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SSOR-LIKE METHOD FOR AUGMENTED SYSTEM†

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Abstract. This paper proposes a new generalized iterative method (SSOR-like method) for solving augmented system. A functional equation relating two involved parameters is obtained, and some convergence conditions for this method are derived. This paper generalizes some foregone results. Numerical examples show that, this method is efficient by suitable choices of the involved parameters.

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1. Introduction

Let $A \in R^{m\times m}$ be a symmetric and positive definite matrix, $B \in R^{m\times n}$ be a full column rank matrix, obviously $m \geq n$. Then the augmented system is of the form

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix}, \tag{1.1}$$

where B^T stands for the transpose of matrix B, and $p \in R^m$, $q \in R^n$. In these cases, the system (1.1) has unique solution.

The augmented system arises in a wide variety of scientific and engineering applications, such as computational fluid dynamics [1], optimization and control [2], linear elasticity and mixed finite element method of elliptic partial differential equations [3], or the generalized least squares problems [4], electronic networks and others. The system of (1.1) is also termed as a Karush-Kuhn-Tucker (KKT) system, or an equilibrium system, or a saddle point problem [5,6,7].

The matrices A and B are frequently large and sparse. Hence, the iterative methods are more efficient because of storage requirements and preservation

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sparsity than direct method. Recently, Golub et al. [8] presented SOR-like algorithm, based on well-known SOR method, to solve the augment system (1.1). Bai el al. presented generalized SOR-like method in [9], which includes the classical Uzawa method [10] and the preconditioned Uzawa method [11]. Motivated by Bai [9] and SOR method, Wu el al. [12] gave out modified symmetric SOR (MSSOR) method. The authors derived associated convergence under suitable restrictions on the iteration parameter, optimal iteration parameter, and corresponding optimal convergence factor under certain conditions. More related works can be found in [13,14,15,16] and references therein.

In this paper, we propose a new generalized iterative method (i.e., SSOR-like method), under another splitting of the coefficient matrix of the system (1.1). An equation relating the parameters and the eigenvalues of the iteration matrix is obtained. And some convergence conditions for this method are also studied. This paper extends some results such as that in [12]. We should point out that the SSOR-like method is different from GSSOR method [17].

The remainder of this paper is organized as follows. In section 2, An new iterative algorithm will be given to solving the augmented system (1.1). In section 3, we will obtain the equation relating the parameters and eigenvalues of the iteration matrix, and the the convergence conditions. In section 4, some numerical examples are used to examine the feasibility and effectiveness of the SSOR-like method.

2. The SSOR-like method

We first recall the SOR-like method for the augmented system (1.1) in [8]. Make the splitting of the coefficient matrix as follows:

$$\begin{aligned}\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} &= \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ B^T & 0 \end{pmatrix} - \begin{pmatrix} 0 & -B \\ 0 & Q \end{pmatrix} \\ &:= D - L - U, \end{aligned} \tag{2.1}$$

where Q is symmetric positive definite and the preconditioning matrix. Then the SOR-like method is defined by

$$(D-\omega L)\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = [(1-\omega)D + \omega U]\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \omega \begin{pmatrix} p \\ -q \end{pmatrix},$$

where ω ia a real parameters.

In order to derive our new iterative method, make the splitting:

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = D - L - U, \tag{2.2}$$

where

$$D = \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ B^T & \alpha Q \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -B \\ 0 & \beta Q \end{pmatrix},$$

Q is symmetric positive definite matrix, and α, β are real parameters which satisfy that $\alpha + \beta = 1$. In the following text, we always see D, L and U as in (2.2).

Obviously, if $\alpha = 0$ meaning $\beta = 1$, the splitting (2.2) becomes (2.1). Motivated by the classical and efficient symmetric successive over-relaxation iterative (SSOR) method, we define the following iteration procedures

$$\begin{pmatrix} x^{(k+\frac{1}{2})} \\ y^{(k+\frac{1}{2})} \end{pmatrix} = \mathcal{L}_{\omega,\alpha} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \omega(D-\omega L)^{-1} \begin{pmatrix} p \\ -q \end{pmatrix}, \tag{2.3}$$

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \mathcal{U}_{\omega,\alpha} \begin{pmatrix} x^{(k+\frac{1}{2})} \\ y^{(k+\frac{1}{2})} \end{pmatrix} + \omega(D-\omega U)^{-1} \begin{pmatrix} p \\ -q \end{pmatrix}, \tag{2.4}$$

where
$$\begin{aligned}\mathcal{L}_{\omega,\alpha} &= (D-\omega L)^{-1}[(1-\omega)D+\omega U)] \\ &= \begin{pmatrix} (1-\omega)I_m & -\omega A^{-1}B \\ \frac{\omega(1-\omega)}{1-\omega\alpha}Q^{-1}B^T & I_n - \frac{\omega^2}{1-\omega\alpha}Q^{-1}B^TA^{-1}B \end{pmatrix}\end{aligned}$$

and
$$\begin{aligned}\mathcal{U}_{\omega,\alpha} &= (D-\omega U)^{-1}[(1-\omega)D+\omega L)] \\ &= \begin{pmatrix} (1-\omega)I_m - \frac{\omega^2}{1-\omega\beta}A^{-1}BQ^{-1}B^T & -\omega A^{-1}B \\ \frac{\omega}{1-\omega\beta}Q^{-1}B^T & I_n \end{pmatrix}.\end{aligned}$$

Combining (2.3) with (2.4), then we get the SSOR-like method:

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1} \end{pmatrix} = \mathcal{T}_{\omega,\alpha} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + C \tag{2.5}$$

with

$$\begin{aligned}\mathcal{T}_{\omega,\alpha} &= \mathcal{U}_{\omega,\alpha}\mathcal{L}_{\omega,\alpha} \\ &= \begin{pmatrix} (1-\omega)^2 I_m - \frac{\omega^2(1-\omega)(2-\omega)}{(1-\omega\alpha)(1-\omega\beta)}A^{-1}BQ^{-1}B^T & \frac{\omega^3(2-\omega)}{(1-\omega)(1-\beta)}A^{-1}BQ^{-1}B^TA^{-1}B - \omega(2-\omega)A^{-1}B \\ \frac{\omega(1-\omega)(2-\omega)}{(1-\omega\alpha)(1-\omega\beta)}Q^{-1}B^T & I_n - \frac{\omega^2(2-\omega)}{(1-\omega\alpha)(1-\omega\beta)}Q^{-1}B^TA^{-1}B \end{pmatrix}\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}C &= \omega\mathcal{U}_{\omega,\alpha}(D-\omega L)^{-1}\begin{pmatrix} p \\ -q \end{pmatrix} + \omega(D-\omega U)^{-1}\begin{pmatrix} p \\ -q \end{pmatrix} \\ &= \omega(2-\omega)\begin{pmatrix} A^{-1}p - \frac{\omega^2}{(1-\omega\alpha)(1-\omega\beta)}A^{-1}BQ^{-1}B^TA^{-1}p + \frac{\omega}{(1-\omega\alpha)(1-\omega\beta)}A^{-1}BQ^{-1}q \\ \frac{\omega}{(1-\omega\alpha)(1-\omega\beta)}Q^{-1}B^TA^{-1}p - \frac{1}{(1-\omega\alpha)(1-\omega\beta)}Q^{-1}q \end{pmatrix}.\end{aligned}$$

Furthermore, the SSOR-like method can be rewritten concretely as follows:

$$\begin{aligned}y^{(k+1)} &= y^{(k)} + \frac{\omega(2-\omega)}{(1-\omega\alpha)(1-\omega\beta)}Q^{-1}B^T[(1-\omega)x^{(k)} - \omega A^{-1}By^{(k)} + \omega A^{-1}p] \\ &\quad - \frac{\omega(2-\omega)}{(1-\omega\alpha)(1-\omega\beta)}Q^{-1}q,\end{aligned}$$

$$x^{(k+1)} = (1-\omega)^2x^{(k)} - \omega A^{-1}B[y^{(k+1)} + (1-\omega)y^{(k)}] + \omega(2-\omega)A^{-1}p.$$

where Q is an approximate (preconditioning) matrix of the Schur complement of matrix $B^TA^{-1}B$.

The SSOR-like method includes two parameters ω, α, one preconditioning matrix Q, which is different from the GSSOR method in [17]. Particularly, if let $\alpha = \beta = \frac{1}{2}$, the SSOR-like method becomes the MSSOR method [12]. Therefore, the method proposed in present paper is the generalization of that in [12].

In the sequel, we will analysis the convergence of SSOR-like method for solving the augmented system (1.1). From (2.2) we have

$$(D-\omega L) = \begin{pmatrix} A & 0 \\ -\omega B^T & (1-\omega\alpha)Q \end{pmatrix}, \quad (D-\omega U) = \begin{pmatrix} A & \omega B \\ 0 & (1-\omega\beta)Q \end{pmatrix},$$

it follows that

$$det(D-\omega L) = (1-\omega\alpha)^n\, det(A)\cdot det(Q) \neq 0,$$
$$det(D-\omega U) = (1-\omega\beta)^n\, det(A)\cdot detdet(Q) \neq 0$$

if and only if $\quad 1-\omega\alpha \neq 0$ and $1-\omega\beta \neq 0. \quad (2.7)$

Where the symbol $det(\cdot)$ denotes the determinant of matrix. So the hypothesis of (2.7) is necessary for the iterative method.

As is well known, the SSOR-like method (2.5) is convergent if and only if the spectral radius of its iteration matrix $\rho(\mathcal{T}_{\omega,\alpha}) < 1$. Naturally, $\omega \neq 0$ from (2.6). For the need of convergence analysis, let λ be an eigenvalue of the iterative matrix $\mathcal{T}_{\omega,\alpha}$ and $(x^T, y^T)^T$ be the corresponding eigenvector, that is,

$$\mathcal{T}_{\omega,\alpha}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}.$$

Noting that (2.7) and substituting (2.6) into the above equation, by simply computation, we obtain

$$\begin{cases} [(1-\omega)^2-\lambda]x = \omega(1-\omega+\lambda)A^{-1}By, \\ \frac{\omega(1-\omega)(2-\omega)}{(1-\omega\alpha)(1-\omega\beta)}Q^{-1}B^Tx = \frac{\omega^2(2-\omega)}{(1-\omega\alpha)(1-\omega\beta)}Q^{-1}B^TA^{-1}By - (1-\lambda)y. \end{cases} \quad (2.8)$$

Remark 2.1. It is clear that the eigenvalues of $\mathcal{T}_{\omega,\alpha}$ will not equal to 1, and $\omega \neq 2$. Otherwise, it will be contradict to the definition of eigenvector.

The next lemma further describes the properties of $\mathcal{T}_{\omega,\alpha}$.

Lemma 2.1. *Suppose that $\lambda \neq \omega - 1$, and α, β satisfy (2.7). Then*
(1) $\lambda = (1-\omega)^2$ is an eigenvalue of $\mathcal{T}_{\omega,\alpha}$ with at least multiplicity of m while $\omega = 1$.
(2) if $\omega \neq 1$, $\lambda = (1-\omega)^2$ is an eigenvalue of which with multiplicity not less than $m-n$, if and only if $m > n$.
Proof. Provided that $\lambda = (1-\omega)^2$ is an eigenvalue of $\mathcal{T}_{\omega,\alpha}$, which follows from (2.8) that

$$\begin{cases} (1-\omega+\lambda)A^{-1}By = 0, \\ \frac{1-\omega}{(1-\omega\alpha)(1-\omega\beta)}Q^{-1}B^Tx = \frac{\omega}{(1-\omega\alpha)(1-\omega\beta)}Q^{-1}B^TA^{-1}By - y. \end{cases} \quad (2.9)$$

Since $\lambda \neq \omega - 1$, and B is full column rank, we obtain from (2.9) that $y = 0$ and $(1-\omega)B^T x = 0$. Furthermore, if $\omega = 1$ we know that $(1-\omega)B^T x = 0$ holds for any $x \in R^m$. That is to say that $\lambda = (1-\omega)^2$ is an eigenvalue with multiplicity not less than m. Otherwise, if $\omega \neq 1$ and $m > n$, it means that $\lambda = (1-\omega)^2$ is an eigenvalue with at least multiplicity $m - n$. Whereas $\lambda = (1-\omega)^2$ is not one when $m = n$. The proof is completed. □

Based on Lemma 2.1, we have the following theorem. It gives out an equation including several parameters and being essential for studying the convergence of the SSOR-like method (2.5).

Theorem 2.1. *Let $A \in R^{m\times m}$, $Q \in R^{n\times n}$ be symmetric positive definite, $B \in R^{m\times n}$ be of full column rank, with $m \geq n$. If λ ($\neq (1-\omega)^2$) is a nonzero eigenvalue of the iterative matrix $\mathcal{T}_{\omega,\alpha}$ in (2.5), and μ satisfies*

$$(1-\omega\alpha)(1-\omega\beta)(1-\lambda)(\lambda-(1-\omega)^2) = \mu\lambda\omega^2(2-\omega)^2, \tag{2.10}$$

then μ is an eigenvalue of matrix $Q^{-1}B^TA^{-1}B$.Conversely, If μ is an eigenvalue of $Q^{-1}B^TA^{-1}B$ and λ satisfies (2.10), then λ is an eigenvalue of $\mathcal{T}_{\omega,\alpha}$.

Proof. Let λ ($\neq (1-\omega)^2$) and $(x^T, y^T)^T$ be the eigenvalue and corresponding eigenvector of $\mathcal{T}_{\omega,\alpha}$, respectively, then they content (2.8). So we obtain from the first equation of (2.8) that $x = \frac{\omega(1-\omega+\lambda)}{(1-\omega)^2-\lambda}A^{-1}By$. Taking it into the second equation yields

$$\frac{\lambda\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)(\lambda-(1-\omega)^2)}Q^{-1}B^TA^{-1}By = (1-\lambda)y.$$

The above equation and (2.10) imply that μ is an eigenvalue of matrix $Q^{-1}B^TA^{-1}B$. In fact, the eigenvalue μ can be positive real number for the particularity of the matrices Q, A and B. The second assertion can be proved by reversing the above process. We complete the proof. □

From Theorem 2.1, we can see that if $\alpha = \beta = \frac{1}{2}$, the equation (2.10) becomes (5) of Theorem 2 proposed in [12].

3. The convergence analysis of iterative method (2.5)

In this section we analysis the convergence of the SSOR-like method of (2.5). The convergence conditions will be derived. Meanwhile, the optimal parameter and factor will also be considered.

Lemma 3.1 ([18]). *Consider the quadratic equation $x^2 - bx + c = 0$, where b and c are real numbers. Booth roots of the quadratic equation are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

By making use of Lemma 3.1, we can obtain the convergence conditions of the SSOR-like method, which are expressed as follows:

Theorem 3.1. *Let $A \in R^{m\times m}$, $Q \in R^{n\times n}$ be symmetric positive definite, $B \in$*

$R^{m\times n}$ be of full column rank, with $m \geq n$. Then the SSOR-like method of (2.5) is convergent if and only if

$$0 < \omega < 2,\ (1-\omega\alpha)(1-\omega\beta) > 0,\ \textit{and}\ \frac{\rho\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)} < 2 + 2(1-\omega)^2, \quad (3.1)$$

where ρ denotes the spectral radius of the matrix $Q^{-1}B^TA^{-1}B$.

Proof. The equation (2.10) can be rewritten equivalently as

$$\lambda^2 - [(1-\omega)^2 + 1 - \frac{\mu\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}]\lambda + (1-\omega)^2 = 0. \quad (3.2)$$

For the convenience of statement, set

$$b = (1-\omega)^2 + 1 - \frac{\mu\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}$$

and

$$c = (1-\omega)^2.$$

Then using Lemma 3.1 we obtain that $|\lambda| < 1$ if and only if

$$|(1-\omega)^2| < 1,$$

and

$$|(1-\omega)^2 + 1 - \frac{\mu\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}| < 1 + (1-\omega)^2.$$

Equivalently,

$$0 < \omega < 2,$$

and

$$0 < \frac{\mu\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)} < 2 + 2(1-\omega)^2. \quad (3.3)$$

Because of the particularity of matrices Q, A and B, we can claim that $\mu > 0$. Then resorting to (3.3) shows that the last two terms of (3.1) are true. □

From Theorem 3.1, we can obtain the necessary conditions of convergence for the SSOR-like method.

Corollary 1. *Suppose $(1-\omega\alpha)(1-\omega\beta) > 0$, let the eigenvalues μ of matrix $Q^{-1}B^TA^{-1}B$ are real and positive, then we have*

(1) If $0 < \mu \leq \frac{1}{4}$, the SSOR-like method (2.5) is convergent, then $0 < \omega < 2$.

(2) If $\mu = \frac{1}{2}$, the method is convergent, then all $0 < \omega < 1$.

(3) If $\mu > \frac{1}{4}$ but $\mu \neq \frac{1}{2}$, it is convergent, then

$$0 < \omega < \frac{2}{1+\sqrt{4\rho - 1}} < 2,$$

where ρ denotes the spectral radius of $Q^{-1}B^TA^{-1}B$.

Proof. Noting that the hypothesis $\alpha + \beta = 1$, and $(1-\omega\alpha)(1-\omega\beta) \leq \frac{(2-\omega)^2}{4}$ for some given ω, the equality is attainable when $\alpha = \beta = \frac{1}{2}$, it follows from (3.3) that $(1-2\mu)\omega^2 - 2\omega + 2 > 0$. The following discussion is similar to the proof of Theorem 1 in [12]. □

We should point out that the above necessary conditions will be sufficient if $\alpha = \beta = \frac{1}{2}$.

4. The optimal relaxation factor of SSOR-like method

Let $\rho = \rho(Q^{-1}B^TA^{-1}B)$ and $0 < \mu_0 = min_{\mu \neq} u$, where μ is a positive eigenvalue of the matrix product $Q^{-1}B^TA^{-1}B$.

Theorem 4.1. *Provided that α, β satisfies (2.7), $\alpha + \beta = 1$ and $0 < \omega < 2$. Then*

Case I. If $0 < \alpha\beta - \mu < \frac{1}{4}$, $0 < \alpha, \beta < 1$, $0 < \mu < \alpha\beta < \frac{1}{4}$, we have

$$\rho(\mathcal{T}_{\omega,\alpha}) = \begin{cases} |1-\omega|, & if \quad \frac{2}{1+\sqrt{1-4(\alpha\beta-\rho)}} \leq \omega \leq \frac{2(1-4\rho)}{(1-4\rho)+\sqrt{(1-4\rho)(1-4\alpha\beta)}} < 2, \\ 0.5\Big\{|(1-\omega)^2 + 1 - \frac{\rho\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)} & \\ +\omega(2-\omega)\sqrt{[1-\frac{\rho\omega^2}{(1-\omega\alpha)(1-\omega\beta)}][1-\frac{\rho(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}]}\Big\}, & \\ \quad if \quad 0 < \omega \leq \frac{2}{1+\sqrt{1-4(\alpha\beta-\rho)}} \; or \; \frac{2(1-4\rho)}{(1-4\rho)+\sqrt{(1-4\rho)(1-4\alpha\beta)}} \leq \omega < 2. \end{cases} \tag{4.1}$$

In this case, the optimal parameter ω_{opt} and $\rho(\mathcal{T}_{\omega_{opt},\alpha})$ are given respectively by $\omega_{opt} = \frac{2}{1+\sqrt{1-4(\alpha\beta-\rho)}}$ *and* $\rho(\mathcal{T}_{\omega_{opt},\alpha}) = \frac{1-\sqrt{1-4(\alpha\beta-\rho)}}{1+\sqrt{1-4(\alpha\beta-\rho)}}$.

Case II. (1) If $\alpha\beta < \mu < \frac{1}{4}$, we have

$$\rho(\mathcal{T}_{\omega,\alpha}) = \begin{cases} |1-\omega|, & if \quad \frac{2(1-4\rho)}{(1-4\rho)+\sqrt{(1-4\rho)(1-4\alpha\beta)}} \leq \omega \leq \frac{2}{1+\sqrt{1-4(\alpha\beta-\rho)}} < 1, \\ 0.5\Big\{|(1-\omega)^2 + 1 - \frac{\rho\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}| & \\ +\omega(2-\omega)\sqrt{[1-\frac{\rho\omega^2}{(1-\omega\alpha)(1-\omega\beta)}][1-\frac{\rho(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}]}\Big\}, & \\ \quad if \quad 0 < \omega \leq \frac{2(1-4\rho)}{(1-4\rho)+\sqrt{(1-4\rho)(1-4\alpha\beta)}} \; or \; \frac{2}{1+\sqrt{1-4(\alpha\beta-\rho)}} \leq \omega < 2, \end{cases} \tag{4.2}$$

and the optimal parameter ω_{opt} and $\rho(\mathcal{T}_{\omega_{opt},\alpha})$ are
$\omega_{opt} = \frac{2(1-4\rho)}{(1-4\rho)+\sqrt{(1-4\rho)(1-4\alpha\beta)}}$ *and* $\rho(\mathcal{T}_{\omega_{opt},\alpha}) = 1 - \frac{2(1-4\rho)}{(1-4\rho)+\sqrt{(1-4\rho)(1-4\alpha\beta)}}$.

(2) If $\alpha\beta < \frac{1}{4} < \mu$, we have

$$\rho(\mathcal{T}_{\omega,\alpha}) = \begin{cases} |1-\omega|, & if \quad 0 < \omega \leq \frac{2}{1+\sqrt{1-4(\alpha\beta-\rho)}} < 1, \\ 0.5\Big\{|(1-\omega)^2 + 1 - \frac{\rho\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}| & \\ +\omega(2-\omega)\sqrt{[1-\frac{\rho\omega^2}{(1-\omega\alpha)(1-\omega\beta)}][1-\frac{\rho(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}]}\Big\}, & \\ \quad if \quad \frac{2}{1+\sqrt{1-4(\alpha\beta-\rho)}} \leq \omega < 2, \end{cases} \tag{4.3}$$

In this case, any ω satisfies (3.1) is the needed optimal.

Proof. By using the equivalent form of (3.2) of the equation (2.10), we have

$$\lambda = 0.5\{(1-\omega)^2 + 1 - \frac{\mu\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)} \pm \sqrt{[(1-\omega)^2 + 1 - \frac{\mu\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}]^2 - 4(1-\omega)^2}\}.$$

Let $\Delta = [(1-\omega)^2 + 1 - \frac{\mu\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}]^2 - 4(1-\omega)^2$

$$= \omega^2(2-\omega)^2[(1 - \frac{\mu\omega^2}{(1-\omega\alpha)(1-\omega\beta)}][(1 - \frac{\mu(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}].$$

So both the solutions of equation (2.10) are complex if $\Delta < 0$, i.e.,

$$[(\alpha\beta-\mu)\omega^2 - \omega + 1][(\alpha\beta-\mu)\omega^2 - (1-4\mu)\omega + (1-4\mu)] < 0, \tag{4.4}$$

which is equivalent to the following two equations

$$\begin{cases} (\alpha\beta-\mu)\omega^2 - \omega + 1 < 0, \\ (\alpha\beta-\mu)\omega^2 - (1-4\mu)\omega + (1-4\mu) > 0, \end{cases} \tag{4.5}$$

or

$$\begin{cases} (\alpha\beta-\mu)\omega^2 - \omega + 1 > 0, \\ (\alpha\beta-\mu)\omega^2 - (1-4\mu)\omega + (1-4\mu) < 0. \end{cases} \tag{4.6}$$

It is clear that $0 < \omega < 2$, $\omega \neq 1$ if $\alpha\beta - \mu = 0$ but $\mu \neq \frac{1}{4}$.

Now, we discuss the solution of the unequal equation of (4.4) by the following two aspects.

Case I. $\alpha\beta - \mu > 0$, $0 < \mu < \frac{1}{4}$, $0 < \alpha\beta - \mu < \frac{1}{4}$, $0 < \alpha, \beta < 1$, $\alpha\beta < \frac{1}{4}$. From (4.5) and (4.6), we obtain

$$|\lambda| = \begin{cases} |1-\omega|, & \text{if } \frac{2}{1+\sqrt{1-4(\alpha\beta-\mu)}} < \omega < \frac{1-4\mu-\sqrt{(1-4\mu)(1-4\alpha\beta)}}{2(\alpha\beta-\mu)} < 2, \\ 0.5\Big\{|(1-\omega)^2 + 1 - \frac{\mu\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}| \\ +\omega(2-\omega)\sqrt{[1 - \frac{\mu\omega^2}{(1-\omega\alpha)(1-\omega\beta)}](1 - \frac{\mu(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}]}\Big\}, \\ \quad \text{if } 0 < \omega \le \frac{2}{1+\sqrt{1-4(\alpha\beta-\mu)}} \text{ or } \frac{2(1-4\mu)}{(1-4\mu)+\sqrt{(1-4\mu)(1-4\alpha\beta)}} \le \omega < 2. \end{cases} \tag{4.7}$$

Case II. $\alpha\beta - \mu < 0$.

(1) $\alpha\beta < \frac{1}{4}$, $0 < \mu < \frac{1}{4}$. Then

$$|\lambda| = \begin{cases} |1-\omega|, & \text{if } \frac{1-4\mu-\sqrt{(1-4\mu)(1-4\alpha\beta)}}{2(\alpha\beta-\mu)} < \omega < \frac{2}{1+\sqrt{1-4(\alpha\beta-\mu)}} < 1, \\ 0.5\Big\{|(1-\omega)^2 + 1 - \frac{\mu\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}| \\ +\omega(2-\omega)\sqrt{[1 - \frac{\mu\omega^2}{(1-\omega\alpha)(1-\omega\beta)}][1 - \frac{\mu(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}]}\Big\}, \\ \quad \text{if } 0 < \omega \le \frac{1-4\mu-\sqrt{(1-4\mu)(1-4\alpha\beta)}}{2(\alpha\beta-\mu)} \text{ or } \frac{2}{1+\sqrt{1-4(\alpha\beta-\mu)}} \le \omega < 2. \end{cases} \tag{4.8}$$

(2) $\alpha\beta < \frac{1}{4}$, $\mu > \frac{1}{4}$. Then we have

$$|\lambda| = \begin{cases} |1-\omega|, \quad \text{if} \quad 0 < \omega < \frac{2}{1+\sqrt{1-4(\alpha\beta-\mu)}} < 1, \\ 0.5\Big\{|(1-\omega)^2 + 1 - \frac{\mu\omega^2(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}| \\ +\omega(2-\omega)\sqrt{[1-\frac{\mu\omega^2}{(1-\omega\alpha)(1-\omega\beta)}][1-\frac{\mu(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}]}\Big\}, \\ \quad \text{if} \quad \frac{2}{1+\sqrt{1-4(\alpha\beta-\mu)}} \le \omega < 2. \end{cases} \tag{4.9}$$

In addition, we know that $|\lambda| = |1-\omega|$ holds when the $\Delta = 0$, and note that the monotonicity of the functions $\frac{2}{1+\sqrt{1-4(\alpha\beta-\mu)}}$ and $\frac{2(1-4\mu)}{(1-4\mu)+\sqrt{(1-4\mu)(1-4\alpha\beta)}}$, it is easy to obtain (4.1),(4.2) and (4.3) from (4.7), (4.8) and (4.9), respectively.

For the optimal parameters, similar to the Varga's analysis[19], write the equation of (2.10) as the form of

$$\frac{(1-\lambda)(\lambda-(1-\omega)^2)}{\omega^2} = \frac{\mu(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}\lambda.$$

Define two functions:

$$f_\omega(\lambda) = \frac{\mu(2-\omega)^2}{(1-\omega\alpha)(1-\omega\beta)}\lambda := k_\mu(\omega)\lambda, \ 0 < \mu \le \rho < 1,$$

and

$$g_\omega(\lambda) = \frac{(1-\lambda)(\lambda-(1-\omega)^2)}{\omega^2}, \ \omega \ne 0.$$

Obviously, $f_\omega(\lambda)$ is a straight line with parameters ω, α and μ, through the point (0,0). And $g_\omega(\lambda)$ is a quadratical function.

By using the ways and means of Varga, the optimal relaxation factor ω_{opt} occurs when $f_\omega(\lambda)$ becomes tangent to $g_\omega(\lambda)$, i.e., the slope of $f_\omega(\lambda)$ equals to that of tangent of $g_\omega(\lambda)$ with respect to some λ, i.e., $g'_\omega(\lambda) = k_\mu(\omega)$. Substituting the respective expressions to it yields

$$\frac{(2-\omega)^2\mu}{(1-\omega\alpha)(1-\omega\beta)} = -\frac{2}{\omega^2}\lambda + \frac{1+(1-\omega)^2}{\omega^2},$$

which implies that

$$\omega_{opt} = \frac{2}{1+\sqrt{1-4(\alpha\beta-\rho)}}$$

and

$$\rho(\mathcal{T}_{\omega_{opt},\alpha}) = \frac{1-\sqrt{1-4(\alpha\beta-\rho)}}{1+\sqrt{1-4(\alpha\beta-\rho)}},$$

under the conditions of the Case I. Furthermore

$$\omega_{opt} = \frac{2(1-4\rho)}{(1-4\rho)+\sqrt{(1-4\rho)(1-4\alpha\beta)}}$$

and

$$\rho(\mathcal{T}_{\omega_{opt},\alpha}) = 1 - \frac{2(1-4\rho)}{(1-4\rho) + \sqrt{(1-4\rho)(1-4\alpha\beta)}},$$

with Case II (1). Otherwise it meets Case II (2), this completes the proof. □

5. Numerical example

In this section, we will give numerical example to illustrate our results. In our computations of the example, we set $m = 2p^2$, $n = p^2$, so the total variable numbers are $m + n = 3p^2$. We choose right-hand-side vector $(p^T, q^T)^T \in R^{m+n}$ such that the exact solution of the augmented system (1.1) is $({x^*}^T, {y^*}^T)^T = (1, 1, \dots, 1)^T \in R^{m+n}$, and the initial iterative vector $({x^{(0)}}^T, {y^{(0)}}^T)^T = 0$.

Choose the given matrices A and B as in [20], we now consider the augmented system (1.1). That is,

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in R^{m \times n}, B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in R^{m \times n},$$

and

$$T = \tfrac{1}{h^2} \cdot tridiag(-1, 2, -1) \in R^{p \times p},\ F = \tfrac{1}{h} \cdot tridiag(-1, 1, 0) \in R^{p \times p},$$

in which $\otimes$ stands for the Kronecker product and $h = \frac{1}{p+1}$ represents the discretization mesh size. Moreover, we choose matrix Q as an approximation to the matrix $B^T A^{-1} B$, according to the cases listed in Table 1.

Table 1. Choices of matrix Q

Case no.	Matrix Q	Description
I	$B^T \hat{A}^{-1} B$	$\hat{A}$=$tridiag(A)$
II	$B^T \hat{A}^{-1} B$	$\hat{A}$=$diag(A)$

Because of the roundoff errors, all runs with respect to the iteration methods will be terminated if norm of absolute error vectors ERR $< 1.0e - 009$, where

$$\text{ERR} = \frac{\sqrt{\|x^{(k)} - x^*\|^2 + \|y^{(k)} - y^*\|^2}}{\sqrt{\|x^{(0)} - x^*\|^2 + \|y^{(0)} - y^*\|^2}}$$

with $({x^{(k)}}^T, {y^{(k)}}^T)^T$ the final approximation solution.

By the SSOR-like method (2.5), the number of iterations (denoted by IT) and norm of absolute residual vectors (denoted by RES) are reported in Table 2. Here, the least residual is RES= $\sqrt{\|p - Ax^{(k)} - By^{(k)}\|^2 + \|q - B^T x^{(k)}\|^2}$.

We can see from Table 2 that the iteration algorithm proposed in this paper may be efficient by choosing proper parameters.

TABLE 2. Optimal parameter, spectral radius, IT and RES for (2.5)

m			128	512	1152
n			64	256	576
$m+n$			192	768	1728
Case I	SOR-like	ω_{opt}	0.5958	0.3657	0.2620
		$\rho(M_{\omega_b})$	0.6358	0.7964	0.8591
		IT	62	130	200
		RES	3.92e-8	7.31e-8	1.16e-7
	MSSOR	ω_0	0.3081	0.1848	0.1316
		$\rho(\Omega_{\omega_0})$	0.6919	0.8152	0.8684
		IT	78	147	218
		RES	3.2945e-008	5.9969e-008	1.3892e-007
	SSOR-like		$\alpha = 0.0294$	$\alpha = 0.7919$	$\alpha = 0.0185$
			$\beta = 0.9706$	$\beta = 0.2081$	$\beta = 0.9815$
		ω_{opt}	0.3134	0.1987	0.2282
		$\rho(\mathcal{T}_{\omega_{opt},\alpha})$	0.6866	0.8013	0.7718
		IT	65	109	205
		RES	6.9261e-008	4.3787e-008	3.4844e-008
Case II	SOR-like	ω_{opt}	0.4664	0.2720	0.1915
		$\rho(M_{\omega_b})$	0.7305	0.8533	0.8992
		IT	92	191	293
		RES	3.69e-8	7.46e-8	1.13e-7
	MSSOR	ω_0	0.2375	0.1367	0.0960
		$\rho(\Omega_{\omega_0})$	0.7625	0.8633	0.9040
		IT	108	208	311
		RES	2.8099e-008	4.9659e-008	6.9060e-008
	SSOR-like		$\alpha = 0.4057$	$\alpha = 0.2028$	$\alpha = 0.4860$
			$\beta = 0.5943$	$\beta = 0.7972$	$\beta = 0.5140$
		ω_{opt}	0.1763	0.1389	0.1517
		$\rho(\mathcal{T}_{\omega_{opt},\alpha})$	0.8237	0.2720	0.1915
		IT	111	218	296
		RES	5.4644e-008	4.5357e-008	8.7412e-008

6. Conclusions

In this paper, we have proposed a new generalized iteration method, i.e. SSOR-like method (2.5), for solving the augmented system (1.1). Meantime, a functional equation including several parameters is obtained, which is essential for analysis the convergence of this method. In addition, some convergence conditions for the method are derived. This paper generalizes some foregone results, for instance [8,12]. Finally, we offer numerical example, which shows that this method is efficient by suitable choices of the involved parameters. However, the numerical example reflects some faults of our method, such as the optimal choices of α, β etc., this will be the future work.

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