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A MATRIX FORMULATION OF THE MIXED TYPE LINEAR VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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ABSTRACT. In this paper we present an operational method for solving linear Volterra-Fredholm integral equations (VFIE). The method is constructed based on three matrices with simple structures which lead to a simple and high accurate algorithm. We also present an error estimation and demonstrate accuracy of the method by numerical examples.

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1. Introduction

Consider a mixed type VFIE of the form

$$u(t,x) = f(t,x) + \lambda \int_0^t \int_\Omega G(t,s,x,\xi) g(u(s,\xi)) d\xi ds \tag{1}$$

where $\lambda \in \mathbb{R}$, $t \in I := [0,T]$, $x \in \Omega \subset \mathbb{R}^d (d = 1,2,3)$ and Ω is closed and bounded. For $g(x) = x$, equation (1) is linear, otherwise it is a nonlinear equation. Here we assume that the given real valued functions $f := f(t,x)$, $G := G(t,s,x,\xi)$ and $g := g(x)$ are at least continuous on $D = [0,T] \times \Omega$, S (where $S := \{(t,s,x,\xi) : \ 0 \le s \le t \le T, \ \ (x,\xi) \in \Omega)$ and $\mathbb{R}$ respectively and $\Omega := (a,b) \subset \mathbb{R}$ $(d = 1)$.

Equations of type (1) often arise from mathematical modeling of the spreading in space and time, of some contagious disease in a population living in a habit Ω [3] and in many physical and biological models. Existence and uniqueness results for (1) may be found in [3,4,10]. In [5] for the linear case and in [6] for the general case numerical solution of VFIE is carried out by continuose-time and discrete-time spline collocation methods. In [7] certain choice of direct

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quadrature(DQ) applied for discretization in time for VFIE. In [8] the trapezoidal Nystrom method used for space and time and in [9] the Nystrom method and direct quadrature method is used respectively for space and time and an efficient solution of the nonlinear systems arising from discretization is given.

2. Linear equations and approximation with respect to spatial variable

Let us consider the linear VFIE

$$u(t,x) = f(t,x) + \lambda \int_0^t \int_a^b G(t,s,x,\xi)\ u(s,\xi)\ d\xi ds \tag{2}$$

and assume that $X = [1, x, x^2, \cdots, x^n]^T$ be the standard basis and $u_n(t,x)$ be approximate solution of equation (2) for spatial part, that is

$$u_n(t,x) = \sum_{i=0}^{n} u_i(t)\ x^i. \tag{3}$$

We also assume that the given functions f and G are polynomials, otherwise they can be approximated by suitable polynomials. Thus they can be written as

$$f(t,x) = \sum_{i=1}^{n} f_i(t) x^i \tag{4}$$

$$G(t,s,x,\xi) = \sum_{j=0}^{n}\sum_{i=0}^{n} G_{i,j}(t,s) x^i \xi^j \tag{5}$$

where $f_i(t)$ and $G_{i,j}(t,s), i,j = 0,1,\cdots,n$ are also polynomials or approximated by suitable polynomials. Substituting from (4) and (5) into (2) and equating coefficient of x^i for $i = 0,1,\cdots,n$ yields the following system of $(n+1)$ linear Volterra Integral Equations(VIEs) for the unknown functions $u_0(t), u_1(t), \cdots, u_n(t)$

$$u_i(t) = f_i(t) + \lambda \sum_{k=0}^{n}\sum_{j=0}^{n} \int_0^t A_{j,k} G_{i,j}(t,s) u_k(s) ds, \quad i = 0,1,\cdots,n. \tag{6}$$

where

$$A_{j,k} = \frac{b^{j+k+1} - a^{j+k+1}}{j+k+1}, \quad j,k = 0,1,\cdots,n. \tag{7}$$

Now let

$$G'_{i,k}(t,s) = \sum_{j=0}^{n} G_{i,j}(t,s) A_{j,k}$$

then Eq. (6) can be rewritten as

$$u_i(t) = f_i(t) + \lambda \int_0^t \sum_{k=0}^{n} G'_{i,k}(t,s) u_k(s) ds, \quad i = 0,1,\cdots,n \tag{8}$$

which is a system of linear second kind VIEs.

To convert (8) to the corresponding system of linear algebraic equations, let

$$u_{i_m}(t) \simeq \sum_{p=0}^{m} u_{ip} t^p \tag{9}$$

$$f_i(t) \simeq \sum_{p=0}^{m} f_{ip} t^p \tag{10}$$

and

$$G'_{i,k}(t,s) \simeq \sum_{l=0}^{m} \sum_{q=0}^{m} G'_{i,k,q,l} t^l s^q \tag{11}$$

and substitute these approximations in (8) to get

$$\sum_{p=0}^{m} u_{ip} t^p = \sum_{p=0}^{m} f_{ip} t^p + \sum_{k=0}^{n} \sum_{l=0}^{m} \sum_{q=0}^{m} \sum_{p=0}^{m} G'_{i,k,q,l} u_{k,p} \frac{t^{l+q+p+1}}{m+q+1}, \quad i = 0,1,\cdots,n \tag{12}$$

which is summarized by using matrix multiplication in

$$U_{nm}(I - \lambda \underline{\underline{G}})T' = f_{nm} T'$$

or equivalently in

$$U_{nm}(I - \lambda \underline{\underline{G}}) = f_{nm} \tag{13}$$

where

$$U_{nm} = [u_{00}\ u_{01}\ \cdots\ u_{0m}, u_{10}\ u_{11}\ \cdots\ u_{1m},\ \cdots,\ u_{n0}\ u_{n1}\ \cdots\ u_{nm}],$$

$$f_{nm} = [f_{00}\ f_{01}\ \cdots\ f_{0m}, f_{10}\ f_{11}\ \cdots\ f_{1m},\ \cdots,\ f_{n0}\ f_{n1} \cdots\ f_{nm}],$$

and

$$\underline{\underline{G}} = \begin{pmatrix} G''_{00} & G''_{01} & \cdots & G''_{0n} \\ G''_{10} & G''_{11} & \cdots & G''_{1n} \\ \vdots & \vdots & & \vdots \\ G''_{n0} & G''_{n1} & \cdots & G''_{nn} \end{pmatrix}.$$

Note that T' contains n times the basis vector $T = [1, t, \cdots, t^m]^T$ and $\underline{\underline{G}}$ is the matrix representation of the integral part of (2) that contains the $(m+1)\times(m+1)$ submatrices $G''_{i,j},\ i,j = 0,1,\cdots,n$. Thus $\underline{\underline{G}}$ is an $(m+1)(n+1)\times(m+1)(n+1)$ matrix and I is an $(m+1)(n+1)\times(m+1)(n+1)$ identity matrix. To find columns of $G''_{i,j}, i,j = 0,1,\cdots,n$, we equate coefficients of t^i for $i = 0,1,\cdots,m$ on both sides of (12) and obtain

$$(G''_{i,k})_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad (G''_{i,k})_2 = \begin{pmatrix} G'_{i,k,0,0} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{14}$$

and

$$
(G''_{i,k})_q = \begin{pmatrix} \sum_{j=0}^{q-2} \frac{G'_{i,k,q-j-2,j}}{j+1} \\ \sum_{j=1}^{q-2} \frac{G'_{i,k,q-j-2,j-1}}{j+1} \\ \vdots \\ \frac{G'_{i,k,0,0}}{q-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \to (q-1)^{th} row, \qquad q = 3, 4, \cdots m. \\ \\ \\ \\ \end{matrix} \tag{15}
$$

Consequently each block of $\underline{\underline{G}}$ is an upper triangular matrix, which has useful computational affects on solving system (13).

After solving the system (13), we will have an approximate solution for (2) of the form

$$
u_{n,m}(t,x) = \sum_{i=0}^{n} \sum_{j=0}^{m} u_{ij} x^i t^j
$$

where u_{ij} is the $((n+1)\times(i-1)+j)^{th}$ component of U_{nm}.

3. Computing blocks of $\underline{\underline{G}}$ by using operational matrices

The operational matrices which firstly defined by E. Ortiz and M. Samara for solving nonlinear differential equations [2] have useful computational effects on the computing the blocks of $\underline{\underline{G}}$, since these blocks can be computed only by using nonzero elements of these operational matrices. These matrices are

$$
\mu = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \eta = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \iota = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1/2 & 0 & \cdots \\ 0 & 0 & 0 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
$$

Here we only use μ and ι for the matrix representation of integral part of (2). To this end, we recall the following lemmas and theorems from [1].

Lemma 1. *If $y_n(x) = \underline{\underline{a}}_n \underline{X}$ with $\underline{\underline{a}}_n = (a_0, a_1, \cdots, a_n, 0, 0, \cdots)$ and $\underline{X} = (1, x, x^2, \cdots, x^n, \cdots)^T$ then*

$$
\int y_n(x)dx = \underline{\underline{a}}_n \iota \underline{X}, \qquad xy_n(x) = \underline{\underline{a}}_n \mu \underline{X}
$$

Theorem 1. *For $\int k(x,t)y_n(t)dt$ with $k(x,t) = \sum_j \sum_i c_{ij} x^i t^j$ and $y_n(x) = \underline{\underline{a}}_n \underline{X} = \underline{\underline{\hat{a}}}_n \underline{V}$ where $\underline{\underline{a}}_n = (a_0, a_1, \cdots, a_n, 0, 0, \cdots)$, $\underline{\underline{\hat{a}}}_n = (\hat{a}_0, \hat{a}_1, \cdots, \hat{a}_n, 0, 0, ...)$we have*

$$
\int_0^t K(x,t)y_n(t)dt = \underline{\underline{a}}_n \Pi_l \underline{X} = \underline{\underline{\hat{a}}}_n \hat{\Pi}_l \underline{V}
$$

with

$$\Pi_l = \sum_{i=0}^{n}\sum_{j=0}^{n} c_{i,j}\mu^i \iota \mu^j, \quad \hat{\Pi}_l = V\Pi_l V^{-1}$$

where $\underline{V} = \{v_i(x))\}_{i=0}^m$ *is a polynomial basis given by* $\underline{V} = VX$, V *is nonsingular lower triangular matrix and* V^{-1} *is its inverse.*

Lemma 2. *For any given* $n \times n$ *matrix* $M = (m_{k,l})$, *we have*

$$(\mu^j M \mu^i)_{k,l} = \begin{cases} m_{k+j,l-i}, & k = 1,2,...,n+1-j, \quad l = i+1,...,n+1 \\ 0, & \textit{otherwise.} \end{cases}$$

Lemma 3. *For the matrix* ι, *we have*

$$(\mu^j \iota \mu^i)_{k,k+i+j+1} = \begin{cases} \frac{1}{k+j}, & i,j,k = 1,2,...,n, \quad k+i+j+1 \le n \\ 0, & \textit{otherwise.} \end{cases} \tag{16}$$

Theorem 2. *Let* $K(x,t) = \sum_{i=0}^n \sum_{j=0}^m c_{i,j}x^i t^j$, *then we have*

$$\int_a^b K(x,t)y_n(t)dt = \underline{\underline{a}}_n \Pi_F \underline{X} = \underline{\underline{\hat{a}}}_n \hat{\Pi}_F \underline{V}$$

where $\Pi_F = \sum_{i=0}^n \sum_{j=0}^m c_{ij}(\xi^{(ij)}(b) - \xi^{(ij)}(a))\underline{e}_{i+1}^T$, *denotes the matrix representation for the Fredholm integral part of (2) and* $\hat{\Pi}_F = V\Pi_F V^{-1}$. $\xi^{(ij)}$ *is to denote* $\xi(a) = (\xi_1, \xi_2, \cdots, \xi_n)^T$ *corresponding to the term* $x^i t^j$ *in the kernel. The first* n *entries of it in* $x = a$ *are as follows*

$$\xi_k = \begin{cases} \frac{a^{k+j}}{k+j}, & k = 1,2,...,n-1-j \\ 0, & \textit{otherwise.} \end{cases}$$

$\underline{e}_{i+1}$ *is the* $(i+1)^{th}$ *coordinate of the unit vector.*

Theorem 3. *Let* $K_2(x,t) = \sum_{i=0}^n \sum_{j=0}^m d_{i,j}x^i t^j$, *then we have*

$$\int_a^x k_2(x,t)y_n(t)dt = \underline{\underline{a}}_n \Pi_V \underline{X} = \underline{\underline{\hat{a}}}_n \hat{\Pi}_V \underline{V}$$

where $\Pi_V = \sum_{i=0}^n \sum_{j=0}^m d_{ij}(\mu^j \iota \mu^i - \xi^{(ij)}(a)\underline{e}_{i+1}^T)$, *stands for the matrix representation of the Volterra integral part of (2). For other basis such as* V, $\hat{\Pi}_V = V\Pi_V V^{-1}$ *must be calculated.*

Now, in equation (2) for the matrix representation of interior integral, the Fredholm part of equation, theorem 2 yields

$$G' = \sum_{i=0}^{n}\sum_{j=0}^{m} G_{i,j}(t,s)(\xi^{(ij)}(b) - \xi^{(ij)}(a))\underline{e}_{i+1}^T.$$

If we let $G'_{i,k} = \sum_{l=0}^m \sum_{q=0}^m G'_{i,k,q,l}t^l s^q = \sum_{l=0}^m \sum_{q=0}^m c_{l,q}t^l s^q$ then the blocks of $\underline{\underline{G}}$ (i.e. $G''_{i,j}$ for $i,j = 0,1,\cdots,n$) are computed by $\Pi_{V,i,k}$ with $a = 0$, since from theorem 3, we have

$$\int_0^t G'_{i,k}(t,s)u_k(s)ds = \underline{\underline{u}}_k \Pi_{l,i,k} T,$$

with

$$\Pi_{l,i,k} = \sum_{i=0}^{m}\sum_{j=0}^{m} c_{i,j}\mu^i \iota \mu^j, \qquad \underline{\underline{u}}_k = (u_{k0}, u_{k1}, \cdots, u_{km}, 0, 0, \cdots),$$

where $T = (1, t, t^2, \cdots, t^m, ..)^T$.

4. Solving the system (13)

Since the diagonal blocks of the coefficient matrix, $I - \lambda\underline{\underline{G}}$, are upper triangular with diagonal entries 1 and its other blocks are strictly upper triangular matrices of size $(m+1)\times(m+1)$,it takes the following form

$$(I - \lambda\underline{\underline{G}}) = \left(\begin{array}{ccccccccc}
1 & -\lambda G''_{0001} & \cdots & -\lambda G''_{000m} & & 0 & -\lambda G''_{0n00} & \cdots & -\lambda G''_{0n0m} \\
 & 1 & \cdots & -\lambda G''_{001m} & \cdots & & 0 & \cdots & -\lambda G''_{0n1m} \\
 & & \ddots & \vdots & & & & \ddots & \vdots \\
 & & & 1 & & & & \&0 & \\
 & & \vdots & & \ddots & & & \vdots & \\
0 & -\lambda G''_{n001} & \cdots & -\lambda G''_{n00m} & & 1 & -\lambda G''_{nn01} & \cdots & -\lambda G''_{nn0m} \\
 & 0 & \cdots & -\lambda G''_{n01m} & \cdots & & 1 & \cdots & -\lambda G''_{nn1m} \\
 & & \ddots & \vdots & & & & \ddots & \vdots \\
 & & & 0 & & & & & 1
\end{array}\right).$$

Hence we have the following simple formulas for solving (13)

$$u_{im} = f_{im}, \quad i = 0, 1, \cdots, n$$

and

$$u_{ij} = f_{ij} - \sum_{p=0}^{i-1}\sum_{q=0}^{n} \lambda G''_{piqj} u_{pq}, \quad j = 0, \cdots m-1, \quad i = 0, \cdots n.$$

Let us consider the linear operator K in the form of

$$K(u) = \int_0^t \int_a^b G(t,s,x,\xi)\ u(s,\xi)\, d\xi ds \tag{17}$$

when the kernel of eq. (2) is not polynomial, its polynomial approximation is substituted in (2) and corresponding operator is called $K_{n,m}$ such that

$$\lim_{n,m\to\infty} \|K - K_{n,m}\| = 0.$$

Thus from [12], we conclude $(I - \lambda K_{n,m})^{-1}$ exists for sufficiently large m, n and it is a uniformly bounded operator which guarantees stability of eq. (2) for small perturbation.

5. Convergence

Consider linear Volterra integral equation(VIE)

$$u(t) = f(t) + \int_0^t K(t,s)u(s)ds, \quad t \in [0,T], \tag{18}$$

where $K(t,s)$ and $f(t)$ assumed to be sufficiently smooth. Suppose that $u_n(t)$ is the approximate solution of degree n for (18) which exactly satisfies in

$$u_n(t) = f_n(t) + \int_0^t K_n(t,s)u_n(s)ds, \tag{19}$$

where $f_n(t)$ and $K_n(t,s)$ are polynomial approximations of degree n for $f(t)$ and $K(t,s)$, respectively such that $\|f - f_n\|_\infty \to 0$ and $\|K - K_n\|_\infty \to 0$ as $n \to \infty$. On the other hand the exact solution of eq. (18) can be written as

$$u(t) = f(t) + \int_0^t \Gamma(t,s)f(s)ds,$$

where $\Gamma(t,s)$ denotes the resolvent kernel for $K(t,s)$:

$$\begin{aligned}
\Gamma(t,s) &= \sum_{m=0}^{\infty} K^{[m+1]}(t,s), \\
K^{[m+1]}(t,s) &= \int_0^t K(t,y)K^{[m]}(y,s)dy, \ m \in \mathbb{N}, \\
K^{[1]}(t,s) &= K(t,s).
\end{aligned}$$

Similarly for the solution of eq. (19), we have

$$u_n(t) = f_n(t) + \int_0^t H_n(t,s)f_n(s)ds$$

with

$$\begin{aligned}
H_n(t,s) &= \sum_{m=0}^{\infty} K_n^{[m+1]}(t,s), \\
K_n^{[m+1]}(t,s) &= \int_0^t K_n(t,y)K_n^{[m]}(y,s)dy, \ m \in \mathbb{N}, \\
K_n^{[1]}(t,s) &= K_n(t,s).
\end{aligned}$$

We show that $\lim_{n\to\infty} u_n(x) = u(x)$. First we prove by induction on m that $\|K_n^{[m]} - K^{[m]}\| \to 0$ as $n \to \infty$ for every $m \in \mathbb{N}$. For $m = 1$, it is obvious. Let $K_n^{[m]}(t,s)$ converges uniformly to $K^{[m]}(t,s)$. We show $K_n^{[m+1]} \to K^{[m+1]}$ uniformly as $n \to \infty$.

Since $K_n(t,y)$ and $K_n^{[m]}(y,s)$ are bounded sequences on $[0,T]$, we can conclude that $K_n(t,y)K_n^{[m]}(y,s)$ converge to $K(t,y)K^{[m]}(y,s)$ uniformly, in other word uniform convergence of $K_n^{[m+1]}(t,s) - K^{[m+1]}(t,s)$ to 0 is concluded. This result also implies uniform convergence of $K_n^{[m+1]}(t,s)f_n(s) - K^{[m+1]}(t,s)f(s)$ to 0 as $n \to \infty$. Thus

$$\begin{aligned}
|u_n(t) - u(t)| &\leq |f_n(t) - f(t)| + \int_0^t |H_n(t,s)f_n(s) - \Gamma(t,s)f(s)|ds \\
&= |f_n(t) - f(t)| + \int_0^t |\sum_{m=0}^{\infty} K_n^{[m+1]}(t,s)f_n(s) - K^{[m+1]}(t,s)f(s)|ds \\
&\leq |f_n(t) - f(t)| + \int_0^t \sum_{m=0}^{\infty} |K_n^{[m+1]}(t,s)f_n(s) - K^{[m+1]}(t,s)f(s)|ds \\
&= |f_n(t) - f(t)| + \sum_{m=0}^{\infty} \int_0^t |K_n^{[m+1]}(t,s)f_n(s) - K^{[m+1]}(t,s)f(s)|ds.
\end{aligned}$$

Since the sequence

$$S_N := \sum_{m=0}^{N} |K_n^{[m+1]}(t,s)f_n(s) - K^{[m+1]}(t,s)f(s)|$$

is increasing and it is uniformly convergent to 0, we used the monotone convergence theorem (see [11], p. 49) to change order of summation and integral. Therefore, by the uniform convergence of $K_n^{[m+1]}(t,s)f_n(s) - K^{[m+1]}(t,s)f(s)$ to 0, we have $u_n(t) \to u(t)$ as $n \to 0$.

A similar argument for convergence of approximate solution is proved for linear Fredholm integral equation.

The approximated solution, $u_{n,m}(t,x)$ is obtained by two times application of operational matrices. First we approximate the solution of (2) with respect to fredholm part, so we have

$$\|u_n(t,x) - u(t,x)\|_\infty \to 0 \quad as\ n \to \infty.$$

Then by application of method on the system of linear VIEs in (8), we obtain

$$\|u_{i_m}(t) - u_i(t)\|_\infty \to 0 \quad as\ n \to \infty.$$

thus $\|u_{n,m}(t,x) - u(t,x)\|_\infty \to 0$ is yield.

6. Error estimation for linear case

In this section, we give an error estimation for the approximate solution of (2). Let $u_{n,m}(t,x)$ be the approximate solution and $e_{n,m}(t,x) = u_{n,m}(t,x) - u(t,x)$, be the error function associated with $u_{n,m}(t,x)$, where $u(t,x)$ is the exact solution of (2). Since $u_{n,m}(t,x)$ is an approximate solution, it satisfies in

$$u_{n,m}(t,x) = f(t,x) + \lambda \int_0^t \int_a^b G(t,s,x,\xi)u_{n,m}(s,\xi)d\xi ds + H_{n,m}(t,x) \qquad (20)$$

where $H_{n,m}(t,x)$ is a perturbation term and it is obtained from

$$H_{n,m}(t,x) = u_{n,m}(t,x) - f(t,x) - \lambda \int_0^t \int_a^b G(t,s,x,\xi) u_{n,m}(s,\xi) d\xi ds, \qquad (21)$$

it is evident that $H_{n,m}(t,x) \to 0$ as $m,n \to \infty$.

Thus subtracting (20) from (2), yields the equation

$$e_{n,m}(t,x) = H_{n,m}(t,x) + \lambda \int_0^t \int_a^b G(t,s,x,\xi) e_{n,m}(s,\xi) d\xi ds \qquad (22)$$

for the error function $e_{n,m}(t,x)$. To find an a approximation $\hat{e}_{n,m}(t,x)$ to $e_{n,m}(t,x)$, we can solve Eq. (22) by the same way as we did for (2). In this case only the function $f(t,x)$ replaces by the perturbation term and the matrix representation of integral part remains unchanged.

Now we obtain an upper bound for the norm of error depending on the problem data, by (22) we have

$$\|e_{n,m}\|_\infty \leq \|H_{n,m}\||_\infty + \lambda \|e_{n,m}\||_\infty \gamma,$$

where

$$\gamma = \max_{[0,T]\times[a,b]} \left| \int_0^t \int_a^b G(t,s,x,\xi)\, d\xi ds \right|,$$

and $H_{n,m}$ is computed by (21). Hence by assuming $|\lambda| < 1/\gamma$, we have

$$\|e_{n,m}\||_\infty \leq \frac{\|H_{n,m}\||_\infty}{1 - |\lambda|\gamma}.$$

This implies uniform convergence of $e_{n,m}(t,x)$ to 0.

7. Numerical Examples

In this section, we give some examples to clarify accuracy of the presented method.

Remark. Note that as we mentioned previously, whenever $G(t,s,x,\xi)$ or $f(t,x)$ are not polynomials, they must be approximated by polynomials of suitable degree. Therefore in the following examples, we approximate all non-polynomial parts of $G(t,s,x,\xi)$ and $f(t,x)$ by polynomials of degree m and n with respect to t,s and x,ξ respectively. We obtain expansion of the exact solution exactly when the exact solution is a bivariate polynomial of degree less than or equal to (m,n).

Example 1. Consider the linear Volterr-Fredholm integral equation

$$u(t,x) = f(t,x) - \lambda \int_0^t \int_0^2 \cos(x-\xi) e^{s-t} u(s,\xi) d\xi ds \qquad (23)$$

where $t \in [0,2]$, $\lambda = 1/3$ and $f(t,x)$ is chosen in such a way that the exact solution of (23) to be $u(t,x) = \cos(x) exp(-t)$. Table 1 shows the absolute errors at the selected points of $[0,2] \times [0,2]$.

Table 1

	m=n=10		m=n=15		m=n=20	
(t,x)	e(t,x)	ê(t,x)	e(t,x)	ê(t,x)	e(t,x)	ê(t,x)
(0.0, 0.0)	0	0	0	0	0	0
(0.5,0.5)	7.093e-8	1.041e-8	8.732e-12	1.005e-11	2.090e-17	9.653e-16
(0.5,2.0)	1.828e-5	2.703e-6	6.371e-10	2.343e-11	1.148e-14	1.859e-15
(1.0,1.0)	3.951e-7	1.180e-7	2.675e-12	1.068e-11	1.441e-16	4.034e-16
(1.0, 1.5)	2.390e-7	9.356e-8	4.275e-12	7.036e-11	1.915e-16	5.404e-15
(1.5, 0.0)	4.032e-6	5.175e-6	1.092e-10	1.292e-10	5.183e-16	7.903e-16
(1.5, 1.5)	1.538e-6	2.384e-6	3.478e-11	6.307e-11	9.705e-17	4.728e-16
(2.0, 0.5)	1.113e-5	1.419e-4	1.022e-8	1.362e-8	1.797e-13	2.034e-13
(2.0, 2.0)	5.274e-5	4.472e-5	2.961e-9	3.724e-9	5.248e-14	6.502e-14
error bound	2.4503e-4		2.6095e-8		4.5183e-13	

Example 2. The second example is linear problem in the form

$$u(t,x) = f(t,x) + \lambda \int_0^t \int_0^1 \left(e^{\xi} + s\right) u(s,\xi) d\xi ds, \tag{24}$$

where $t \in [0,1],\ \lambda = 1/5$ and $f(t,x)$ is chosen such that $u(t,x) = exp(x) - t$. Absolute errors for this example reported in Table 2 at the randomly selected points from $(t,x) = (0.25i, 0.25j), i,j = 0, \cdots, 4$.

Table 2

	m=n=10		m=n=15	
(t,x)	e(t,x)	ê(t,x)	e(t,x)	ê(t,x)
(0.00, 0.00)	0	0	0	0
(0.25, 0.50)	9.3088e-10	1.1842e-9	1.2527e-15	2.9792e-15
(0.25, 1.00)	4.1522e-8	7.4227e-9	8.0051e-14	2.9792e-15
(0.50, 0.25)	1.9977e-9	1.5812e-9	2.6492e-15	6.0007e-15
(0.50, 0.75)	3.6722e-9	1.8485e-9	3.4389e-15	6.0007e-15
(0.75, 0.00)	3.0905e-9	2.0088e-9	4.2460e-15	9.1493e-15
(0.75, 0.75)	4.8342e-9	1.3278e-9	5.0357e-15	9.1493e-15
(1.00,0.50)	4.4659e-9	2.8246e-9	6.1105e-15	1.2552e-14
(1.00, 1.00)	4.5750e-8	3.6704e-8	8.4909e-14	1.2552e-14
error bound	8.6643e-8		1.5893e-13	

Example 3. Consider the following linear problem

$$u(t,x) = f(t,x) + \lambda \int_0^t \int_0^1 \left(x\xi^2 + cos(s)\right) u(s,\xi) d\xi ds \tag{25}$$

where $t \in [0,1],\ \lambda = 1/2$ and $f(t,x)$ is such that $u(t,x) = x\sin(t)$ to be the exact solution. Table 3 shows the absolute errors at $(t,x) = (0.25i, 1 + 0.25j), i,j = 0, \cdots, 4$, randomly.

Table 3

	m=n=10		m=n=15	
(t,x)	e(t,x)	$\hat{e}$(t,x)	e(t,x)	$\hat{e}$(t,x)
(0.00, 0.00)	0	0	0	0
(0.25, 0.50)	1.0167e-12	2.2362e-12	2.6045e-18	2.2745e-17
(0.25, 1.00)	1.1454e-12	1.3899e-12	2.9789e-18	2.3199e-17
(0.50, 0.25)	7.1362e-9	7.2694e-9	1.5360e-15	6.1469e-16
(0.50,0.75)	1.1169e-9	6.9624e-10	1.5022e-15	5.7304e-16
(0.75, 0.00)	9.3295e-8	5.8427e-8	9.9482e-13	3.7326e-13
(0.75,0.75)	9.0968e-8	5.4569e-8	9.6358e-13	3.3183e-13
(1.00, 0.50)	2.0480e-6	1.2030e-6	9.4635e-11	3.1758e-11
(1.00,0.00)	2.0142e-6	1.1437e-6	9.2664e-11	2.9030e-11
error bound	4.2222e-6		1.8409e-10	

Example 4. The following linear example is solved in [9] by discretization in space and time by numerical integration. There the accuracy of method is defined by the number of correct digits *cd* at the end point (the maximal absolute end point error is written as 10^{-cd}). for this problem when the number of mesh points is 32, the reported *cd* value is equal to 3.12.

$$u(t,x) = f(t,x) + \int_0^t \int_1^2 \left(\log(x\xi + 1)\cos(t-s)\right) u(s,\xi) d\xi ds, \qquad (26)$$

where $t \in [0,1]$ and $f(t,x)$ such that $u(t,x) = x^3 \cos(t)$ is the exact solution of problem. Here we solve this problem by new method. Absolute errors for this example reported at $(t,x) = (0.25i, 1+0.25j), i,j = 0,\cdots,4$, randomly, in Table 4.

Table 4

	m=n=10		m=n=15	
(t,x)	e(t,x)	$\hat{e}$(t,x)	e(t,x)	$\hat{e}$(t,x)
(0.00, 1.00)	0	0	0	0
(0.25, 1.25)	9.5587e-7	3.3901e-6	3.5402e-8	1.0074e-8
(0.25, 2.00)	6.6070e-6	2.0386e-5	2.7990e-7	9.1344e-7
(0.50, 1.00)	1.8920e-6	7.3305e-6	9.1032e-8	1.9914e-7
(0.50, 1.75)	2.4349e-6	9.6125e-6	2.3542e-8	2.2352e-8
(0.75, 1.25)	2.7962e-6	1.2966e-5	1.5543e-7	3.5878e-7
(0.75, 1.75)	3.5118e-6	1.3699e-5	7.6930e-7	1.1845e-7
(1.00, 1.25)	3.3679e-6	1.8711e-5	1.9567e-7	7.9876e-8
(1.00, 2.00)	1.9250e-5	6.9471e-5	9.1777e-7	2.0952e-7

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