J. Appl. Math. & Informatics Vol. 29(2011), No. 5 - 6, pp. 1229 - 1243
Website: http://www.kcam.biz

MULTIPLICITY OF SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS WITH CRITICAL EXPONENTIAL GROWTH†

YANQIN FANG, JIHUI ZHANG*

ABSTRACT. In this paper we consider a system of N-Laplacian elliptic equations with critical exponential growth. The existence and multiplicity results of solutions are obtained by a limit index method and Trudinger-Moser inequality.

AMS Mathematics Subject Classification : 35A15, 35B33.
Key words and phrases : critical exponential growth, limit index, multiple solutions.

1. Introduction and main result

In this paper, we study the existence of multiple solutions for the following equations with exponential critical growth

$$\begin{cases} \Delta_N u = f(x,u) + R_u(x,u,v), \quad x \quad \text{in} \quad \Omega, \\ -\Delta_N v = g(x,v) + R_v(x,u,v), \quad x \quad \text{in} \quad \Omega, \\ u = v = 0, \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^N$ $(N \geq 2)$ is a smooth bounded domain and $R : \bar{\Omega} \times R^2 \to R$ is a C^1 function.

In the previous decades, there has been a number of activities in the study of the elliptic equations leading to indefinite functionals. For example, when N=2, this class of system is called noncooperative and many recent studies have focused on it. Results relating to these problems can be found in [1, 3, 4, 5, 6, 7, 11, 13, 15, 19] and the references therein.

Received November 8, 2010. Revised December 6, 2010. Accepted May 2, 2011.
*Corresponding author. †This work was supported by grant 10871096 from the National Natural Science Foundation of China.

In a recent paper, Lin and Li [13] had considered the following system

$$\begin{cases} \Delta u = |u|^{2^*-2}u + F_s(x,u,v) \text{ in } \Omega, \\ -\Delta v = |v|^{2^*-2}v + F_t(x,u,v) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

By applying the Limit Index Theory, they obtained the existence of multiple solutions under some assumptions on nonlinear part.

In [10], Huang and Li applied the Principle of Symmetric Criticality and the Limit Index Theory to study the system of elliptic equations involving the p-Laplacian in the unbounded domain in R^N

$$\begin{cases} \Delta_p u - |u|^{p-2}u = F_u(|x|,u,v) \text{ in } R^N, \\ -\Delta_p v + |v|^{p-2}v = F_v(|x|,u,v) \text{ in } R^N, \\ u,v \in W^{1,p}(R^N), \end{cases} \tag{3}$$

where $1 < p < N$, and they extended some results of [15].

In [8], Fang and Zhang dealt with the existence and multiplicity of solutions to the following systems

$$\begin{cases} \Delta_p u = |u|^{p^*-2}u + F_u(x,u,v) \text{ in } \Omega, \\ -\Delta_q v = |v|^{q^*-2}v + F_v(x,u,v) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0, \end{cases} \tag{4}$$

where $\Omega \subset R^N$ is an open-bounded domain with smooth boundary, $F = F(x,u,v)$, $F_u = \frac{\partial F}{\partial u}$, $F_v = \frac{\partial F}{\partial v}$, $1 < p,q < N$, $p^* = pN/(N-p)$ and $q^* = qN/(N-q)$ denote the critical Sobolev exponent.

We would like to emphasize that in the literature rather less attention has been paid to noncooperative systems involving exponential critical growth to the case $N \geq 2$. In [1], Alves and Soares considered N-Laplacian and they proved the existence of nontrivial solution for the corresponding system (1) with critical Sobolev exponent and critical exponential growth on bounded domain of R^N for $N \geq 2$. The proof is based on a linking theorem without the Palais-Smale condition. And we should also mention the article [9], where a class of Hamiltonian systems with exponential critical growth has been considered.

Motivated by works just described, a natural question arises whether the existence of multiple solutions can be obtained when we consider the N-Laplacian operator and assume that the nonlinearities have a critical exponential growth. In this paper we deal with the problem (1). The functional Φ is strongly indefinite in the sense that it is neither bounded from above nor from below. We can not apply the symmetric Mountain Pass Theorem in considering the existence of infinitely many critical points of the functional Φ. Here, we employ a limit index and Trudinger-Moser inequality. Then main difficulties are related to verify the condition of limit index and $(PS)_c^*$. In our paper, we must overcome these difficulties.

In order to treat variationally (1) in $W^{1,N}(\Omega)\times W^{1,N}(\Omega)$, we use the inequalities of Trudinger and Moser (See [17, 20]), which provide

$$\exp(\alpha|u|^{N/(N-1)}) \in L^1(\Omega), \quad \text{for all} \quad u \in W_0^{1,N}(\Omega) \quad \text{and} \quad \alpha > 0 \tag{5}$$

and there exists a constant $C(\Omega) > 0$ such that

$$\sup_{\|u\|\leq 1} \int_\Omega \exp(\alpha|u|^{N/(N-1)})dx \leq C(\Omega), \quad \text{for all} \quad u \in W_0^{1,N}(\Omega) \quad \text{and} \quad \alpha \leq \alpha_N, \tag{6}$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the $N-1$-dimensional surface of the unit sphere.

Now, we give the following assumptions.

(F_1) There exists a constant $C > 0$ such that

$$|f(x,s)| \leq C\exp(\alpha_N|s|^{N/(N-1)}), \quad \text{for all} \quad x \in \bar{\Omega}, \quad s \in R.$$

(F_2) There exists $\nu \in (0, N)$ such that

$$0 \leq \nu F(x,s) \leq f(x,s)s, \quad \text{for all} \quad x \in \Omega, \quad s \in R,$$

where $F(x,s) = \int_0^s f(x,t)dt$.

(G_1) There exists a continuous function b verifying

$$g(x,s) = b(x,s)\exp(\alpha_N|s|^{N/(N-1)}),$$

with

$$c_p|s|^{p-2}s \leq b(x,s) \leq d_p|s|^{p-2}s \quad \text{for all} \quad x \in \bar{\Omega}, \quad s \in R,$$

for some $p > N$ and constants $c_p, d_p > 0$.

(G_2) There exists $\mu > N$ such that

$$0 \leq \mu B(x,s) \leq b(x,s)s, \quad \text{for all} \quad x \in \Omega, \quad s \in R,$$

where $B(x,s) = \int_0^s b(x,t)dt$.

(G_3) The constants c_p, ν, μ given by conditions (F_2), (G_1) and (G_2) satisfy

$$\max\left\{\frac{\nu N}{N-\nu}, \frac{\mu N}{\mu - N}\right\}\left(\frac{p-N}{pN}\right)\frac{1}{c_p^{\frac{N}{p-N}}}\left(\frac{1}{r}\right)^{\frac{N}{p-N}} < 1,$$

where r will be given later.

Related to function R, we assume that the following conditions hold.

(R_1) $R_u(x,0,0) = R_v(x,0,0) = 0$ and $R(x,u,v) \geq 0$ for all $(x,u,v) \in \bar{\Omega}\times R^2$.

(R_2) For any $\alpha, \beta > 0$

$$\lim_{|(u,v)|\to+\infty} \frac{R_u(x,u,v)}{\exp(\alpha|u|^{N/(N-1)}) + \exp(\beta|v|^{N/(N-1)})} = 0$$

and

$$\lim_{|(u,v)|\to+\infty} \frac{R_v(x,u,v)}{\exp(\alpha|u|^{N/(N-1)}) + \exp(\beta|v|^{N/(N-1)})} = 0.$$

(R_3) For ν and μ given by condition (F_2) and (G_2), we assume that

$$0 \leq R(x,s,t) \leq \frac{1}{\nu} R_u(x,s,t)s + \frac{1}{\mu} R_v(x,s,t)t, \text{ for all } x \in \Omega, \ (s,t) \in R^2.$$

We note that the hypotheses (R_1)-(R_3) are satisfied by the function given by $R(u,v) = |u|^s e^{|u|^\alpha} |v|^t e^{|v|^\beta}$, where $1 < \alpha, \beta < 2$, s and t are positive real numbers such that $\frac{s}{\nu} + \frac{t}{\mu} \geq 1$, where ν and μ are given by conditions (F_2) and (G_2).

By $X = E \times E$ we denote the space $W_0^{1,N}(\Omega) \times W_0^{1,N}(\Omega)$ endowed with the norm

$$\|(u,v)\|^N = \|u\|^N + \|v\|^N,$$

where $\|\cdot\|$ denotes the usual norm in $W_0^{1,N}(\Omega)$ and we write $\Phi : X \to R$ the functional given by

$$\begin{aligned}\Phi(u,v) = &-\frac{1}{N}\int_\Omega |\nabla u|^N dx + \frac{1}{N}\int_\Omega |\nabla v|^N dx - \int_\Omega F(x,u)dx \\ &- \int_\Omega G(x,v)dx - \int_\Omega R(x,u,v)dx. \end{aligned} \tag{7}$$

Under the assumptions (F_1) and (R_2), the functional Φ is well defined, belongs to $C^1(X,R)$ and

$$\begin{aligned}\langle \Phi'(u,v), (\phi,\psi)\rangle = &-\int_\Omega |\nabla u|^{N-2}\nabla u \nabla \phi dx + \int_\Omega |\nabla v|^{N-2}\nabla v \nabla \psi dx - \int_\Omega f(x,u)\phi dx \\ &- \int_\Omega g(x,v)\psi dx - \int_\Omega R_u(x,u,v)\phi dx - \int_\Omega R_v(x,u,v)\psi dx \end{aligned} \tag{8}$$

for all $(u,v), (\phi,\psi) \in X$.

Now we give the main result of this paper.

Theorem 1.1. *Suppose that the assumptions $(F_1) - (F_2)$, $(G_1) - (G_3)$ and $(R_1) - (R_3)$ hold. Then the functional Φ possesses $k_0 - 1$ critical values such that $0 < c_{-k_0+1} \leq \cdots \leq c_{-1} \leq \beta$, where $k_0 > 1$ and $\beta > 0$. That is, the system (1) possesses at least $k_0 - 1$ pairs weak nontrivial solutions.*

2. Preliminaries

First of all, we recall the Limit Index Theory due to Li [15]. In order to do that, we introduce the following definitions.

Definition 2.1 ([15, 22])**.** The action of a topological group G on a normed space Z is a continuous map

$$G \times Z \to Z : [g,z] \longmapsto gz$$

such that

$$1 \cdot z = z, \ \ (gh)z = g(hz), \ \ z \longmapsto gz \text{ is linear}, \ \ \forall g,h \in G.$$

The action is isometric if

$$\|gz\| = \|z\|, \ \ \forall g \in G, \ \ z \in Z,$$

and in this case Z is called G-space.

The set of invariant points is defined by

$$\text{Fix}\, G := \{z \in Z; \ \ gz = z, \ \ \forall g \in G\}.$$

A set $A \subset Z$ is invariant if $gA = A$ for every $g \in G$. A function $\varphi : Z \to R$ is invariant $\varphi \circ g = \varphi$ for every $g \in G, z \in Z$. A map $f : Z \longrightarrow Z$ is equivariant if $g \circ f = f \circ g$ for every $g \in G$.

Suppose Z is a G-Banach space, that is, there is a G isometric action on Z. Let

$$\Sigma = \{A \subset Z; A \ \text{ is } \ \text{closed} \ \text{and} \ gA = A, \ \forall g \in G\}$$

be a family of all G-invariant closed subset of Z, and let

$$\Gamma = \{h \in C^0(Z, Z); h(gu) = g(h(u)), \ \forall g \in G\}$$

be the class of all G-equivariant mapping of Z. Finally, we call the set

$$O(u) := \{gu; g \in G\}$$

G-orbit of u.

Definition 2.2 ([18]). An index for (G, Σ, Γ) is a mapping $i : \Sigma \longrightarrow \mathcal{Z}_+ \bigcup \{+\infty\}$ (where $\mathcal{Z}_+$ is the set of all nonnegative integers) such that for all $A, B \in \Sigma$, $h \in \Gamma$ the following conditions are satisfied: 1

(1) $i(A) = 0 \Longleftrightarrow A = \emptyset$;
(2) (Monotonicity) $A \subset B \Longrightarrow i(A) \leq i(B)$;
(3) (Subadditivity) $i(A \bigcup B) \leq i(A) + i(B)$;
(4) (Supervariance) $i(A) \leq i(\overline{h(A)})$, $\forall h \in \Gamma$;
(5) (Continuity) If A is compact and $A \cap$ Fix $G = \emptyset$, then $i(A) < +\infty$ and there is a G-invariant neighborhood N of A such that $i(\bar{N}) = i(A)$;
(6) (Normalization) If $x \notin$ Fix G, then $i(O(x)) = 1$.

Definition 2.3 ([5]). An index theory is said to satisfy the d-dimension property if there is a positive integer d such that

$$i(V^{dk} \bigcap B_1(0)) = k$$

for all dk-dimensional subspaces $V^{dk} \in \Sigma$ such that $V^{dk} \bigcap$ Fix $G = \{0\}$, where $B_1(0)$ is the unit sphere in Z.

Suppose U and V are G-invariant closed subspaces of Z such that

$$Z = U \bigoplus V,$$

where V is infinite dimensional and

$$V = \overline{\bigcup_{j=1}^{\infty} V_j},$$

where V_j is a dn_j-dimensional G-invariant subspace of V, $j = 1, 2, \cdots$, and $V_1 \subset V_2 \subset \cdots V_n \subset \cdots$. Let

$$Z_j = U \bigoplus V_j,$$

and $\forall A \in \Sigma$, let

$$A_j = A \bigcap Z_j.$$

Definition 2.4 ([15]). Let i be an index theory satisfying the d-dimension property. A limit index with respect to (Z_j) induced by i is a mapping

$$i^\infty : \Sigma \longrightarrow \mathcal{Z} \bigcup \{-\infty, +\infty\},$$

given by

$$i^\infty(A) = \limsup_{j \to \infty}(i(A_j) - n_j).$$

Proposition 2.1 ([15]). *Let $A,\ B \in \Sigma$. Then i^∞ satisfies:*

(1) $A = \emptyset \Longrightarrow i^\infty = -\infty$;

(2) *(Monotonicity)* $A \subset B \Longrightarrow i^\infty(A) \leq i^\infty(B)$;

(3) *(Subadditivity)* $i^\infty(A \bigcup B) \leq i^\infty(A) + i^\infty(B)$;

(4) *If* $V \bigcap Fix\, G = \{0\}$, *then* $i^\infty(B_\rho(0) \bigcap V) = 0$, *where* $B_\rho(0) = \{z \in Z, \|z\| = \rho\}$;

(5) *If* Y_0 *and* $\tilde{Y_0}$ *are* G*-invariant closed subspaces of* V *such that* $V = Y_0 \bigoplus \tilde{Y_0}$, $\tilde{Y_0} \subset V_{j_0}$ *for some* j_0 *and* $\dim \tilde{Y_0} = dm$, *then* $i^\infty(B_\rho(0) \bigcap Y_0) \geq -m$.

Definition 2.5 ([14]). A functional $J \in C^1(Z, R)$ is said to satisfy the condition $(PS)_c^*$ if any sequence $\{u_{n_k}\}$, $u_{n_k} \in Z_{n_k}$ such that

$$J(u_{n_k}) \to c, \ \ dJ_{n_k}(u_{n_k}) \to 0, \ \text{ as } \ k \to \infty$$

possesses a convergent subsequence, where Z_{n_k} is the n_k-dimension subspace of Z, $J_{n_k} = J|_{Z_{n_k}}$.

Theorem 2.1 ([15]). *Assume that*

(B_1) $J \in C^1(Z, R)$ *is* G*-invariant;*

(B_2) *There are* G*-invariant closed subspaces* U *and* V *such that* V *is infinite dimensional and* $Z = U \bigoplus V$;

(B_3) *There is a sequence of* G*-invariant finite-dimensional subspaces*

$$V_1 \subset V_2 \subset \cdots V_j \subset \cdots, \dim V_j = dn_j,$$

such that $V = \overline{\bigcup_{j=1}^\infty} V_j$;

(B_4) *There is an index theory* i *on* Z *satisfying the* d*-dimension property;*

(B_5) *There are* G*-invariant subspaces* Y_0, $\tilde{Y_0}$, Y_1, *of* V *such that* $V = Y_0 \bigoplus \tilde{Y_0}$, Y_1, $\tilde{Y_0} \subset V_{j_0}$ *for some* j_0, *and* $\dim \tilde{Y_0} = dm < dk = \dim Y_1$;

(B_6) *There are* α *and* β, $\alpha < \beta$ *such that* J *satisfies* $(PS)_c^*$, $\forall c \in [\alpha, \beta]$;

$$(B_7) \begin{cases} (a) \text{ either } Fix\, G \subset U \oplus Y_1, \ \text{ or } \ Fix\, G \cap V = \{0\}, \\ (b) \text{ there is } \rho > 0, \ \text{ such that } \forall u \in Y_0 \cap B_\rho(0), \ \ J(u) \geq \alpha, \\ (c)\ \forall z \in U \oplus Y_1, \ \ J(z) \leq \beta, \end{cases} \tag{9}$$

if i^∞ is the limit index corresponding to i, then the numbers

$$c_j = \inf_{i^\infty(A)\geq j} \sup_{z\in A} J(z), \quad -k+1 \leq j \leq -m,$$

are critical values of J, and $\alpha \leq c_{-k+1} \leq \cdots \leq c_{-m} \leq \beta$. Moreover, if $c = c_l = \cdots = c_{l+r}$, $r \geq 0$, then $i(\iota_c) \geq r+1$ where $\iota_c = \{z \in Z; dJ(z) = 0, \ \ J(z) = c\}$.

According to [21] (Section 4.9.4) there exists a Schauder basis $\{e_n\}_{n=1}^\infty$ for $E = W_0^{1,N}(\Omega)$. Furthermore, since E is reflexive, $\{e_n^*\}_{n=1}^\infty$, the biorthogonal functionals associated to the basis $\{e_n\}_{n=1}^\infty$ (which are characterized by the relations $\langle e_m^*, e_n\rangle = \delta_{m,n}$), form a basis for E^* with the following properties (cf. [16] Propositon 1.b.1 and Theorem 1.b.5). Denote

$$E_n = \mathrm{span}\{e_1, e_2, \cdots e_n\}, \quad E_n^\perp = \overline{\mathrm{span}\{e_{n+1}, \cdots\}}$$

and

$$E_m^* = \mathrm{span}\{e_1^*, e_2^*, \cdots e_m^*\}.$$

Let $P_n : E \to E_n$ be the projector corresponding to the decomposition $E = E_n \oplus E_n^\perp$ and $P_n^* : E^* \to E_n^*$ the projector corresponding to the decomposition $E^* = E_n^* \oplus (E_n^*)^\perp$. Then $P_n u \to u$, $P_n^* v^* \to v^*$ for any $u \in E$, $v^* \in E^*$ as $n \to \infty$ and

$$\langle P_n^* v^*, u\rangle = \langle v^*, P_n u\rangle.$$

Let $\tau : E \to E^*$ be the mapping given by

$$\langle \tau u, \tilde{u}\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \tilde{u} dx.$$

It is easy to check that the operator τ is bounded, continuous. And if $u_n \rightharpoonup \tilde{u}$ in E and $\langle \tau u_n - \tau\tilde{u}, u_n - \tilde{u}\rangle \to 0$, then $u_n \to \tilde{u}$ in E (See [10, 15]).

Now, we set

$$X = U \oplus V, \ \ U = E \times \{0\}, \ \ V = \{0\} \times E, \tag{10}$$

$$Y_0 = \{0\} \times E_1^\perp, \ \ V = Y_0 \oplus \tilde{Y_0}, \tag{11}$$

$$Y_1 = \{0\} \times E_{k_0}, E_{k_0} = \mathrm{span}\{e_1, e_2, \cdots e_{k_0}\}, \tag{12}$$

then $\dim \tilde{Y_0} = 1$, $\dim Y_1 = k_0$.

We define a group action $G = \{1, \tau\} \cong Z_2$ by setting $\tau(u,v) = (-u,-v)$, then Fix $G = \{0\} \times \{0\}$ (also denote $\{0\}$). It is clear that U and V are G-invariant closed subspaces of X, and Y_0, $\tilde{Y_0}$ and Y_1 are G-invariant subspace of V.

Set

$$\Sigma = \{A \subset X; A \ \text{ is closed and } \ (u,v) \in A \Rightarrow (-u,-v) \in A\}. \tag{13}$$

Define an index γ on Σ by:

$$\gamma(A) = \begin{cases} \min\{N \in Z_+; \exists h \in C(A, R^N\backslash\{0\}) \text{ such that } h(-u,-v) = h(u,v)\}, \\ 0, \ \text{ if } \ A = \emptyset, \\ +\infty, \ \text{ if such } h \text{ does not exist.} \end{cases} \tag{14}$$

Then we have the following proposition: γ is an index satisfying the properties given in Definition 2.2. Moreover, γ satisfies the one-dimension property. According to Definition 2.4 we can obtain a limit index γ^∞ with respect to (X_n) from γ.

The following Proposition 2.2 and Lemma 2.2 play an important part in our proofs.

Proposition 2.2. *Let (φ_j) be a sequence of functions in $W_0^{1,N}(\Omega)$ converging to φ weakly in $W_0^{1,N}(\Omega)$. Assume that $\|\varphi_j\|^{N/(N-1)} \leq \delta < 1$ and $l \in C(\bar{\Omega} \times R, R)$ satisfies*

$$|l(x,s)| \leq C\exp(\alpha_N |s|^{N/(N-1)}), \ \ \textit{for all} \ \ (x,s) \in (\bar{\Omega} \times R, R)$$

and for some $C > 0$. Then,

$$\lim_{j\to+\infty} \int_\Omega l(x,\varphi_j)\omega dx = \int_\Omega l(x,\varphi)\omega dx, \tag{15}$$

for every $\omega \in W_0^{1,N}(\Omega)$, and

$$\lim_{j\to+\infty} \int_\Omega l(x,\varphi_j)\varphi_j dx = \int_\Omega l(x,\varphi)\varphi dx. \tag{16}$$

Proof. The proof is similar to [1]. Consider $q > 1$ so that $q\delta < 1$. From the hypothesis on l,

$$\int_\Omega |l(x,\varphi_j)|^q dx \leq C\int_\Omega e^{q\alpha_N|\varphi_j|^{N/(N-1)}}dx \tag{17}$$

$$= C\int_\Omega e^{q\alpha_N\|\varphi_j\|^{N/(N-1)}(\frac{|\varphi_j|}{\|\varphi_j\|})^{N/(N-1)}}dx \tag{18}$$

$$\leq C\int_\Omega e^{q\alpha_N\delta(\frac{|\varphi_n|}{\|\varphi_j\|})^{N/(N-1)}}dx. \tag{19}$$

By Trudinger and Moser inequality, there exists $M_1 > 0$ such that

$$\int_\Omega |l(x,\varphi_j)|^q dx \leq M_1, \ \ \forall n \in N. \tag{20}$$

Combing Sobolev embeddings with Egoroff theorem, given $\epsilon > 0$ there exists $E \subset \Omega$ such that $|E| < \epsilon$ and $\varphi_j(x) \to \varphi(x)$ uniformly on $\Omega\backslash E$. By Hölder inequality and using (20), we get

$$|\int_\Omega (l(x,\varphi_j) - l(x,\varphi))\omega dx| \leq \int_{\Omega\backslash E} |l(x,\varphi_j) - l(x,\varphi)||\omega|dx + o_\epsilon(1)$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$. As $\epsilon > 0$ is arbitrary and $l(x,\varphi_j) \to l(x,\varphi)$ uniformly on $\Omega\backslash E$, we conclude the proof of (15). Similar argument shows that the limit (16) hold. $\square$

Lemma 2.2. *Suppose that R satisfies the condition (R_2) and let (φ_j, ϕ_j) be a sequence weakly convergent to (φ, ϕ) in $W_0^{1,N}(\Omega) \times W_0^{1,N}(\Omega)$. Then,*

$$\int_\Omega R(x, \varphi_j, \phi_j)dx \to \int_\Omega R(x, \varphi, \phi)dx,$$

$$\int_\Omega R_u(x, \varphi_j, \phi_j)\varphi_j dx \to \int_\Omega R_u(x, \varphi, \phi)\varphi dx,$$

$$\int_\Omega R_u(x, \varphi_j, \phi_j)\xi dx \to \int_\Omega R_u(x, \varphi, \phi)\xi dx,$$

$$\int_\Omega R_v(x, \varphi_j, \phi_j)\phi_j dx \to \int_\Omega R_v(x, \varphi, \phi)\phi dx,$$

$$\int_\Omega R_v(x, \varphi_j, \phi_j)\psi dx \to \int_\Omega R_v(x, \varphi, \phi)\psi dx,$$

for all $\xi, \psi \in W_0^{1,N}(\Omega)$.

Proof. Since (φ_j, ϕ_j) is weakly convergent, there is $M > 0$ such that

$$\|\varphi_j\|, \|\phi_j\| \leq M \text{ for all } j \in N.$$

Now from (R_2), given $0 < \alpha, \beta < M^{-N/(N-1)}\alpha_N$, there exists a constant $C > 0$ such that

$$|R_u\left(x, \varphi_j, \phi_j)\right| \leq C(e^{\alpha|\varphi_j|^{N/(N-1)}} + e^{\beta|\phi_j|^{N/(N-1)}}\Big) \tag{21}$$

$$|R_v\left(x, \varphi_j, \phi_j)\right| \leq C(e^{\alpha|\varphi_j|^{N/(N-1)}} + e^{\beta|\phi_j|^{N/(N-1)}}\Big) \tag{22}$$

As a consequence,

$$|R\left(x, \varphi_j, \phi_j)\right| \leq C(e^{\alpha|\varphi_j|^{N/(N-1)}} + e^{\beta|\phi_j|^{N/(N-1)}}\Big)(|\varphi_j| + |\phi_j|). \tag{23}$$

Taking $q > 1$ such that $q\alpha|M|^{N/(N-1)}$, $q\beta M^{N/(N-1)} < \alpha_N$, from Trudinger and Moser inequality there exists $K > 0$ such that

$$\int_\Omega e^{\alpha q|\varphi_j|^{N/(N-1)}}, \int_\Omega e^{\beta q|\phi_j|^{N/(N-1)}} \leq K \ \ \forall n \in N.$$

This combing with (21)-(23) and Sobolev embeddings imply that the above limits hold. This concludes the proof. □

Lemma 2.3. *Suppose that the assumptions $(F_1)-(F_2)$, $(G_1)-(G_2)$ and $(R_1)-(R_3)$ hold. Then*
(i) there is $\alpha, \rho > 0$ such that $\forall (0, v) \in Y_0 \cap B_\rho(0)$, $\Phi(0, v) \geq \alpha$;
(ii)there is $\beta > 0$ such that $\forall (u, v) \in U \oplus Y_1$, $\Phi(u, v) \leq \beta$.

Proof. We start observing that, from (G_1),

$$|G(x, t)| \leq d_p|t|^p e^{\alpha_N |t|^{N/(N-1)}}, \text{ for all } x \in \bar{\Omega} \ \ t \in R.$$

Thus if $(0, v) \in Y_0 \cap B_\rho(0)$, by (5),

$$\Phi(0, v) = \frac{1}{N}\|v\|^N - \int_\Omega G(x, v)dx$$

$$\geq \frac{1}{N}\|v\|^N - d_p\int_\Omega |v|^p e^{\alpha_N|v|^{N/(N-1)}}dx \tag{24}$$

$$\geq \frac{1}{N}\|v\|^N - d_p|v|_{2p}^p\{\int_\Omega e^{2\alpha_N|v|^{N/(N-1)}}dx\}^{\frac{1}{2}} \tag{25}$$

$$= \frac{1}{N}\|v\|^N - d_p|v|_{2p}^p\{\int_\Omega e^{2\alpha_N\|v\|^{N/(N-1)}(\frac{|v|}{\|v\|})^{N/(N-1)}}dx\}^{\frac{1}{2}}. \tag{26}$$

By Trudinger and Moser inequality (6), if $\|v\|^{N/(N-1)} < \frac{1}{2}$, then

$$\Phi(0,v) \geq \frac{1}{N}\|v\|^N - C|v|_{2p}^p \geq \frac{1}{N}\|v\|^N - C\|v\|^p.$$

Since $p > N$, there exists $0 < \rho < (\frac{1}{2})^{(N-1)/N}$ such that $\Phi(0,v) \geq \alpha$ for every $\|v\| = \rho$, that is (i).

Now, we give the proof of (ii). From (G_1),

$$G(x,s) \geq \frac{c_p}{p}|s|^p, \text{ for all } (x,s) \in \bar{\Omega}\times R.$$

Thus,

$$\begin{aligned}\Phi(u,v) &\leq \frac{1}{N}\|v\|^N - \frac{c_p}{p}|v|_p^p\\ &\leq \max_{v\in E_{k_0}}\{\frac{1}{N}\|v\|^N - \frac{c_p}{p}|v|_p^p\}\\ &= \max_{\{t\geq 0,\ \nu\in\partial B_1(0)\cap E_{k_0}\}}\{\frac{1}{N}t^N - \frac{t^p c_p}{p}|\nu|_p^p\}\\ &= (\frac{1}{N}-\frac{1}{p})\frac{1}{c_p^{\frac{N}{p-N}}}(\frac{1}{|\nu|_p^p})^{\frac{N}{p-N}}\end{aligned} \tag{27}$$

We set $r = \min\{\int_\Omega |\nu|^p dx :\ \nu \in \partial B_1(0)\cap E_{k_0}\}$. Since $p > N$, we obtain that

$$\Phi(u,v) \leq (\frac{1}{N}-\frac{1}{p})\frac{1}{c_p^{\frac{N}{p-N}}}(\frac{1}{r})^{\frac{N}{p-N}}$$

Let $\beta = (\frac{1}{N}-\frac{1}{p})\frac{1}{c_p^{\frac{N}{p-N}}}(\frac{1}{r})^{\frac{N}{p-N}}$. □

Lemma 2.4. *Suppose that the assumptions $(F_1)-(F_2)$, $(G_1)-(G_2)$ and $(R_1)-(R_3)$ hold. Let $\{(u_{n_k}, v_{n_k})\}$ be a sequence such that $(u_{n_k}, v_{n_k}) \in X_{n_k}$ and*

$$\Phi(u_{n_k}, v_{n_k}) \to c \in [\alpha,\beta],\ \ d\Phi_{n_k}(u_{n_k}, v_{n_k}) \to 0, \textit{as } k\to\infty, \tag{28}$$

then (u_{n_k}, v_{n_k}) is bounded in X_{n_k}. Moreover, there is $k_0 \in N$ and $m \in (0,1)$ such that

$$\|v_{n_k}\|^{N/(N-1)}, \|u_{n_k}\|^{N/(N-1)} \leq m, \textit{ for all } k \geq k_0. \tag{29}$$

Proof. We start observing that the condition (G_2) implies that

$$0 \leq \mu G(x,s) \leq g(x,s)s, \text{ for all } s\in R \text{ and } x\in\Omega. \tag{30}$$

From (28),

$$\Phi(u_{n_k}, v_{n_k}) - \langle \Phi'_{n_k}(u_{n_k}, v_{n_k}), (\frac{1}{\nu}u_{n_k}, \frac{1}{\mu}v_{n_k})\rangle = c + o_k(1)\|(u_{n_k}, v_{n_k})\|. \quad (31)$$

By $(G_1)-(G_2)$, (F_2), (R_3) and (30),

$$\begin{aligned}
&(\frac{1}{N}-\frac{1}{\mu})\|v_{n_k}\|^N + (\frac{1}{\nu}-\frac{1}{N})\|u_{n_k}\|^N \\
&\leq \Phi(u_{n_k}, v_{n_k}) - \langle \Phi'_{n_k}(u_{n_k}, v_{n_k}), (\frac{1}{\nu}u_{n_k}, \frac{1}{\mu}v_{n_k})\rangle \\
&\leq \beta, \qquad (32)
\end{aligned}$$

from where it follows that (u_{n_k}, v_{n_k}) is bounded in X_{n_k}. Consequently,

$$\limsup_{k\to\infty} \|v_{n_k}\|^N \leq \frac{\mu N\beta}{\mu - N}$$

and

$$\limsup_{k\to\infty} \|u_{n_k}\|^N \leq \frac{\nu N\beta}{\nu - N}.$$

From (G_3) and Lemma 2.3, we get

$$\limsup_{k\to\infty} \|v_{n_k}\|^N, \limsup_{k\to\infty} \|u_{n_k}\|^N < 1. \quad (33)$$

Therefore, there are $k_0 \in N$ and $m \in (0,1)$ such that

$$\|v_{n_k}\|^{N/(N-1)}, \|u_{n_k}\|^{N/(N-1)} \leq m, \text{ for all } k \geq k_0,$$

which proves the lemma. □

Lemma 2.5. Φ *satisfies* $(PS)^*_c$, $\forall c \in [\alpha, \beta]$.

Proof. By (28), we have

$$\begin{aligned}
\Phi(u_{n_k}, v_{n_k}) = &-\frac{1}{N}\int_\Omega |\nabla u_{n_k}|^N dx + \frac{1}{N}\int_\Omega |\nabla v_{n_k}|^N dx - \int_\Omega F(x, u_{n_k})dx \quad (34)\\
&-\int_\Omega G(x, v_{n_k})dx - \int_\Omega R(x, u_{n_k}, v_{n_k})dx \\
&\to c \in [\alpha, \beta],
\end{aligned}$$

$$\begin{aligned}
\langle d\Phi_{n_k}(u_{n_k}, v_{n_k}), (\tilde{u}, \tilde{v})\rangle = &-\int_\Omega |\nabla u_{n_k}|^{N-2}\nabla u_{n_k}\nabla\tilde{u}dx - \int_\Omega f(x, u_{n_k})\tilde{u}dx \quad (35)\\
&+\int_\Omega |\nabla v_{n_k}|^{N-2}\nabla v_{n_k}\nabla\tilde{v}dx - \int_\Omega g(x, v_{n_k})\tilde{v}dx \\
&-\int_\Omega R_u(x, u_{n_k}, v_{n_k})\tilde{u}dx - \int_\Omega R_v(x, u_{n_k}, v_{n_k})\tilde{v}dx \\
&\to 0, \text{ as } k \to \infty.
\end{aligned}$$

By Lemma 2.4, since (u_{n_k}, v_{n_k}) is bounded, we assume

$$u_{n_k} \rightharpoonup u \text{ in } W_0^{1,N}(\Omega),$$

$$\begin{aligned} v_{n_k} &\rightharpoonup v \ \text{ in } \ W_0^{1,N}(\Omega), \\ u_{n_k} &\rightharpoonup u, \ \ a.e. \ \text{ on } \ \Omega, \\ v_{n_k} &\rightharpoonup v, \ \ a.e. \ \text{ on } \ \Omega. \end{aligned}$$

According to (See [1, 2, 12, 23])

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq |x-y|^p \ \text{ for } \ p \geq 2, \tag{36}$$

we have

$$\int_\Omega |\nabla u_{n_k}|^{N-2}\nabla u_{n_k}\nabla \tilde{u} dx \to \int_\Omega |\nabla u|^{N-2}\nabla u \nabla \tilde{u} dx \tag{37}$$

and

$$\int_\Omega |\nabla v_{n_k}|^{N-2}\nabla v_{n_k}\nabla \tilde{v} dx \to \int_\Omega |\nabla v|^{N-2}\nabla v \nabla \tilde{v} dx. \tag{38}$$

Let m be the constant given by Lemma 2.4. Since m is independent of n_k, the weak convergent implies that

$$\|u\|^{N/(N-1)}, \ \ \|v\|^{N/(N-1)} \leq m. \tag{39}$$

On the other hand,

$$\|u_{n_k}\|^{N/(N-1)}, \|v_{n_k}\|^{N/(N-1)} \leq m < 1, \ \text{ for all } \ k \geq k_0,$$

thus, by Proposition 2.2 and Lemma 2.2 it follows that

$$\begin{aligned} \int_\Omega f(x, u_{n_k}) u dx &\to \int_\Omega f(x,u) u dx, \\ \int_\Omega f(x, u_{n_k}) u_{n_k} dx &\to \int_\Omega f(x,u) u dx, \end{aligned} \tag{40}$$

$$\begin{aligned} \int_\Omega g(x, v_{n_k}) v dx &\to \int_\Omega g(x,v) v dx, \\ \int_\Omega g(x, v_{n_k}) v_{n_k} dx &\to \int_\Omega g(x,v) v dx, \end{aligned} \tag{41}$$

$$\int_\Omega R_u(x, u_{n_k}, v_{n_k}) u_{n_k} dx \to \int_\Omega R_u(x,u,v) u dx, \tag{42}$$

$$\begin{aligned} \int_\Omega R_u(x, u_{n_k}, v_{n_k}) u dx &\to \int_\Omega R_u(x,u,v) u dx, \\ \int_\Omega R_v(x, u_{n_k}, v_{n_k}) v_{n_k} dx &\to \int_\Omega R_v(x,u,v) v dx, \end{aligned} \tag{43}$$

and

$$\int_\Omega R_v(x, u_{n_k}, v_{n_k}) v dx \to \int_\Omega R_v(x,u,v) v dx$$

as $k \to \infty$.

It follows from (35) that

$$\begin{aligned} &- \int_\Omega |\nabla u|^{N-2}\nabla u \nabla \tilde{u} dx + \int_\Omega |\nabla v|^{N-2}\nabla v \nabla \tilde{v} dx - \int_\Omega f(x,u)\tilde{u} dx \\ &- \int_\Omega g(x,v)\tilde{v} dx - \int_\Omega R_u(x,u,v)\tilde{u} dx - \int_\Omega R_v(x,u,v)\tilde{v} dx = 0. \end{aligned} \tag{44}$$

By setting $(\tilde{u}, \tilde{v}) = (u, 0)$, we get

$$\int_\Omega |\nabla u|^N dx + \int_\Omega f(x,u)udx + \int_\Omega R_u(x,u,v)udx = 0; \tag{45}$$

and then setting $(\tilde{u}, \tilde{v}) = (0, v)$, we have

$$\int_\Omega |\nabla v|^N dx - \int_\Omega g(x,v)vdx - \int_\Omega R_v(x,u,v)vdx = 0. \tag{46}$$

Note that

$$\begin{aligned} \langle d\Phi_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k})\rangle = &\int_\Omega |\nabla v_{n_k}|^N dx - \int_\Omega g(x, v_{n_k})v_{n_k}dx \\ &- \int_\Omega R_v(x, u_{n_k}, v_{n_k})v_{n_k}dx \\ &\to 0, \end{aligned} \tag{47}$$

$$\begin{aligned} \langle d\Phi_{n_k}(u_{n_k}, v_{n_k}), (u_{n_k}, 0)\rangle = &- \int_\Omega |\nabla u_{n_k}|^N dx - \int_\Omega f(x, u_{n_k})u_{n_k}dx \\ &- \int_\Omega R_u(x, u_{n_k}, v_{n_k})u_{n_k}dx \\ &\to 0. \end{aligned} \tag{48}$$

Let $\omega_{n_k} = u_{n_k} - u$, $\zeta_{n_k} = v_{n_k} - v$. By Brézis-Lieb Lemma [22], (47)-(48) can be changed to

$$\begin{aligned} &\int_\Omega |\nabla \zeta_{n_k}|^N dx + \int_\Omega |\nabla v|^N dx - \int_\Omega g(x, v_{n_k})v_{n_k}dx \\ &\qquad - \int_\Omega R_v(x, u_{n_k}, v_{n_k})v_{n_k}dx \to 0, \end{aligned} \tag{49}$$

$$\begin{aligned} &-\int_\Omega |\nabla \omega_{n_k}|^N dx - \int_\Omega |\nabla u|^N dx - \int_\Omega f(x, u_{n_k})u_{n_k}dx \\ &\qquad - \int_\Omega R_u(x, u_{n_k}, v_{n_k})u_{n_k}dx \to 0, \end{aligned} \tag{50}$$

By (45)-(46), it is easy to obtain

$$\begin{aligned} &\int_\Omega |\nabla \zeta_{n_k}|^N dx + \int_\Omega g(x,v)vdx + \int_\Omega R_v(x,u,v)vdx \\ &\qquad - \int_\Omega g(x, v_{n_k})v_{n_k}dx - \int_\Omega R_v(x, u_{n_k}, v_{n_k})v_{n_k}dx \to 0, \end{aligned} \tag{51}$$

$$\begin{aligned} &\int_\Omega |\nabla \omega_{n_k}|^N dx - \int_\Omega f(x,u)udx - \int_\Omega R_u(x,u,v)udx \\ &\qquad + \int_\Omega f(x, u_{n_k})u_{n_k}dx + \int_\Omega R_u(x, u_{n_k}, v_{n_k})u_{n_k}dx \to 0, \end{aligned} \tag{52}$$

By (40)-(43), we obtain

$$\int_\Omega |\nabla \zeta_{n_k}|^N dx \to 0$$

and

$$\int_\Omega |\nabla \omega_{n_k}|^N dx \to 0.$$

That is

$$u_{n_k} \to u, \text{ in } W_0^{1,N}$$

and

$$v_{n_k} \to v, \text{ in } W_0^{1,N}.$$

Then we complete the proof of Lemma 2.5 □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Now we shall verify the conditions of Theorem 2.1. It is clear that (B_1), (B_2), (B_4) in Theorem 2.1 are satisfied. Set $V_j = E_j = \operatorname{span}\{e_1, e_2, \cdots e_j\}$, then (B_3) is also satisfied. Since $1 = \dim \tilde{Y_0} < k_0 = \dim Y_1$, (B_5) is satisfied. Since Fix $G \bigcap V = 0$, that is (a) of (B_7) holds. $(b)-(c)$ of (B_7) can be obtained by Lemma 2.3. By Lemma 2.5, (B_6) in Theorem 2.1 hold. So according to Theorem 2.1,

$$c_j = \inf_{i^\infty(A)\geq j} \sup_{(u,v)\in A} \Phi(u,v), \quad -k_0+1 \leq j \leq -1$$

are critical values of Φ, $\alpha \leq c_{-k_0+1} \leq \cdots \leq c_{-1} \leq \beta$, and Φ has at least $k_0 - 1$ pairs critical points.

Acknowledgements

We would like to thank the referees for their careful reading of the original manuscript.

This research was supported by the NSFC (10871096) and The Institute of Mathematical Sciences, The Chinese University of Hong Kong.

References

1. C. O. Alves, S. H. Monari, *Existence of solution for a class of quasilinear systems*, Adv. Nonlinear Stud. **9** (2009), 537-564.
2. C. O. Alves, G. M. Figueiredo, *On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in R^N*, J. Differential Equations **246** (2009), 1288-1311.
3. L. Boccardo, D. G. Defigueiredo, *Some remarks on a system of quasilinear elliptic equations*, NoDEA Nonlinear Differ. Equ. Appl. **9** (3) (2002), 309-323.
4. J. F. Bonder, S. Martínezi, J. D. Rossi, *Existence results for Gradient elliptic systems with nonlinear boundary conditions*, NoDEA Nonlinear Differ. Equ. Appl. **14** (2007), 153-179.
5. V. Benci, *On critical point theory for indefinite functional in presence of symmetries*, Tran. Amer. Math. Soc. **24** (1982), 533-572.
6. D. G. Costa, *Multiple solutions for a class of stongly indefinite problems*, Mat. Contemp. **15** (1998), 87-103.

7. M. Clapp, Y. H. Ding, S. Hernández-Linares, *Strongly indefinite functional with perturbed symmetries and multiple solutions of nonsymmetric elliptic systems*, Electron. J. Differential Equations **100** (2004), 1-18.
8. Y. Q. Fang, J. H. Zhang, *Multiplicity of solutions for a class of elliptic systems with critical Sobolev exponent*, accepted by Nonlinear Anal. (2010).
9. D. G. de Figueiredo, J. M. Bezerra do Ó, B. Ruf, *Critical and subcritical elliptic systems in dimension two*, Indiana Univ. Math. J. **53** (2004), no.4, 1037-1054.
10. D. Huang, Y. Li, *Multiple solutions for a noncooperative p-Laplacian elliptic system in* R^N , J. Differential Equations **215** (2005), 206-223.
11. W. Kryszewski, A. Szulkin, *An infinte dimensional Morse theory with applications*, Trans. Amer. Math. Soc. **349** (1997), 3181-3234.
12. V. Kondvatev, M. Shubin, *Discreteness of spectrum for the Schrödinger operators on manifolds of bounded geometry*, Oper. Theory Adv. Appl. **110** (1999), 185-226.
13. F. Lin, Y. Li, *Multiplcity of solutions for a noncooperative elliptic system with critical Sobolev exponent*, Z. Angew. Math. Phys. **60** (2009), 402-415.
14. S. J. Li, M. willem, *Applications of local linking to critical point thery*, J. Math. Anal. Appl. **189** (1995), 6-32.
15. Y. Q. Li, *A limit index theory and its application*, Nonlinear Anal. **25** (1995), 1371-1389.
16. J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces* I. Springer, Berlin (1977).
17. J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077-1092.
18. M. Struwe, *Variational Methods, Springer*, Berlin 1990.
19. A. Szulkin, *Index theories for indefinite functionals and applications, in: P. Drábek (Ed.), Pitman Research Notes in Mathematics, Series 365, Topological and Variational Methods for Nonlinear Boundary Value Problems*, Longman, Harlow 1997, pp. 89-121.
20. N. S. Trudinger, *On the imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473-484.
21. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators. North-Holland*, Amsterdam (1978).
22. M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.
23. J. F. Yang, *Positive solutions of quasilinear elliptic obstacle problems with critical exponents*, Nonlinear Anal. **25** (1995), 1283-1306.

Yanqin Fang received her M.Sc. from Nanjing Normal University. She is current a doctoral student since 2009. Her research interests include Nonlinear PDEs and functional analysis.

School of Mathematics Sciences Nanjing Normal University, Nanjing 210046, Jiangsu, People's Republic of China.
e-mail: nj_yqfang@hotmail.com

Jihui Zhang received his M.Sc. from Xibei Normal University in 1988, and Ph.D. from Lanzhou University in 1997. He is currently a professor at Nanjing Normal University. His research interests are Nonlinear PDEs and functional analysis.

School of Mathematics Sciences Nanjing Normal University, Nanjing 210046, Jiangsu, People's Republic of China.
e-mail: jihuiz@jlonline.com