals of d-algebras are introduced, and related properties are investigated. Characterizations of an $\mathcal{N}$-subalgebra and a (positive implicative) $\mathcal{N}$-ideals of d-algebras are given. Relations between an $\mathcal{N}$-subalgebra, an $\mathcal{N}$-ideal and a positive implicative $\mathcal{N}$-ideal of d-algebras are discussed.

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1. Introduction

J. Neggers and H. S. Kim ([13]) introduced the idea of a d-algebra as a generalization of BCK-algebras introduced by Y. Imai and K. Iséki ([7]). This class of algebras has been studied in rather great detail and is of current interest to many researchers ([2,3,9,10,11,12]). Beside d-algebras, other generalizations of BCK-algebras include the class of BCI-algebras, also introduced by Y. Imai and K. Iséki ([8]), the class of BCH-algebras, introduce by Q. P. Hu and X. Li ([5,6]), of which the class of BCI-algebras is a proper subclass. The class of BCK-algebras is a proper subclass of the class of d-algebras, and it has been shown that many constructions on the class of d-algebras leave this class invariant, but not so the class of BCK-algebras. For example, the mirror algebra of a d-algebra is a d-algebra, but the mirror algebra of a BCK-algebra is not necessarily a BCK-algebra ([2]). J. Neggers, Y. B. Jun and H. S. Kim ([12]) have studied the ideal theory of d-algebras, introducing the notions of d-subalgebra, d-ideal, $d^{\#}$-ideal and d^{*}-ideal, and various relations among them. After that some further aspects were studied in [3,9,10,11].

In this paper, we introduce the notions of $\mathcal{N}$-subalgebra, (positive implicative) $\mathcal{N}$-ideals of d-algebras, investigate their related properties. We also find

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characterizations of an $\mathcal{N}$-subalgebra and a (positive implicative) $\mathcal{N}$-ideals of d-algebras. We discuss relations between an $\mathcal{N}$-subalgebra, an $\mathcal{N}$-ideal and a positive implicative $\mathcal{N}$-ideal of d-algebras.

2. Preliminaries

A *d-algebra* ([13]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y \in X$.

For brevity we also call X a *d-algebra*. In X, we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$. Note that if X is a d-algebra with $x * 0 = 0$ for any $x \in X$, then $x = 0$.

A BCK-algebra is a d-algebra $(X; *, 0)$ satisfying the additional axioms:

(IV) $((x * y)) * (x * z)) * (z * y) = 0$,
(V) $(x * (x * y)) * y = 0$, for all $x, y, z \in X$.

Example 2.1 ([12])**.** Let $\mathbb{R}$ be the set of all real numbers and define $x * y := -x(x-y), x, y \in \mathbb{R}$, where "$\cdot$" and "$-$" are the ordinary product and subtraction of real numbers. Then $(\mathbb{R}; \cdot, 0)$ is a d-algebra, not a BCK-algebra, since $5 * 0 = -5^2 = -25 \neq 5$.

Definition 2.2 ([12])**.** Let $(X; *, 0)$ be a d-algebra and $\emptyset \neq I \subseteq X$. I is a *d-subalgebra* of X if $x * y \in I$ whenever $x \in I$ and $y \in I$. I is called a *BCK-ideal* (briefly, an *ideal*) of X if it satisfies:

(D_0) $0 \in I$,
(D_1) $x * y \in I$ and $y \in I$ imply $x \in I$.

I is a called a *d-ideal* of X if it satisfies (D_1) and

(D_2) $x \in I$ and $y \in X$ imply $x * y \in I$, i.e., $I * X \subseteq I$.

I is called a *$d^{\#}$-ideal* of X if it satisfies $(D_1), (D_2)$ and

(D_3) $x * z \in I$ whenever $x * y \in I$ and $y * z \in I$, for any $x, y, z \in X$.

I is called a *d^*-ideal* of X if it satisfies (D_1), (D_2), (D_3) and

(D_4) $x * y \in I$ and $y * x \in I$ imply $(x * z) * (y * z) \in I$ and $(z * x) * (z * y) \in I$, for any $x, y, z \in X$.

Note that d^*-ideal $\subsetneq$ $d^{\#}$-ideal $\subsetneq$ d-ideal in a d-algebra(See [13]).

Definition 2.3 ([13])**.** Let $(X; *, 0)$ be a d-algebra and $x \in X$. Define $x * X := \{x * a | a \in X\}$. X is said to be *edge* if for any $x \in X$, $x * X = \{x, 0\}$.

Lemma 2.4 ([13])**.** *Let $(X; *, 0)$ be an edge d-algebra. Then $x * 0 = x$ for any $x \in X$.*

Definition 2.5 ([12])**.** A d-algebra X is called a d^*-algebra if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$.

Clearly, a BCK-algebra is a d^*-algebra, but the converse need not be true (see [12]).

Theorem 2.6 ([12]). *In a d^*-algebra, every BCK-ideal is a d-ideal.*

Corollary 2.7 ([12]). *In a d^*-algebra, every BCK-ideal is a d-subalgebra.*

Definition 2.8 ([1]). A d-algebra X is *positive implicative* if for all $x, y, z \in X, (x * y) * z = (x * z) * (y * z)$.

Example 2.9 ([1]). Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	c	0

Then $(X; *, 0)$ is a positive implicative d-algebra.

Proposition 2.10. *If X is a positive implicative d^*-algebra, then $x * 0 = x$ for any $x \in X$.*

Proof. Since X is a d^*-algebra, we have $(x * 0) * x = 0$ for any $x \in X$. Using (I) and (II), we obtain

$$\begin{aligned}(x * (x * 0)) * 0 =& (x * (x * 0)) * (0 * (x * 0)) \\ =& (x * 0) * (x * 0) = 0.\end{aligned}$$

Hence $x * (x * 0) = 0$. By (III), we have $x * 0 = x$. □

3. $\mathcal{N}$-subalgebras and (positive implicative) $\mathcal{N}$-ideals

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, *$\mathcal{N}$-function* on X). By an *$\mathcal{N}$-structure* we mean an ordered pair (X, φ) of a d-algebra X and an $\mathcal{N}$-function on X. In what follows, let X denote a d-algebra and φ an $\mathcal{N}$-function on X unless specified.

Definition 3.1. An $\mathcal{N}$-structure (X, φ) is called a *subalgebra of X based on $\mathcal{N}$-function φ* (briefly, *$\mathcal{N}$-subalgebra* of X) if φ satisfies the following assertion:

(N1) $\varphi(x * y) \leq \max\{\varphi(x), \varphi(y)\}, \forall x, y \in X$.

For any $\mathcal{N}$-function φ on X and $t \in [-1, 0)$, the set

$$C(\varphi; t) := \{x \in X | \varphi(x) \leq t\}$$

is called a *closed (φ, t)-cut* of φ, and the set

$$O(\varphi; t) := \{x \in X | \varphi(x) < t\}$$

is called an *open (φ, t)-cut* of φ.

Theorem 3.2. *Let (X, φ) be an $\mathcal{N}$-structure of X and φ. Then (X, φ) is an $\mathcal{N}$-subalgebra of X if and only if every non-empty closed (φ, t)-cut of φ is a subalgebra of X for all $t \in [-1, 0)$.*

Proof. Assume that (X, φ) is an $\mathcal{N}$-subalgebra of X and let $t \in [-1, 0)$ be such that $C(\varphi; t) \neq \emptyset$. Let $x, y \in C(\varphi, t)$. Then $\varphi(x) \leq t$ and $\varphi(y) \leq t$. It follows from (N1) that

$$\varphi(x * y) \leq \max\{\varphi(x), \varphi(y)\} \leq t$$

so that $x * y \in C(\varphi; t)$. Hence $C(\varphi; t)$ is a subalgebra of X.

Conversely, suppose that every non-empty closed (φ, t)-cut of φ is a subalgebra of X for all $t \in [-1, 0)$. If (X, φ) is not an N-subalgebra of X, then there exist $a, b \in X$ and $t_0 \in [-1, 0)$ such that $\varphi(a * b) > t_0 \geq \max\{\varphi(a), \varphi(b)\}$. Hence $a, b \in C(\varphi; t_0)$ and $a * b \notin C(\varphi; t_0)$. This is a contradiction. Thus $\varphi(x * y) \leq \max\{\varphi(x), \varphi(y)\}$ for all $x, y \in X$. □

Corollary 3.3. *If (X, φ) is an $\mathcal{N}$-subalgebra of X, then every non-empty open (φ, t)-cut of φ is a subalgebra of X for all $t \in [-1, 0)$.*

Proof. Straightforward. □

Lemma 3.4. *Every $\mathcal{N}$-subalgebra of X satisfies the following inequality:*

$$\varphi(x) \geq \varphi(0), \forall x \in X.$$

Proof. Using (N1), we have

$$\varphi(0) = \varphi(x * x) \leq \max\{\varphi(x), \varphi(x)\} = \varphi(x)$$

for all $x \in X$. □

Proposition 3.5. *Let X be an edge d-algebra. If every $\mathcal{N}$-subalgebra (X, φ) of X satisfies the following inequality:*

$$\varphi(x * y) \leq \varphi(y) \text{ for any } x, y \in X,$$

then φ is a constant function.

Proof. Let $x \in X$. Using Lemma 2.4 and assumption, we have $\varphi(x) = \varphi(x * 0) \leq \varphi(0)$. It follows from Lemma 3.4 that $\varphi(x) = \varphi(0)$, and so φ is a constant function. □

Definition 3.6. An $\mathcal{N}$-structure (X, φ) is called an *ideal of X based on $\mathcal{N}$-function φ* (briefly, *$\mathcal{N}$-ideal* of X) if φ satisfies the following assertion:

(N2) $\varphi(0) \leq \varphi(x), \forall x \in X$.
(N3) $\varphi(x) \leq \max\{\varphi(x * y), \varphi(y)\}, \forall x, y \in X$.

Example 3.7. Let $X := \{0, a, b, c\}$ be a set with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	a
c	c	c	c	0

Then $(X;*,0)$ is a d-algebra, which is not a BCK/BCI-algebra. Define a $\mathcal{N}$-function φ by

$*$	0	a	b	c
φ	-0.8	-0.4	-0.4	-0.3

It is easy to check that (X,φ) is both an $\mathcal{N}$-subalgebra and an $\mathcal{N}$-ideal of X.

Proposition 3.8. *If (X,φ) is an $\mathcal{N}$-ideal of X, then*

$$x \leq y \Rightarrow \varphi(x) \leq \varphi(y) \text{ for any } x,y \in X.$$

Proof. Let $x,y \in X$ be such that $x \leq y$. Then $x*y=0$, and so

$$\begin{aligned}\varphi(x) &\leq \max\{\varphi(x*y),\varphi(y)\}\\ &= \max\{\varphi(0),\varphi(y)\}\\ &= \varphi(y).\end{aligned}$$

This completes the proof. □

Definition 3.9. Let $(X;*,0)$ be a d-algebra, and let $\emptyset \neq I \subseteq X$. I is called a *positive implicative ideal* if it satisfies, for all $x,y,z \in X$,

(i) $0 \in I$,
(ii) $(x*y)*z \in I$ and $y*z \in I$ imply $x*z \in I$.

Obviously, X is always a positive implicative ideal of X.

Proposition 3.10. *Any positive implicative ideal of a d-algebra X with $x*0=x$ for any $x \in X$ is a BCK-ideal of X.*

Proof. Straightforward. □

The converse of Proposition 3.10 is not true as seen in the following example.

Example 3.11. Let $X := \{0,1,2,3,4,5\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	1	0	1	0	1
3	3	1	1	0	0	1
4	4	1	1	1	0	1
5	5	5	5	5	5	0

Then X is a d-algebra with $x*0=x$ for all $x \in X$ and also it not a BCK/BCI-algebra. Let $I := \{0\}$. Then I is a BCK-ideal (briefly, an ideal) of X, but not a positive implicative ideal of X since $(2*3)*3=0 \in I$, but $2*3=1 \notin I$. Let $J := \{0,1,2,3,4\}$. Then J is a positive implicative ideal of X.

Definition 3.12. An $\mathcal{N}$-structure (X,φ) is called a *positive implicative ideal of X based on $\mathcal{N}$-function φ* (briefly, *positive implicative $\mathcal{N}$-ideal* of X) if φ satisfies (N2) and

(N4) $\varphi(x * z) \leq \max\{\varphi((x * y) * z), \varphi(y * z)\}, \forall x, y, z \in X$.

Example 3.13. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	2
2	2	2	0	0	2	1
3	3	3	3	0	0	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Then $(X; *, 0)$ is a d-algebra which is not a BCK/BCI-algebra. Define an $\mathcal{N}$-function φ by

$*$	0	1	2	3	4	5
φ	-0.9	-0.7	-0.7	-0.7	0.7	-0.2

It is easy to show that (X, φ) is an $\mathcal{N}$-ideal of X and a positive implicative $\mathcal{N}$-ideal of X. Define an $\mathcal{N}$-function v by

$*$	0	1	2	3	4	5
v	-0.8	-0.4	-0.4	-0.4	-0.4	-0.4

It is easily checked that (X, v) is an $\mathcal{N}$-subalgebra, an $\mathcal{N}$-ideal, and not a positive implicative $\mathcal{N}$-ideal of X, since $v(2 * 4) = v(2) = -0.4 \not\leq -0.8 = v(0) = \max\{v((2 * 3) * 4), v(3 * 4)\}$.

Theorem 3.14. *For any subalgebra (resp., (positive implicative) ideal) U of X, there exists an $\mathcal{N}$-function φ such that (X, φ) is an $\mathcal{N}$-subalgebra (resp., (positive implicative) $\mathcal{N}$-ideal) of X and $C(\varphi; t) = U$ for some $t \in [-1, 0)$.*

Proof. Let U be a subalgebra (resp., (positive implicative) ideal) of X and let φ be an $\mathcal{N}$-function on X defined by

$$\varphi(x) = \begin{cases} 0 & \text{if } x \notin U, \\ t & \text{otherwise} \end{cases}$$

where t is fixed in $[-1, 0)$. Then (X, φ) is an $\mathcal{N}$-subalgebra (resp., (positive implicative) $\mathcal{N}$-ideal) of X and $C(\varphi; t) = U$. $\square$

Theorem 3.15. *Let (X, φ) be an $\mathcal{N}$-structure of X and φ. Then (X, φ) is a positive implicative $\mathcal{N}$-ideal of X if and only if it satisfies:*

(1) *$C(\varphi; t) \neq \emptyset \Rightarrow C(\varphi; t)$ is a positive implicative ideal of X for all $t \in [-1, 0)$.*

Proof. Assume that (X, φ) is a positive implicative $\mathcal{N}$-ideal of X and let $t \in [-1, 0)$ be such that $C(\varphi; t) \neq \emptyset$. Obviously, $0 \in C(\varphi; t)$. Let $x, y, z \in X$ be such that $(x * y) * z, y * z \in C(\varphi; t)$. Then $\varphi(y * z) \leq t$ and $\varphi((x * y) * z) \leq t$. It follows from (N4) that $\varphi(x*z) \leq \max\{\varphi((x*y)*z), \varphi(y*z)\} \leq t$ so that $x*z \in C(\varphi;t)$. Hence $C(\varphi;t)$ is a positive implicative ideal of X.

Conversely, suppose that (1) is valid. If there exists $a \in X$ such that $\varphi(0) > \varphi(a)$, then $\varphi(0) > t_a \geq \varphi(a)$ for some $t_a \in [-1,0)$. Then $0 \notin C(\varphi;t_a)$ which is a contradiction. Now assume that there exist $a,b,c \in X$ such that $\varphi(a*c) > \max\{\varphi((a*b)*c), \varphi(b*c)\}$. Then there exists $s \in [-1,0)$ such that $\varphi(a*c) > s \geq \max\{\varphi((a*b)*c), \varphi(b*c)\}$. It follows that $(a*b)*c \in C(\varphi;s)$ and $b*c \in C(\varphi;s)$, but $a*c \notin C(\varphi;s)$. This is impossible, and so $\varphi(x*z) \leq \max\{\varphi((x*y)*z), \varphi(y*z)\}$ for all $x,y,z \in X$. Therefore (X,φ) is a positive implicative $\mathcal{N}$-ideal of X. □

Theorem 3.16. *Let (X,φ) be an $\mathcal{N}$-structure of X and φ. Then (X,φ) is an $\mathcal{N}$-ideal of X if and only if it satisfies:*

(2) $C(\varphi;t) \neq \emptyset \Rightarrow C(\varphi;t)$ *is an ideal of X for all $t \in [-1,0)$.*

Proof. Straightforward. □

Corollary 3.17. *If (X,φ) is a (positive implicative) $\mathcal{N}$-ideal of X, then every non-empty open $(\varphi;t)$-cut of φ is a (positive implicative) ideal of X for all $t \in [-1,0)$.*

Proposition 3.18. *Let (X,φ) be an $\mathcal{N}$-ideal of X. If X satisfies the following assertion:*

(3) $x*y \leq z$ *for all $x,y,z \in X$,*

then $\varphi(x) \leq \max\{\varphi(y), \varphi(z)\}$ for all $x,y,z \in X$.

Proof. Assume that (3) is valid in X. Then

$$\begin{aligned}\varphi(x*y) &\leq \max\{\varphi((x*y)*z), \varphi(z)\}\\ &= \max\{\varphi(0), \varphi(z)\}\\ &= \varphi(z), \qquad \forall x,y,z \in X.\end{aligned}$$

Hence $\varphi(x) \leq \max\{\varphi(x*y), \varphi(y)\} \leq \max\{\varphi(z), \varphi(y)\}$. Thus
$\varphi(x) \leq \max\{\varphi(y), \varphi(z)\}$ for all $x,y,z \in X$. □

Theorem 3.19. *For any d^*-algebra X, every $\mathcal{N}$-ideal is an $\mathcal{N}$-subalgebra of X.*

Proof. Let (X,φ) be an $\mathcal{N}$-ideal of a d^*-algebra X and let $x,y \in X$. Then

$$\begin{aligned}\varphi(x*y) &\leq \max\{\varphi((x*y)*x), \varphi(x)\}\\ &= \max\{\varphi(0), \varphi(x)\}\\ &\leq \max\{\varphi(x), \varphi(y)\}.\end{aligned}$$

Thus (X,φ) is an $\mathcal{N}$-subalgebra of X. □

The converse of Theorem 3.19 may not be true in general as seen in the following example.

Example 3.20. Let $X := \{0, 1, 2, 3, 4\}$ be a set with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	c	a	0

It is easily verified that $(X; *, 0)$ is a d^*-algebra which is not a BCK/BCI-algebra. Define a $\mathcal{N}$-function φ by

$*$	0	a	b	c
φ	-0.7	-0.5	-0.5	-0.2

Then (X, φ) is an $\mathcal{N}$-subalgebra, not an $\mathcal{N}$-ideal, since $\varphi(c) = -0.2 \not\leq -0.5 = \max\{\varphi(c * b), \varphi(b)\}$.

Proposition 3.21. *If (X, φ) is a positive implicative $\mathcal{N}$-ideal of an edge d-algebra X, then it is an $\mathcal{N}$-ideal of X.*

Proof. Straightforward □

Proposition 3.22. *Let (X, φ) be a positive implicative $\mathcal{N}$-ideal of an edge d-algebra X.*

(i) *If $x \leq y$, then $\varphi(x) \leq \varphi(y)$ for any $x, y \in X$.*
(ii) *If $\varphi(x * y) = \varphi(0)$, then $\varphi(x) \leq \varphi(y)$ for any $x, y \in X$.*

Proof. (i) Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$. Hence

$$\begin{aligned}\varphi(x) = \varphi(x * 0) &\leq \max\{\varphi((x * y) * 0), \varphi(y * 0)\}\\ &= \max\{\varphi(x * y), \varphi(y)\}\\ &= \max\{\varphi(0), \varphi(y)\}\\ &= \varphi(y).\end{aligned}$$

Thus $\varphi(x) \leq \varphi(y)$.
(ii) For any $x, y \in X$, we have

$$\begin{aligned}\varphi(x) = \varphi(x * 0) &\leq \max\{\varphi((x * y) * 0), \varphi(y * 0)\}\\ &= \max\{\varphi(x * y), \varphi(y)\}\\ &= \max\{\varphi(0), \varphi(y)\}\\ &= \varphi(y).\end{aligned}$$

Thus $\varphi(x) \leq \varphi(y)$. □

Proposition 3.23. *Let X be an edge d^*-algebra. If (X, φ) is a positive implicative $\mathcal{N}$-ideal of X, then*

(i) *$\varphi(x * y) \leq \varphi(x)$ for any $x, y \in X$.*
(ii) *$\varphi(x * y) \leq \max\{\varphi(x), \varphi(y)\}$ for any $x, y \in X$.*
(iii) *$\varphi(x * (y * z)) \leq \max\{\varphi(x), \varphi(y), \varphi(z)\}$ for any $x, y, z \in X$.*

Proof. (i) Since X is a d^*-algebra, we have $(x * y) * x = 0$ and so $x * y \leq x$ for any $x, y \in X$. Using Proposition 3.22 (i), we have $\varphi(x * y) \leq \varphi(x)$.
(ii) It is easily verified from Theorem 3.19 and Proposition 3.21.
(iii) For any $x, y, z \in X$, we have

$$\begin{aligned}\varphi(x * (y * z)) &\leq \max\{\varphi(x), \varphi(y * z)\} \\ &\leq \max\{\varphi(x), \varphi(y), \varphi(z)\}.\end{aligned}$$

□

For any element w of X, we consider the set

$$X_w := \{x \in X | \varphi(x) \leq \varphi(w)\}.$$

Obviously, $w \in X_w$, and so X_w is a non-empty subset of X.

Theorem 3.24. *Let $w \in X$. If (X, φ) is a positive implicative $\mathcal{N}$-ideal of X, then the set X_w is a positive implicative ideal of X.*

Proof. Using Corollary 3.17, it is evident. □

Theorem 3.25. *Let $w \in X$. If (X, φ) is an $\mathcal{N}$-ideal of X, then the set X_w is an ideal of X.*

Proof. Straightforward. □

Theorem 3.26. *Let $w \in X$ and let (X, φ) be an $\mathcal{N}$-structure of X and φ.*
(i) If X_w is an ideal of X, then (X, φ) satisfies the following assertion:

(4) $\varphi(x) \geq \max\{\varphi(y * z), \varphi(z)\} \Rightarrow \varphi(x) \geq \varphi(y), \forall x, y, z \in X.$

(ii) If (X, φ) satisfies $\varphi(0) \leq \varphi(x)$ for all $x \in X$ and (4), then X_w is an ideal of X.

Proof. (i) Assume that X_w is an ideal of X for each $w \in X$. Let $x, y, z \in X$ be such that $\varphi(x) \geq \max\{\varphi(y * z), \varphi(z)\}$. Then $y * z \in X_x$ and $z \in X_x$. Since X_x is an ideal of X, it follows that $y \in X_x$, i.e., $\varphi(y) \leq \varphi(x)$.
(ii) Suppose that (X, φ) satisfies $\varphi(0) \leq \varphi(x)$ for all $x \in X$ and (4). For each $w \in X$, let $x, y \in X$ be such that $x * y \in X_w$ and $y \in X_w$. Then $\varphi(x * y) \leq \varphi(w)$ and $\varphi(y) \leq \varphi(w)$, which imply that $\max\{\varphi(x * y), \varphi(y)\} \leq \varphi(w)$. Using (4), we have $\varphi(w) \geq \varphi(x)$ and so $x \in X_w$. Since $\varphi(0) \leq \varphi(w)$ for any $w \in X$, we have $0 \in X_w$. Therefore X_w is an ideal of X. □

Proposition 3.27. *Let (X, φ) be an $\mathcal{N}$-structure of X and φ. Then X_w is an ideal of X for any $w \in X$ if and only if*

(i) $\varphi(0) \leq \varphi(w)$
(ii) $\varphi(x * y) \leq \varphi(w)$ *and* $\varphi(y) \leq \varphi(w)$ *imply* $\varphi(x) \leq \varphi(y)$ *for any* $x, y \in X$.

Proof. Assume that X_w is an $\mathcal{N}$-ideal of X for any $w \in X$. Then $\varphi(0) \leq \varphi(w)$. Let $x, y, z \in X$ be such that $\varphi(x * y) \leq \varphi(w)$ and $\varphi(y) \leq \varphi(w)$. Then $x * y, y \in X_w$. Since X_w is an ideal of X, we have $x \in X_w$. Hence $\varphi(x) \leq \varphi(w)$.

Conversely, consider X_w for any $w \in X$. Obviously, $0 \in X_w$ for any $w \in X$. Assume that $x * y, y \in X_w$. Then $\varphi(x * y) \leq \varphi(w)$ and $\varphi(y) \leq \varphi(w)$. It follows from hypothesis that $\varphi(x) \leq \varphi(w)$. Hence $x \in X_w$. Thus X_w is an ideal of X. □

Theorem 3.28. *Let X be a d-algebra. Then an $\mathcal{N}$-subalgebra (X, φ) of X is a positive implicative $\mathcal{N}$-ideal of X if and only if for any $x, y, z \in X, x * z \in C(\varphi; t), y * z \notin C(\varphi; t)$ imply $(y * x) * z \notin C(\varphi; t)$.*

Proof. Assume that an $\mathcal{N}$-subalgebra (X, φ) of X is a positive implicative $\mathcal{N}$-ideal of X and let $x, y, z \in X$ be such that $x * z \in C(\varphi; t), y * z \notin C(\varphi; t)$. If $(y * x) * z \in C(\varphi; t)$, then $\varphi(y * z) \leq \max\{\varphi((y * x) * z), \varphi(x * z)\} \leq t$. Hence $y * z \in C(\varphi; t)$, which is a contradiction.

Conversely, suppose that for any $x, y, z \in X$, $x * z \in C(\varphi; t), y * z \notin C(\varphi; t)$ imply $(y * x) * z \notin C(\varphi; t)$. Since (X, φ) is an $\mathcal{N}$-subalgebra of X, we have $\varphi(0) \leq \varphi(x)$ for any $x \in X$ by Lemma 3.4. Hence $0 \in C(\varphi; t)$. For every $x * z \in C(\varphi; t)$, let $(y * x) * z \in C(\varphi; t)$. If $y * z \notin C(\varphi; t)$, then $(y * x) * z \notin C(\varphi; t)$ by hypothesis. This is a contradiction. Hence $y * z \in C(\varphi; t)$ and so $C(\varphi; t)$ is a positive implicative ideal of X. By Theorem 3.15, (X, φ) is a positive implicative $\mathcal{N}$-ideal of X. □

Theorem 3.29. *Let X be a d^*-algebra. Then a $\mathcal{N}$-subalgebra (X, φ) of X is an $\mathcal{N}$-ideal of X if and only if for any $x, y \in X, x \in C(\varphi; t), y \notin C(\varphi; t)$ imply $y * x \notin C(\varphi; t)$.*

Proof. Straightforward. □

For any a, b in a d^*-algebra X, the set

$$A(a, b) := \{x \in X | (x * a) * b = 0\} \text{ for any } a, b \in X.$$

Clearly, $0, a, b \in A(a, b)$ for all $a, b \in X$. Note $A(a, b)$ is not an ideal of X in general as seen the following example.

Example 3.30. Let $X := \{0, 1, 2, 3\}$ be a set with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	1
3	3	3	3	0

Then $(X; *, 0)$ is a d^*-algebra which is not a BCK/BCI-algebra. Also $A(0, 3) = \{0, 1, 3\}$ is not an ideal of X, since $2 * 3 = 1 \in A(0, 3)$ and $2 \notin A(0, 3)$.

For every $a, b \in X$, let (X, φ_a^b) be an $\mathcal{N}$-structure in which (X, φ_a^b) is given by

$$\varphi_a^b(x) = \begin{cases} \alpha & \text{if } x \in A(a, b) \\ \beta & \text{otherwise} \end{cases}$$

for all $x \in X$ and $\alpha, \beta \in [-1, 0]$ with $\alpha < \beta$.

The following example shows that there exist $a, b \in X$ such that φ_a^b is not an $\mathcal{N}$-ideal of X.

Example 3.31. Let $X := \{0, 1, 2, 3\}$ be a d^*-algebra as in Example 3.30. Then (X, φ_0^3) is not an $\mathcal{N}$-ideal of X since $\varphi_0^3(2) = \beta > \alpha = \max\{\varphi_0^3(2 * 3) = \varphi_0^3(1), \varphi_0^3(3)\}$.

We provide a condition for an $\mathcal{N}$-structure (X, φ_a^b) to be an $\mathcal{N}$-ideal of X.

Theorem 3.32. *If X is a positive implicative d^*-algebra, then (X, φ_a^b) is an $\mathcal{N}$-ideal of X for all $a, b \in X$.*

Proof. Let $a, b \in X$. For every $x \in X$, if $x \notin A(a, b)$, then $\varphi_a^b(x) = \beta \geq \varphi_a^b(0) = \alpha$. If $x \in A(a, b)$, then $\varphi_a^b(x) = \alpha$. Since $(0*a)*b = 0*b = 0$, we have $0 \in A(a, b)$. Hence $\varphi_a^b(0) = \alpha = \varphi_a^b(x)$ for all $x \in X$.
Now let $x, y \in X$. If $y \notin A(a, b)$, then $\varphi_a^b(y) = \beta$. Thus

$$\varphi_a^b(x) \leq \max\{\varphi_a^b(x * y), \varphi_a^b(y)\} = \beta.$$

If $x, y \in A(a, b)$, then $\varphi_a^b(x) = \varphi_a^b(y) = \alpha$. Hence $\varphi_a^b(x) \leq \max\{\varphi_a^b(x*y), \varphi_a^b(y)\}$. If $x \notin A(a, b)$ and $y \in A(a, b)$, then $\varphi_a^b(x) = \beta$ and $\varphi_a^b(y) = \alpha$ and hence $(x * a) * b \neq 0$ and $(y * a) * b = 0$. Therefore we have

$$\begin{aligned} ((x * y) * a) * b &= ((x * a) * (y * a)) * b \\ &= ((x * a) * b) * ((y * a) * b) \\ &= ((x * a) * b) * 0 = (x * a) * b \neq 0. \end{aligned}$$

Hence $x * y \notin A(a, b)$, i.e., $\varphi_a^b(x * y) = \beta$. Therefore $\varphi_a^b(x) = \beta = \max\{\varphi_a^b(x * y), \varphi_a^b(y)\}$. Thus (X, φ_a^b) is an $\mathcal{N}$-ideal of X for all $a, b \in X$. □

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References

1. S. S. Ahn and K. S. So, *On kernels and annihilators of left-regular mappings in d-algebras*, Honam Math. J. **30**(2008), 645-658.
2. P. J. Allen, H. S. Kim and J. Neggers, *L-up and mirror algebras*, Sci. Math. Japo., **59**(2004), 605-612.
3. P. J. Allen, H. S. Kim and J. Neggers, *On companion d-algebras*, Math. Slovaca, **57**(2007), 93-106.
4. W. H. Cornish, *A multiplier approach to implicative BCK-algebras*, Math. Seminar Notes, **8**(1980), 157-159.
5. Q. P. Hu and X. Li, *On BCH-algebras*, Math. Seminar Notes, **11**(1983), 313-320.
6. Q. P. Hu and X. Li, *On proper BCH-algebras*, Math. Japon., **30**(1985), 659-661.
7. K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japon.,**23**(1978), 1-26.
8. K. Iséki. *On BCI-algebras*, Math. Seminar Notes, **8**(1980), 125-130.
9. Y. B. Jun, J. Neggers and H. S. Kim, *Fuzzy d-ideals of d-algebras*, J. Fuzzy Math., **8**(1999), 123-130.
10. Y. C. Lee and H. S. Kim, *On d-subalgebras of d-transitive d^*-algebras*, Math. Slovaca, **49**(1999), 27-33.
11. J. Neggers, A. Dvurecenskij and H. S. Kim, *On d-fuzzy functions in d-algebras*, Found. of Physics, **30**(2000), 889-894.

12. J. Neggers, Y. B. Jun and H. S. Kim, *On d-ideals in d-algebras*, Math. Slovaca, **49**(1999), 243-251.
13. J. Neggers and H. S. Kim, *On d-algebras*, Math. Slovaca, **49**(1999), 19-26.

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