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GENERALIZED HYERES–ULAM STABILITY OF A QUADRATIC FUNCTIONAL EQUATION WITH INVOLUTION IN QUASI-β-NORMED SPACES

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Abstract. In this paper, using a fixed point approach, the generalized Hyeres–Ulam stability of the following quadratic functional equation

$$f(x+y+z)+f(x+\sigma(y))+f(y+\sigma(z))+f(x+\sigma(z)) = 3(f(x)+f(y)+f(z))$$

will be studied, where f is a function from abelian group G into a quasi-β-normed space and σ is an involution on the group G. Next, we consider its pexiderized equation of the form

$$f(x+y+z)+f(x+\sigma(y))+f(y+\sigma(z))+f(x+\sigma(z)) = g(x)+g(y)+g(z)$$

and its generalized Hyeres–Ulam stability.

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1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?" If there exists an affirmative answer we say that the equation $\mathcal{E}$ is stable [11]. During the last decades several stability problems for various functional equations have been investigated by numerous mathematicians. We refer the reader to the survey articles [11, 12, 22] and monographs [9, 13, 15, 18, 23] and references therein.

Let $\mathcal{G}$, be a abelian group and $\mathfrak{X}$ be a normed space. A function $f : \mathcal{G} \to \mathfrak{X}$ satisfying the functional equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y) \tag{1}$$

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is called the quadratic functional equation. The Hyers-Ulam stability of the quadratic equation (1) on normed spaces has been studied in [8]. A mapping $\sigma : \mathcal{G} \to \mathcal{G}$ is called an involution if σ is a group homomorphism for which $\sigma^2 = I$. For $f : \mathcal{G} \to \mathfrak{X}$, and an involution σ on $\mathcal{G}$, the Hyers-Ulam stability of the quadratic functional equation with involution

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y) \tag{2}$$

have been studied by many authors in various cases and by different methods, (for example see [5], [16] and [25]).

Consider the following functional equations,

$$f(x+y+z) + f(x-y) + f(y-z) + f(x-z) = 3f(x) + 3f(y) + 3f(z), \tag{3}$$

and its pexiderized form as follows

$$f(x+y+z) + f(x-y) + f(y-z) + f(x-z) = g(x) + g(y) + g(z). \tag{4}$$

This equations, their solutions and their Hyers-Ulam stability is studied in [1], [2], [3] and [4].

In this paper we adopt the ideas of Cădariu and Radu [7], S.-M. Jung and Z.-H. Lee [16] to investigate the Hyers-Ulam-Rassias stability of the equation

$$f(x+y+z)+f(x+\sigma(y))+f(y+\sigma(z))+f(x+\sigma(z)) = 3f(x)+3f(y)+3f(z), \tag{5}$$

and its pexiderized form as follows

$$f(x+y+z)+f(x+\sigma(y))+f(y+\sigma(z))+f(x+\sigma(z)) = g(x)+g(y)+g(z). \tag{6}$$

on abelian groups using a fixed point method, so we need some preliminaries in this area.

For a nonempty set $\mathcal{M}$, a function $d : \mathcal{M} \times \mathcal{M} \to [0, \infty]$ is called a generalized metric on $\mathcal{M}$ if d satisfies
(M_1) $d(x, y) = 0$ if and only if $x = y$;
(M_2) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{M}$;
(M_3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{M}$.
Trivially the only one difference of the generalized metric from the usual metric is that the range of the former is permitted to include infinity.
We now introduce one of fundamental results of fixed point theory. For the proof, we refer to [17]. This theorem will play an important role in proving our main theorem.

Theorem 1. *Let $(\mathcal{M}, d)$ be a generalized complete metric space. Assume that $\Lambda : \mathcal{M} \to \mathcal{M}$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in \mathcal{M}$, then the following are true:*
(a) The sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ;
(b) x^* is the unique fixed point of Λ in

$$\mathcal{M}^* = \{y \in X : d(\Lambda^k x, y) < \infty\};$$

(c) If $y \in \mathcal{M}^*$, then $d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y)$.

We consider some basic concept concerning quasi-β-normed spaces and some preliminary result. We fixed a real number β with $0 < \beta \leq 1$ and $\mathbb{K}$ denote either $\mathbb{C}$ or $\mathbb{R}$. Let $\mathfrak{X}$ be a linear spaces over $\mathbb{K}$. A quasi-β-norm on $\mathfrak{X}$ is a function $\|.\|_\beta : \mathfrak{X} \to [0, \infty)$, for which
(N_1) $\|x\|_\beta = 0$, if and only if $x = 0$;
(N_2) $\|\lambda x\|_\beta = |\lambda|^\beta.\|x\|_\beta$, for all $\lambda \in \mathbb{K}$ and all $x \in \mathfrak{X}$;
(N_3) There is a constant $k \geq 1$ such that $\|x + y\|_\beta \leq k(\|x\|_\beta + \|y\|_\beta)$, for all $x, y \in \mathfrak{X}$.

It follows from condition (N_3) that

$$\|\sum_{i=1}^{2n} x_i\|_\beta \leq k^n \sum_{i=1}^{2n} \|x_i\|_\beta, \quad \|\sum_{i=1}^{2n+1} x_i\| \leq k^{n+1} \sum_{i=1}^{2n+1} \|x_i\|_\beta \tag{7}$$

for all $n \in \mathbb{N}$ and all$x_1, x_2, ..., x_{2n+1} \in \mathfrak{X}$

The pair $(\mathfrak{X}, \|.\|_\beta)$ is called a quasi-β-normed space if $\|.\|_\beta$ is a quasi-β-norm on $\mathfrak{X}$. The smallest possible k is called the *modulus of concavity* of $\|.\|_\beta$. A quasi-β-Banach space is a complete quasi-β-normed space.

A quasi-β-normed $\|.\|_\beta$ is called a (β, p)-norm $(0 < p \leq 1)$ if

$$\|x + y\|_\beta^p \leq \|x\|_\beta^p + \|y\|_\beta^p$$

for all $x, y \in \mathfrak{X}$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.

Given a p-norm, the formula $d(x, y) := \|x-y\|^p$ gives us a translation invariant metric on X. By the Aoki–Rolewicz Theorem, each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms [6]. In [26], J. Tabor has investigated stability of the Cauchy functional equations (see [19]) in quasi-Banach spaces. One can see [10], [14], [20, 21], and [27] for some other works on stability in quasi-normed spaces.

Recently generalized stability of additive functional equations in quasi-β-normed spaces has been studied by J. Rassias and H.-M. Kim [24].

2. Stability Results

Throughout this section $\mathcal{G}$ is an abelian group, $\mathfrak{X}$, is a (β, p)-Banach space with p-normed $\|.\|_\beta$ and $\sigma : \mathcal{G} \to \mathcal{G}$ is an involution on $\mathcal{G}$, (i.e. σ is a group homomorphism for which $\sigma^2 = \sigma$). With $f : \mathcal{G} \to \mathfrak{X}$, consider the following quadratic functional equation

$$f(x+y+z) + f(x+\sigma(y)) + f(y+\sigma(z)) + f(x+\sigma(z)) = 3(f(x) + f(y) + f(z))$$

and suppose

$$\begin{aligned} \mathfrak{D}f(x, y, z) &= f(x+y+z) + f(x+\sigma(y)) + f(y+\sigma(z)) + f(x+\sigma(z)) \\ &- 3(f(x) + f(y) + f(z)). \end{aligned} \tag{8}$$

Theorem 2. *Let* $0 < L < 1$, $0 < \beta \leq 1$ *and* $\phi : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to [0, \infty)$ *be a mapping for which*

$$\phi(2x, 2y, 2z) \leq 4^{\beta} \sqrt[p]{\frac{L}{2}} \phi(x, y, z), \tag{9}$$

$$\phi(x + \sigma(x), y + \sigma(y), z + \sigma(z)) \leq 4^{\beta} \sqrt[p]{\frac{L}{2}} \phi(x, y, z), \tag{10}$$

$$\tag{11}$$

for all $x, y, z \in \mathcal{G}$. *Also let* $f : \mathcal{G} \to \mathfrak{X}$ *be a mapping for which* $f(0) = 0$ *and*

$$\|\mathfrak{D} f(x, y, z)\|_{\beta} \leq \phi(x, y, z), x, y, z \in \mathcal{G}. \tag{12}$$

Then there exists a unique function $Q : \mathcal{G} \to \mathfrak{X}$ *such that* $\mathfrak{D} Q(x, y, z) = 0$ *and*

$$\|Q(x) - f(x)\|_{\beta} \leq \frac{1}{4^{\beta}} \frac{1}{\sqrt[p]{1 - L}} \phi(x, x, 0). \tag{13}$$

Proof. Obviously $\sigma(0) = 0$. From (12), we have

$$\|\mathfrak{D} f(x, x, 0)\|_{\beta} = \|f(2x) + f(x + \sigma(x)) - 4f(x) - 3f(0)\|_{\beta} \leq \phi(x, x, 0). \tag{14}$$

so

$$\|\frac{f(2x) + f(x + \sigma(x))}{4} - f(x)\|_{\beta} \leq \frac{1}{4^{\beta}} \phi(x, x, 0), \tag{15}$$

Let $\mathcal{M}$ be the set of all functions from $\mathcal{G}$ into $\mathfrak{X}$. Define $d : \mathcal{M} \times \mathcal{M} \to [0, \infty]$ by

$$d(g, h) = \inf\{C \in [0, \infty], \|g(x) - h(x)\|_{\beta} \leq C^{\frac{1}{p}} \phi(x, x, 0), \text{ for all } x \in \mathcal{G}\}.$$

One can verify that $(\mathcal{M}, d)$ is a complete generalized metric space (for example see the proof of Theorem 3.1 in [16]). Now define $\Lambda : \mathcal{M} \to \mathcal{M}$ by

$$\Lambda(g)(x) = \frac{g(2x) + g(x + \sigma(x))}{4}.$$

From (15) one can see that

$$d(\Lambda(f), f) \leq \frac{1}{4^{\beta p}} < \infty. \tag{16}$$

Now we show that Λ is strictly contractive. For given $g, h \in \mathcal{M}$, let $d(f, g) \leq C$, for some $C \in [0, \infty]$. Thus

$$\|g(x) - h(x)\|_{\beta}^{p} \leq C\phi^{p}(x, x, 0).$$

It follows from definition of Λ, (9) and (10) that

$$\begin{aligned} \|(\Lambda g)(x) - (\Lambda h)(x)\|_{\beta}^{p} &\leq \frac{1}{4^{\beta p}} \big[\|g(2x) - h(2x)\|_{\beta}^{p} \\ &+ \|g(x + \sigma(x)) - h(x + \sigma(x))\|_{\beta}^{p}\big] \\ &\leq \frac{C}{4^{\beta p}} \big[\phi^{p}(2x, 2x, 0) + \phi^{p}(x + \sigma(x), x + \sigma(x), 0)\big] \\ &\leq LC\phi^{p}(x, x, 0), \end{aligned} \tag{17}$$

for all $x \in \mathcal{G}$, that is, $d(\Lambda g, \Lambda h) \leq LC$. We hence conclude that $d(\Lambda g, \Lambda h) \leq Ld(g, h)$. Therefor Λ is strictly contractive because $0 < L < 1$. By mathematical induction we obtain

$$\Lambda^n f(x) = \frac{1}{2^{2n}}\big[f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))\big]. \tag{18}$$

Now by Theorem 1, there exists a fixed point Q of Λ which is unique in $\mathcal{M}^* = \{g \in M : d(\Lambda f, g) < \infty\}$. Also $d(\Lambda^n f, Q) \to 0$ as $n \to \infty$. This, by definition of d, implies that there exists a sequence $\{C_n\}_{n\in\mathbb{N}}$ such that $C_n \to 0$, as $n \to \infty$, and for all $x \in \mathcal{G}$,

$$d(\Lambda^n f(x), Q(x)) \leq C_n,$$

therefore for any $x \in G$,

$$\lim_{n\to\infty} \|\Lambda^n f(x) - Q(x)\|_\beta^p = 0.$$

Hence

$$Q(x) = \lim_{n\to\infty} \Lambda^n f(x) = \lim_{n\to\infty} \frac{1}{2^{2n}}\big[f(2^n x) + (2^n - 1)f(2^{n-1}x + 2^{n-1}\sigma(x))\big], \tag{19}$$

for each $x \in \mathcal{G}$. Now by (9), (10), (12) and the fact that $0 < L < 1$ we have

$$\begin{aligned}
&\|\mathfrak{D}Q(x,y,z)\|_\beta^p = \|Q(x+y+z) + Q(x+\sigma(y)) + Q(y+\sigma(z)) + Q(x+\sigma(z))\\
&-3(Q(x)+Q(y)+Q(z))\|_\beta^p\\
&= \lim_{n\to\infty} \frac{1}{2^{2p\beta n}}\|\big[f(2^n(x+y+z))\\
&+(2^n-1)f(2^{n-1}(x+y+z) + 2^{n-1}\sigma(x+y+z))\big]\\
&+\big[f(2^n(x+\sigma(y))) + (2^n-1)f(2^{n-1}(x+\sigma(y)) + 2^{n-1}\sigma(x+\sigma(y)))\big]\\
&+\big[f(2^n(y+\sigma(z))) + (2^n-1)f(2^{n-1}(y+\sigma(z)) + 2^{n-1}\sigma(y+\sigma(z)))\big]\\
&+\big[f(2^n(x+\sigma(z))) + (2^n-1)f(2^{n-1}(x+\sigma(z)) + 2^{n-1}\sigma(x+\sigma(z)))\big]\\
&-3\big[f(2^n x) + (2^n-1)f(2^{n-1}x + 2^{n-1}\sigma(x))\big]\\
&-3\big[f(2^n y) + (2^n-1)f(2^{n-1}y + 2^{n-1}\sigma(y))\big]\\
&-3\big[f(2^n z) + (2^n-1)f(2^{n-1}z + 2^{n-1}\sigma(z))\big]\|_\beta^p\\
&\leq \lim_{n\to\infty} \frac{1}{4^{p\beta n}}\big[\|f(2^n(x+y+z)) + f(2^n(x+\sigma(y))) + f(2^n(y+\sigma(z)))\\
&+f(2^n(x+\sigma(z))) + 3(f(2^n x) + f(2^n y) + f(2^n z))\|_\beta^p\\
&+(2^n-1)^{p\beta}\|f(2^{n-1}(x+y+z) + 2^{n-1}\sigma(x+y+z))\\
&+f(2^{n-1}(x+\sigma(x)) + 2^{n-1}(y+\sigma(y)))\\
&+f(2^{n-1}(y+\sigma(y)) + 2^{n-1}(z+\sigma(z)))\\
&+f(2^{n-1}(x+\sigma(x)) + 2^{n-1}(z+\sigma(z)))\\
&+3\big[f(2^{n-1}x + 2^{n-1}\sigma(x)) + f(2^{n-1}y\\
&+2^{n-1}\sigma(y)) + f(2^{n-1}z + 2^{n-1}\sigma(z))\big]\|_\beta^p\big]
\end{aligned}$$

$$\leq \lim_{n\to\infty}\Big[\frac{1}{4^{\beta np}}\phi^p(2^n x, 2^n y, 2^n z)$$
$$+\frac{(2^n-1)^{\beta p}}{4^{\beta np}}\phi^p(2^{n-1}(x+\sigma(x)), 2^{n-1}(y+\sigma(y))+2^{n-1}(z+\sigma(z)))\Big]$$
$$\leq \lim_{n\to\infty}\frac{1}{4^{p\beta n}}(\frac{4^{p\beta}}{2}L)^n(1+\frac{(2^n-1)^{p\beta}}{4^{pn\beta}})\phi^p(x,y,z)$$
$$= \lim_{n\to\infty}(\frac{L}{2})^n(1+\frac{(2^n-1)^{p\beta}}{4^{pn\beta}})\phi^p(x,y,z)=0.$$

By Theorem 1 and (16), we obtain

$$d(f,Q)\leq\frac{1}{1-L}d(\Lambda f,f)\leq\frac{1}{4^{p\beta}(1-L)}, \tag{20}$$

that is, (13) is true for all $x\in\mathcal{G}$.
For the uniqueness part, it is enough to show that $Q\in\mathcal{M}^*$, i.e. $d(\Lambda(f),Q)<\infty$. The fact that Λ is a contraction implies that

$$\begin{aligned} d(\Lambda(f),Q) &= d(\Lambda(f),\Lambda(Q)) \\ &\leq Ld(f,Q)\leq\frac{L}{1-L}d(\Lambda f,f)\leq\frac{L}{4^{p\beta}(1-L)}<\infty, \end{aligned}$$

hence $Q\in\mathcal{M}^*$, and this completes the proof. □

In a similar way, by applying Theorem 1, we can prove the following theorem.

Theorem 3. *Let $0<L<1$, $0<\beta\leq 1$ and $\phi:\mathcal{G}\times\mathcal{G}\times\mathcal{G}\to[0,\infty)$ be a mapping for which*

$$\phi(x,y,z)\leq\frac{1}{4^\beta}\sqrt[p]{\frac{L}{2}}\phi(2x,2y,2z) \tag{21}$$

$$\phi(x+\sigma(x),y+\sigma(y),z+\sigma(z))\leq 2^\beta\phi(2x,2y,2z) \tag{22}$$

for all $x,y,z\in\mathcal{G}$. Also let $f:\mathcal{G}\to\mathfrak{X}$ be a mapping with

$$\|\mathfrak{D}f(x,y,z)\|_\beta\leq\phi(x,y,z), x,y,z\in\mathcal{G}. \tag{23}$$

Then there exists a unique function $Q:\mathcal{G}\to\mathfrak{X}$ such that $DQ(x,y,z)=0$ and

$$\|Q(x)-f(x)\|_\beta\leq\frac{1}{4^\beta}\sqrt[p]{\frac{L}{1-L}}\phi(x,x,0). \tag{24}$$

Proof. We use the same definitions for $\mathcal{M}$ and d as in the proof of Theorem 2. From (23), we have

$$\|\mathfrak{D}f(\frac{x}{2},\frac{x}{2},0)\|_\beta=\|f(x)+f(\frac{x}{2}+\frac{\sigma(x)}{2})-4f(\frac{x}{2})\|_\beta\leq\phi(\frac{x}{2},\frac{x}{2},0), \tag{25}$$

and

$$\begin{aligned} \|\mathfrak{D}f(\frac{x}{4}+\frac{\sigma(x)}{4},\frac{x}{4}+\frac{\sigma(x)}{4},0)\|_\beta &= \|2f(\frac{x}{2}+\frac{\sigma(x)}{2})-4f(\frac{x}{4}+\frac{\sigma(x)}{4})\|_\beta \\ &\leq \phi(\frac{x}{4}+\frac{\sigma(x)}{4},\frac{x}{4}+\frac{\sigma(x)}{4},0) \end{aligned} \tag{26}$$

Now define $\Lambda : \mathcal{M} \to \mathcal{M}$ by

$$(\Lambda h)(x) = 4\big[h(\frac{x}{2}) - \frac{1}{2}h(\frac{x}{4} + \frac{\sigma(x)}{4})\big].$$

Therefore, it follows by (25), (26), (23), (21) and (22) that for any $x \in \mathcal{G}$,

$$\begin{aligned}
\|f(x) \quad &- \quad (\Lambda f)(x)\|_\beta^p = \|f(x) - 4\big[f(\frac{x}{2}) - \frac{1}{2}f(\frac{x}{4} + \frac{\sigma(x)}{4})\big]\|_\beta^p \\
&\leq \quad \|f(x) + f(\frac{x}{2} + \frac{\sigma(x)}{2}) - 4f(\frac{x}{2})\|_\beta^p \\
&+ \quad \|f(\frac{x}{2} + \frac{\sigma(x)}{2}) - 2f(\frac{x}{4} + \frac{\sigma(x)}{4})\|_\beta^p \\
&\leq \quad \phi^p(\frac{x}{2}, \frac{x}{2}, 0) + \frac{1}{2^{p\beta}}\phi^p(\frac{x}{4} + \frac{\sigma(x)}{4}, \frac{x}{4} + \frac{\sigma(x)}{4}, 0) \\
&\leq \quad \frac{L}{2.4^{p\beta}}\phi^p(x, x, 0) + \frac{1}{2^{p\beta}}\frac{L}{2.4^{p\beta}}2^{p\beta}\phi^p(x, x, 0)) \\
&= \quad \frac{1}{4^{p\beta}}L\phi^p(x, x, 0).
\end{aligned}$$

This means that

$$d(\Lambda f, f) \leq \frac{1}{4^{p\beta}}L. \tag{27}$$

Also Λ is a strictly contractive operator. In fact for given $g, h \in \mathcal{M}$, if $d(f, g) < C$, $C \in [0, \infty]$, then for all $x \in \mathcal{G}$, $\|g(x) - h(x)\|_\beta^p \leq C\phi(x, x, 0)$, thus by (21), (22) and definition of Λ,

$$\begin{aligned}
\|(\Lambda f)(x) - (\Lambda g)x\|_\beta^p \quad &= \quad 4^{p\beta}\|g(\frac{x}{2}) - \frac{1}{2}g(\frac{x}{4} + \frac{\sigma(x)}{4}) - h(\frac{x}{2}) - \frac{1}{2}h(\frac{x}{4} + \frac{\sigma(x)}{4})\|_\beta^p \\
&\leq \quad 4^{p\beta}\|g(\frac{x}{2}) - h(\frac{x}{2})\|_\beta^p \\
&+ \quad 2^{p\beta}\|g(\frac{x}{4} + \frac{\sigma(x)}{4}) - h(\frac{x}{4} + \frac{\sigma(x)}{4})\|_\beta^p \\
&\leq \quad 4^{p\beta}C\phi^p(\frac{x}{2}, \frac{x}{2}, 0) + 2^{p\beta}C\phi^p(\frac{x}{4} + \frac{\sigma(x)}{4}, \frac{x}{4} + \frac{\sigma(x)}{4}, 0) \\
&\leq \quad LC\phi^p(x, x, 0).
\end{aligned}$$

This implies that $d(\Lambda f, \Lambda g) \leq Ld(g, h)$. Thus by Theorem 1, there exists a unique function $Q : \mathcal{G} \to \mathfrak{X}$ which is a fixed point of Λ in $\mathcal{M}^*$ and $Q = \lim_{n\to\infty} \Lambda^n(f)$. But using mathematical induction one may obtain that

$$(\Lambda^n f)(x) = 2^{2n}\big[f(\frac{x}{2^n}) + (\frac{1}{2^n} - 1)f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}})\big].$$

A similar argument to the proof of Theorem 2, shows that $\mathfrak{D}Q(x, y, z) = 0$, $x, y, z \in \mathcal{G}$, and by Theorem 2(c) and (27), we get

$$d(f, Q) \leq \frac{1}{4^{p\beta}}\frac{L}{1 - L}.$$

This completes the proof. $\square$

Remark 1. Suppose $\sigma(x) = x$, so if for $x, y, z \in \mathcal{G}$,

$$\mathfrak{D}f(x,y,z) = f(x+y+z)+f(x+y)+f(y+z)+f(x+z)-3(f(x)+f(y)+f(z)) = 0,$$

then $\mathfrak{D}f(0,0,0) = f(0) = 0$, and for any $x, y \in \mathcal{G}$,

$$\mathfrak{D}f(x,y,0) = 2\big(f(x+y) - f(x) - f(y)\big) = 0.$$

Hence f satisfies in the Cauchy equation and so is an additive function. Now with $\sigma(x) = x$, $0 < L < 1$, and ϕ satisfying (9) and (10), suppose

$$\|\mathfrak{D}f(x,y,z)\|_\beta \leq \phi(x,y,z)$$

so by Theorem 2, and the above assertion, there exists an additive mapping Q such that for all $x \in \mathcal{G}$

$$\|Q(x) - f(x)\| \leq \frac{1}{4^\beta}\sqrt[p]{\frac{L}{1-L}}\phi(x,x,0).$$

A similar argument can be concluded with the conditions of Theorem 3.

In the sequel we consider the pexiderized form of the quadratic functional equation (5). For $f, g : \mathcal{G} \to \mathfrak{X}$ we consider the following pexiderized quadratic equation

$$f(x+y+z)+f(x+\sigma(y))+f(y+\sigma(z))+f(x+\sigma(z)) = g(x)+g(y)+g(z). \quad (28)$$

with the involution σ. Let

$$\begin{aligned} \mathfrak{D}_{f,g}(x,y,z) : \quad &= \quad f(x+y+z) + f(x+\sigma(y)) + f(y+\sigma(z)) + f(x+\sigma(z)) \\ &- \quad g(x) - g(y) - g(z). \end{aligned}$$

Also for $\phi : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to \mathfrak{X}$, put

$$\begin{aligned} \widetilde{\phi}(x,y,z) \quad &= \quad \big(\phi^p(x+y+z,0,0) + \phi^p(x+\sigma(y),0,0) + \phi^p(y+\sigma(z),0,0) \\ &+ \quad \phi^p(x+\sigma(z),0,0) + 3^{p\beta}\phi^p(x,y,z) + \phi^p(0,0,0)\big)^{\frac{1}{p}}, \end{aligned}$$

and

$$\widetilde{\psi}(x,y,z) = \big(\phi^p(x,y,z) + \phi^p(x,0,0) + \phi^p(0,y,0) + \phi^p(0,0,z) + 2^{p\beta}\phi^p(0,0,0)\big)^{\frac{1}{p}}.$$

For studying the Hyer-Ulam stability of (28), we need the following lemma.

Lemma 1. *Suppose $f, g : \mathcal{G} \to \mathfrak{X}$ and $\phi : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to \mathfrak{X}$ are functions for which*

$$\|\mathfrak{D}_{f,g}(x,y,z)\|_\beta \leq \phi(x,y,z), \quad x, y, z \in \mathcal{G} \quad (29)$$

and ϕ satisfies (9) and (10), then for any $x, y, z \in \mathcal{G}$,

$$\begin{aligned} \|\mathfrak{D}_f(x,y,z) + 5f(0)\|_\beta \quad &\leq \widetilde{\psi}(x,y,z), \quad (30) \\ \|\mathfrak{D}_g(x,y,z) + 5g(0)\|_\beta \quad &\leq \widetilde{\phi}(x,y,z). \quad (31) \end{aligned}$$

Proof. It is easy to see that

$$\mathfrak{D}_{f,g}(x+y+z,0,0) = 3f(x+y+z) - g(x+y+z) + f(0) - 2g(0), \tag{32}$$
$$\mathfrak{D}_{f,g}(x+\sigma(y),0,0) = 3f(x+\sigma(y)) - g(x+\sigma(y)) + f(0) - 2g(0), \tag{33}$$
$$\mathfrak{D}_{f,g}(y+\sigma(z),0,0) = 3f(y+\sigma(z)) - g(y+\sigma(z)) + f(0) - 2g(0), \tag{34}$$
$$\mathfrak{D}_{f,g}(x+\sigma(z),0,0) = 3f(x+\sigma(z)) - g(x+\sigma(z)) + f(0) - 2g(0). \tag{35}$$

So these relations and (29) and definition of $\widetilde{\psi}$, imply that

$$\begin{aligned}
&\|\mathfrak{D}_g(x,y,z) - 5g(0)\|_\beta^p \\
&\le \|g(x+y+z) - 3f(x+y+z) - f(0) + 2g(0)\|_\beta^p \\
&+\|g(x+\sigma(y)) - 3f(x+\sigma(y)) - f(0) + 2g(0)\|_\beta^p \\
&+\|g(y+\sigma(z)) - 3f(y+\sigma(z)) - f(0) + 2g(0)\|_\beta^p \\
&+\|g(x+\sigma(z)) - 3f(x+\sigma(z)) - f(0) - 2g(0)\|_\beta^p \\
&+3^{p\beta}\|f(x+y+z) + f(x+\sigma(y)) + f(y+\sigma(z)) \\
&+f(x+\sigma(z)) - g(x) - g(y) - g(z)\|_\beta^p + \|4f(0) - 3g(0)\|_\beta^p \\
&\le \phi^p(x+y+z,0,0) + \phi^p(x+\sigma(y),0,0) + \phi^p(y+\sigma(z),0,0) \\
&+\phi^p(x+\sigma(z),0,0) + 3^{p\beta}\phi^p(x,y,z) + \phi^p(0,0,0) \\
&= \widetilde{\phi}^p(x,y,z).
\end{aligned}$$

This implies the inequality (42). The inequality (41) can be concluded similarly. □

Theorem 4. *Let $0 < L < 1$, $0 < \beta \le 1$ and $\phi : \mathcal{G}\times\mathcal{G}\times\mathcal{G} \to [0,\infty)$ be a mapping for which*

$$\phi(2x,2y,2z) \le 4^\beta \sqrt[p]{\frac{L}{2}}\phi(x,y,z) \tag{36}$$

$$\phi(x+\sigma(x), y+\sigma(y), z+\sigma(z)) \le 4^\beta \sqrt[p]{\frac{L}{2}}\phi(x,y,z) \tag{37}$$

for all $x,y,z \in \mathcal{G}$. Also let $f,g : \mathcal{G} \to \mathfrak{X}$ be mappings with $f(0) = g(0) = 0$ and

$$\|\mathfrak{D}_{f,g}(x,y,z)\|_\beta \le \phi(x,y,z), x,y,z \in \mathcal{G}. \tag{38}$$

Then there exists a unique function $Q : \mathcal{G} \to \mathfrak{X}$ such that $\mathfrak{D}Q(x,y,z) = 0$ and

$$\|Q(x) - f(x)\|_\beta \le \frac{1}{4^\beta}\frac{1}{\sqrt[p]{1-L}}\widetilde{\phi}(x,x,0) \tag{39}$$

$$\|3Q(x) - g(x)\|_\beta \le \frac{1}{4^\beta}\frac{1}{\sqrt[p]{1-L}}\widetilde{\psi}(x,x,0) \tag{40}$$

Proof. Lemma 1 and $f(0) = g(0) = 0$ imply that

$$\|\mathfrak{D}_f(x,y,z)\|_\beta^p \qquad \le \widetilde{\psi}(x,y,z), \tag{41}$$

$$\|\mathfrak{D}_g(x,y,z)\|_\beta^p \qquad \le \widetilde{\phi}(x,y,z). \tag{42}$$

Using (36) and (37) one can easily see that

$$\widetilde{\phi}(2x,2y,2z) \le 4^{\beta}\sqrt[p]{\frac{L}{2}}\widetilde{\phi}(x,y,z)$$

and

$$\widetilde{\phi}(x+\sigma(x),y+\sigma(y),z+\sigma(z)) \le 4^{\beta}\sqrt[p]{\frac{L}{2}}\widetilde{\phi}(x,y,z),$$

also

$$\widetilde{\psi}(2x,2y,2z) \le 4^{\beta}\sqrt[p]{\frac{L}{2}}\widetilde{\psi}(x,y,z)$$

and

$$\widetilde{\psi}(x+\sigma(x),y+\sigma(y),z+\sigma(z)) \le 4^{\beta}\sqrt[p]{\frac{L}{2}}\widetilde{\psi}(x,y,z).$$

Now by Theorem 2, there exist unique functions $Q_1, Q_2 : G \to X$ such that $\mathfrak{D}Q_1(x,y,z) = \mathfrak{D}Q_2(x,y,z) = 0$, for which

$$\|Q_1(x) - f(x)\|_{\beta} \le \frac{1}{4^{\beta}}\frac{1}{\sqrt[p]{1-L}}\widetilde{\phi}(x,x,0)$$

and

$$\|Q_2(x) - g(x)\|_{\beta} \le \frac{1}{4^{\beta}}\frac{1}{\sqrt[p]{1-L}}\widetilde{\psi}(x,x,0).$$

It is enough to show that $3Q_2 = Q_1$. By Theorem 2, with Λ as in the proof of Theorem 2,

$$Q_1(x) = lim_{n\to\infty}\Lambda^n(f)(x) \quad \text{and} \quad Q_2(x) = lim_{n\to\infty}\Lambda^n(g)(x). \tag{43}$$

On the other hand by (18), (38), (36) and (37) we have

$$\begin{aligned}
&\|3\Lambda^n(f)(x) - \Lambda^n(g)(x)\|_{\beta}^p \le \frac{1}{2^{2np\beta}}\|3f(2^n x) - g(2^n x)\|_{\beta}^p \\
&+\frac{(2^n-1)^{p\beta}}{2^{2np\beta}}\|3f(2^{n-1}(x+\sigma(x))) - g(2^{n-1}(x+\sigma(x)))\|_{\beta}^p \\
&\le \frac{1}{2^{2np\beta}}\phi^p(2^n x,0,0) + \frac{(2^n-1)^{p\beta}}{2^{2np\beta}}\phi^p(2^{n-1}(x+\sigma(x)),0,0) \\
&\le \frac{1}{4^{np\beta}}(\frac{4^{p\beta}}{2}L)^n(1+\frac{2(2^n-1)^{p\beta}}{4^{p\beta}L})\phi^p(x,0,0) \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Now (43) implies that $3Q_2 = Q_1$. This completes the proof. $\square$

As in Theorem 3, in a similar way, by applying Lemma 1, and the same argument as in the proof of Theorem 4, we can state the following conclusion.

Theorem 5. *Let $0 < L < 1$, $0 < \beta \le 1$ and $\phi : G \times G \times G \to [0,\infty)$ be a mapping for which*

$$\phi(x,y,z) \le \frac{1}{4^{\beta}}\sqrt[p]{\frac{L}{2}}\phi(2x,2y,2z) \tag{44}$$

$$\phi(x+\sigma(x),y+\sigma(y),z+\sigma(z)) \le 2^{\beta}\phi(2x,2y,2z) \tag{45}$$

for all $x, y, z \in G$. Also let $f, g : G \to X$ be mappings with $f(0) = g(0) = 0$ and

$$\|D_{f,g}(x, y, z)\|_\beta \le \phi(x, y, z), x, y, z \in G. \tag{46}$$

Then there exists a unique function $Q : G \to X$ such that $DQ(x, y, z) = 0$ and

$$\|Q(x) - f(x)\|_\beta \le \frac{1}{4^\beta} \sqrt[p]{\frac{L}{1 - L}} \widetilde{\phi}(x, x, 0) \tag{47}$$

$$\|3Q(x) - g(x)\|_\beta \le \frac{1}{4^\beta} \sqrt[p]{\frac{L}{1 - L}} \widetilde{\psi}(x, x, 0). \tag{48}$$

Remark 2. As a conclusion of Theorems 4 and 5, and 1, with $\sigma(x) = x$, one may obtain that the function Q in Theorems 4 and 5 is additive.

Corollary 1. *Suppose $0 < p < 1$ and $\frac{p+1}{2} < \beta \le 1$. Let $\mathfrak{E}$ be a normed space and let $\mathfrak{X}$ be a complete (β, p)-normed space. If for some positive ε, functions $f, g : \mathfrak{E} \to \mathfrak{X}$ satisfy*

$$\|\mathfrak{D}_{f,g}(x, y, z)\|_\beta \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \tag{49}$$

and $\|x + \sigma(x)\|^p \le 2^p \|x\|^p$, for all $x \in \mathfrak{E}$, then there exists a unique function $Q : \mathfrak{E} \to \mathfrak{X}$ such that $\mathfrak{D}Q(x, y, z) = 0$, for all $x, y, z \in \mathfrak{E}$. Furthermore

$$\|f(x) - Q(x)\|_\beta \le \varepsilon \sqrt[p]{\frac{2^{p^2+1} + 2 + 2^p.3^{p\beta}}{4^{p\beta} - 2^{p^2+1}}} \|x\|^p \tag{50}$$

$$\|g(x) - 3Q(x)\|_\beta \le \varepsilon \sqrt[p]{\frac{2^{p^2} + 2}{4^{p\beta} - 2^{p^2+1}}} \|x\|^p \tag{51}$$

Proof. Put $\phi(x, y, z) = \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$, for all $x, y, z \in \mathfrak{E}$, and set $L = \frac{2^{p^2+1}}{4^p \beta}$. Then $0 < L < 1$ and

$$\phi(2x, 2y, 2z) = 4^\beta \sqrt[p]{\frac{L}{2}} \phi(x, y, z).$$

Moreover we have

$$\phi(x + \sigma(x), y + \sigma(y), z + \sigma(z)) \le 4^\beta \sqrt[p]{\frac{L}{2}} \phi(x, y, z).$$

According to Theorem 4, there exists a unique function $Q : \mathfrak{E} \to \mathfrak{X}$ such that $\mathfrak{D}Q(x, y, z) = 0$, $x, y, z \in \mathfrak{E}$, and (50), (51) holds, for all $x \in \mathfrak{E}$. □

Corollary 2. *For a fixed numbers $q > 1$ and $\varepsilon > 0$, let $0 < \beta < \frac{q-1}{2}$. Suppose $pq > 1\mathfrak{E}$ and X is are normed space and complete (β, p)-normed spaces, respectively. Also let for $f, g : \mathfrak{E} \to \mathfrak{X}$, $\mathfrak{D}_{f,g}(x, y, z) \le \varepsilon(\|x\|^q + \|y\|^q + \|z\|^q)$, for all $x, y, z \in \mathfrak{E}$, and $\|x + \sigma(x)\|^p \le 2^{p+\beta}\|x\|^p$, for all $x \in \mathfrak{E}$. Then there exists a unique function $Q : \mathfrak{E} \to \mathfrak{X}$ such that $\mathfrak{D}Q(x, y, z) = 0$, $x, y, z \in \mathfrak{E}$ and*

$$\|f(x) - Q(x)\|_\beta \le \varepsilon \sqrt[p]{\frac{2^{pq}(1 + 2^{p\beta}) + 2 + 2^p.3^{p\beta}}{2^{pq-1} - 4^{p\beta}}} \|x\|^q \tag{52}$$

$$\|g(x) - 3Q(x)\|_\beta \leq \varepsilon \sqrt[p]{\frac{2^p + 2}{2^{pq-1} - 4^{p\beta}}} \|x\|^q \tag{53}$$

Proof. Set $\phi(x, y, z) = \varepsilon(\|x\|^q + \|y\|^q + \|z\|^q)$, for all $x, y, z \in \mathfrak{E}$, and $L = \frac{4^{p\beta}}{2^{pq-1}}$. Then $0 < L < 1$ and

$$\phi(x, y, z) = \frac{1}{4^\beta} \sqrt[p]{\frac{L}{2}} \phi(2x, 2y, 2z).$$

Moreover we have

$$\phi(x + \sigma(x), y + \sigma(y), z + \sigma(z)) \leq 2^\beta \phi(2x, 2y, 2z).$$

According to Theorem 5, there exists a unique function $Q : \mathfrak{E} \to \mathfrak{X}$ such that $\mathfrak{D}Q(x, y, z) = 0$, $x, y, z \in \mathfrak{E}$, and (52), (53) holds, for all $x \in \mathfrak{E}$. □

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