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BIFURCATIONS AND FEEDBACK CONTROL IN AN EXPLOITED PREY-PREDATOR SYSTEM WITH STAGE STRUCTURE FOR PREY†

T. K. KAR* AND U. K. PAHARI

Abstract. In the present paper we consider a differential-algebraic prey-predator model with stage structure for prey and harvesting of predator species. Stability and instability of the equilibrium points are discussed and it is observed that the model exhibits a singular induced bifurcation when the economic profit is zero. It indicates that the zero economic profit brings impulse, i.e. rapid expansion of the population and the system collapses. For the purpose of stabilizing the system around the positive equilibrium, a state feedback controller is designed. Finally, numerical simulations are given to show the consistency with theoretical analysis.

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1. Introduction

The study of renewable resources such as fisheries, forestry and wildlife is becoming more and more interesting field of research [1, 2, 3, 5, 8]. In the natural world it has been noticed that the life history of many species is composed of at least two stages: immature and mature, with significantly different morphological and behavioral characteristics. So the study of stage structured predator-prey systems has attracted considerable attention in recent years, as a way to overcome the limitations of classical Lotka-Volterra models.
Again the biological resources in the prey-predator ecosystem is commercially harvested and sold with the aim of achieving economic interest. For this reason harvesting plays an important role in the study of biological resources. Furthermore, the harvest effort is usually influenced by the variation of economic interest of harvesting. To formulate a biological economic system from an economic point

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of view and to investigate the dynamical behavior of the model system many scientists use differential-algebraic equations. The differential equations investigate the dynamics of the biological species such as prey, predators etc., and the algebraic equation studies the harvest effort in harvesting from an economic perspective. The pioneering work of Aiello and Freedman [1] on a single species growth model with stage structure represents a mathematically more careful and biologically meaningful formulation approach. They studied a model of single species population growth incorporating stage structure as a reasonable generalization of the classical logistic model. Song and Chen [9] have considered the exploitation of a predator-prey population with stage structure and harvesting for the prey and showed that the nonnegative equilibrium point is globally asymptotically stable under a certain condition. The effects on population size and yield of different levels of harvesting of a predator in a predator-prey system have been explored by Matsuda and Abrams [7] and showed that the predator may increase in population size with increasing fishing effort. Zhang and Zhang [11]; and Zhang et.al [12] have established a class of differential-algebraic biological economic models by several differential equations and an algebraic equation. They have studied the effect of harvest effort on ecosystem from an economic perspective. Zhang and Zhang [13] systematically studied a hybrid predator prey economic model, which is formulated by differential-difference-algebraic equations. They proved that this model exhibits two bifurcation phenomena at the inter sampling instants.

In this paper, we consider a differential-algebraic model for a prey-predator system with a stage-structure for the prey species and harvesting of predator species. We denote the density of immature prey, mature prey and predator by x_1, x_2 and x_3 respectively. Also we make the following assumptions:

(i) The birth rate of the immature population is proportional to the existing mature population with proportionality constant α; for the immature population, the death rate and transformation rate to mature are proportional to the existing immature population with proportionality constants r_1 and β; the death rate of immature population due to interaction among themselves is of logistic type.

(ii) The death rate of the mature population is proportional to the existing mature population with proportionality constant r_2 .

(iii) The predator consumes the prey species only and the death rate of the predator is of logistic type.

According to these assumptions, we can set up the stage-structured predator-prey model as follows:

$$\begin{aligned} \dot{x_1} &= \alpha x_2 - r_1 x_1 - \beta x_1 - \eta x_1^2 - \beta_1 x_1 x_3, \\ \dot{x_2} &= \beta x_1 - r_2 x_2, \\ \dot{x_3} &= x_3 \left(-r + k\beta_1 x_1 - \eta_1 x_3\right), \end{aligned} \tag{1}$$

where $\alpha, r_1, r_2, \beta, \beta_1, \eta, \eta_1, r, k$ are all positive constants. Here k is the digesting constant, η is the intra-specific competition rate and $\dot{x}_i = dx_i/dt,\ i = 1, 2, 3$. Let us take a transformation as

$$y_1 = \frac{k\beta_1}{r_2}x_1,\ y_2 = \frac{k\beta_1}{\beta}x_2,\ y_3 = \frac{\eta_1}{r_2}x_3,\ dt = \frac{1}{r_2}d\tau.$$

Then the model (1) becomes

$$\begin{aligned} \dot{y_1} &= ay_2 - by_1 - cy_1^2 - dy_1y_3, \\ \dot{y_2} &= y_1 - y_2, \\ \dot{y_3} &= y_3(-e + y_1 - y_3), \end{aligned} \tag{2}$$

where

$$\dot{y_i} = dy_i/d\tau,\ a = \frac{\alpha\beta}{r_2^2},\ b = \frac{r_1 + \beta}{r_2},\ c = \frac{\eta}{k\beta_1},\ d = \frac{\beta_1}{\eta_1},\ e = \frac{r}{r_2},\ i = 1, 2, 3.$$

2. Model formulation

We assume that the predator species is subjected to harvesting. The functional form of harvest is generally considered using the phrase catch-per-unit-effort (CPUE) hypothesis [3] to describe an assumption that catch per unit effort is proportional to the stock level. Thus we consider the harvest function $h(t)$ as

$$h(t) = qEy_3, \tag{3}$$

where q is the catchability coefficient and E is the harvesting effort. To investigate the economic interest of the yield we take another equation as

$$\begin{aligned} Net\ Economic\ Revenue\ (v) &= TotalRevenue\ (TR) - TotalCost\ (TC) \\ &= pqy_3E - c_1E \\ &= (pqy_3 - c_1)E, \end{aligned} \tag{4}$$

where p is the price of unit harvested predator and c_1 is the cost of unit harvest effort. Thus considering the economic interest of the system we take the differential algebraic model which consists of three differential equations and an algebraic equation as follows:

$$\begin{aligned} \dot{y_1} &= ay_2 - by_1 - cy_1^2 - dy_1y_3, \\ \dot{y_2} &= y_1 - y_2, \\ \dot{y_3} &= y_3(-e + y_1 - y_3 - qE), \\ 0 &= (pqy_3 - c_1)E - v. \end{aligned} \tag{5}$$

The differential-algebraic model system (5) can be expressed in the following form:

$$A\begin{bmatrix} \dot{X} \\ 0 \end{bmatrix} = \begin{bmatrix} f(X, E, v) \\ g \end{bmatrix} \tag{6}$$

where

$$X = (y_1, y_2, y_3)^T, A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; and$$

$$f(X, E, v) = \begin{bmatrix} f_1(X, E, v) \\ f_2(X, E, v) \\ f_3(X, E, v) \end{bmatrix} = \begin{bmatrix} ay_2 - by_1 - cy_1^2 - dy_1y_3 \\ y_1 - y_2 \\ y_3(-e + y_1 - y_3 - qE \end{bmatrix},$$

$$g = (pqy_3 - c_1)E - v.$$

When the economic profit is zero, the system (5) takes the form:

$$\begin{aligned} \dot{y_1} &= ay_2 - by_1 - cy_1^2 - dy_1y_3, \\ \dot{y_2} &= y_1 - y_2, \\ \dot{y_3} &= y_3(-e + y_1 - y_3 - qE), \\ 0 &= (pqy_3 - c_1)E. \end{aligned} \tag{7}$$

3. Equilibrium and stability analysis

The system (7) has
(i) a trivial equilibrium point $P_0(0, 0, 0, 0)$.
(ii) a boundary equilibrium point$P_1(\frac{a-b}{c}, \frac{a-b}{c}, 0, 0)$ which exists if $a > b$.
(iii) another equilibrium point $P_2(\bar{y_1}, \bar{y_2}, \bar{y_3}, 0)$ which exists if $a > b + ce$ where

$$\bar{y_1} = \frac{a - b + de}{c + d}, \; \bar{y_2} = \bar{y_1}, \; \bar{y_3} = \frac{a - b - ce}{c + d}.$$

(iv) an interior equilibrium point $P_3(y_1^*, y_2^*, y_3^*, E^*)$ which exists if

$$(a - b - ce)pq > c_1(c + d),$$

where

$$y_1^* = \frac{(a - b)pq - c_1d}{cpq}, \; y_2^* = y_1^*, \; y_3^* = \frac{c_1}{pq}, \; E^* = \frac{(a - b - ce)pq - c_1(c + d)}{cpq^2}.$$

From the system (7) we get

$$J_1 = D_Xf - D_Ef(D_Eg)^{-1}D_Xg = \begin{bmatrix} -b - 2cy_1 - dy_3 & a & -dy_1 \\ 1 & -1 & 0 \\ y_3 & 0 & -e + y_1 - 2y_3 + \frac{c_1qE}{pqy_3 - c_1} \end{bmatrix}$$

The characteristic equation of the matrix J_1 is $det(\lambda I - J_1) = 0$, i.e.

$$\lambda^3 + a_1(X, E)\lambda^2 + a_2(X, E)\lambda + a_3(X, E) = 0$$

where

$$a_1(X, E) \;\; = \;\; 1 + e - y_1 + 2y_3 - \frac{c_1qE}{pqy_3 - c_1} + b + 2cy_1 + dy_3,$$

$$a_2(X,E) = (b+2cy_1+dy_3+1)(e-y_1+2y_3-\frac{c_1qE}{pqy_3-c_1})+b+2cy_1+dy_3+dy_1-a,$$
$$a_3(X,E) = (b+2cy_1+dy_3-a)(e-y_1+2y_3-\frac{c_1qE}{pqy_3-c_1})+dy_1.$$

It is easy to verify that, the equilibrium point P_0 is a stable node for $a > b$; P_1 is unstable and P_2 is stable if the conditions $a_1 > 0,\ a_3 > 0$ and $a_1a_2 > a_3$ are satisfied. For the interior equilibrium point P_3 we have the following theorem.

Theorem 1. *The system (5) has a singularity induced bifurcation (SIB) at the interior equilibrium and* $v = 0$ *is a bifurcation value. Furthermore, a stability switch occurs as* v *increases through 0.*

Proof. We see that at the interior equilibrium point P_3,

$$g(X,E,v) = \begin{cases} 0, & \text{if v=0;} \\ nonzero, & \text{if } v \neq 0 \end{cases}$$

which implies that $dimker\,(D_Eg(X,E,v)|_{P_3}) \ = \ 1.$

Let $\triangle \ = \ D_Eg(X,E,v) = pqy_3 - c_1.$

Then it has a simple zero eigen value at P_3 .

Now,

$$\begin{vmatrix} D_Xf & D_Ef \\ D_Xg & D_Eg \end{vmatrix}_{P_3} = \begin{vmatrix} -a-cy_1^* & a & -dy_1^* & 0 \\ 1 & -1 & 0 & 0 \\ y_3^* & 0 & -y_3^* & -qy_3^* \\ 0 & 0 & pqE^* & 0 \end{vmatrix} = cpq^2y_1^*y_3^*E^* \ \neq 0$$

and,

$$\begin{vmatrix} D_Xf & D_Ef & D_vf \\ D_Xg & D_Eg & D_vg \\ D_X\triangle & D_E\triangle & D_v\triangle \end{vmatrix}_{P_3} = \begin{vmatrix} -a-cy_1 & a & -dy_1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ y_3 & 0 & -y_3 & -qy_3 & 0 \\ 0 & 0 & pqE & 0 & -1 \\ 0 & 0 & pq & 0 & 0 \end{vmatrix}_{P_3}$$
$$= cpq^2y_1^*y_3^* \neq 0.$$

Again,

$$\begin{aligned} trace\ (D_Efadj(D_Eg)D_Xg)P_3 &= trace \begin{pmatrix} 0 \\ 0 \\ -qy_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & pqE \end{pmatrix}_{P_3} \\ &= trace \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -pq^2y_3E \end{pmatrix}_{P_3} \\ &= -pq^2y_3^*E^* \neq 0. \end{aligned}$$

Based on the above analysis, three items can be obtained as follows:

(I) $f(X(0), E(0), 0) = 0$ and $g(X(0), E(0), 0) = 0$ and $\triangle = D_E g(X, E, v)$ has a simple zero eigen value at P_3 and *trace* $(D_E f adj(D_E g) D_X g)_{P_3} \neq 0$.

(II) $\begin{pmatrix} D_X f & D_E f \\ D_X g & D_E g \end{pmatrix}$ is non singular at P_3.

(III) $\begin{pmatrix} D_X f & D_E f & D_v f \\ D_X g & D_E g & D_v g \\ D_X \triangle & D_E \triangle & D_v \triangle \end{pmatrix}$ is non singular at P_3.

The conditions for occurrence of singularity induced bifurcation (SIB) are all satisfied for the system (5) (see Venkatasubramanian *et.al.* [10] and hence the system (5) undergoes singular induced bifurcation at the interior equilibrium point $P_3(y_1^*, y_2^*, y_3^*, E^*)$ when the bifurcation parameter $v = 0$. Again it is noted that

$$M = -trace\ (D_E f adj(D_E g) D_X g) P_3 = pq^2 y_3^* E^* > 0$$

and

$$N = \left(D_v \triangle - \begin{pmatrix} D_X \triangle & D_E \triangle \end{pmatrix} \begin{pmatrix} D_X f & D_E f \\ D_X g & D_E g \end{pmatrix}^{-1} \begin{pmatrix} D_v f \\ D_v g \end{pmatrix} \right)_{P_3} = \frac{1}{E^*} > 0.$$

Consequently, we have

$$\frac{M}{N} = pq^2 y_3^* E^{*^2} > 0. \tag{8}$$

Therefore, when v increases through 0, one eigen value (denoted by λ_1) of the system (5) moves from C^- to C^+ along the real axis by diverging through infinity (see Theorem 3 in Venkatasubramanian *et. al.* [10]). It brings impulse, i.e. rapid expansion of the population in biological explanation.

We now calculate the other eigen values (denoted by λ_2 and λ_3) of the differential-algebraic model system (5) at P_3 . We denote the jacobian matrix at P_3 by

$$J_{P_3} = \begin{pmatrix} -a - cy_1^* & a & -dy_1^* & 0 \\ 1 & -1 & 0 & 0 \\ y_3^* & 0 & -y_3^* & -qy_3^* \\ 0 & 0 & pqE^* & 0 \end{pmatrix} \tag{9}$$

According to the leading matrix A in the system (6) and J_{P_3}, we obtain the characteristic equation of the system (5) at P_3 as

$$det\,(\lambda A - J_{P_3}) = 0,$$

which can be expressed as follows:

$$\begin{vmatrix} \lambda + (a + cy_1^*) & -a \\ -1 & \lambda + 1 \end{vmatrix} = 0$$

$$i.e.\ \lambda^2 + (a + cy_1^* + 1)\lambda + cy_1^* \quad = \quad 0.$$

Therefore, $\lambda_1 + \lambda_2 = -(a + cy_1^* + 1) < 0$ and $\lambda_1\lambda_2 = cy_1^* > 0$.

Thus the eigen values λ_2 and λ_3 of the differential-algebraic model system (5) at P_3 are negative. And hence they are continuous, nonzero and can not jump from one half open complex plane to another as v increases through 0. Therefore, they are continuous and bounded in the C^{-1} half plane as v increase through 0 and their movement behaviors have no influence on the stability of the system (5) at the interior equilibrium point P_3. Therefore, it can be concluded that the system (5) is stable at P_3 as $v < 0$ and it is unstable as $v > 0$. Consequently, a stability switch occurs as economic profit v increases through 0. This completes the proof. □

Remark. We have seen from the above discussion of theorem 1 that when economic interest of harvesting becomes positive then the differential-algebraic model system (5) becomes unstable and an impulsive phenomenon occurs, i.e. if we consider a prey-predator fishery, then a rapid expansion of the population occurs. If this phenomenon lasts for a period of time, the species population will be beyond the carrying capacity of the environment and the fishery will be out of balance which may lead to collapse of the sustainable ecosystem of the prey-predator fishery.
On the other hand, government and society or fishery agencies always try to make profit from fishery and so they are usually interested in the case of positive economic interest of harvesting. Since the dynamical behavior of the system (5) is unstable around the interior equilibrium point when the economic interest is positive, it is impossible for sustainable development of harvesting on fishery.

Therefore, it is necessary to eliminate the impulsive phenomenon caused by singularity induced bifurcation to resume the sustainability of the ecosystem and stabilize the system (5) for positive economic interest and for this reason some related measures should be taken.

4. Feedback Control for Singular Induced Bifurcation

In the theorem1 we see that the system (5) has a singularity induced bifurcation around the interior equilibrium point when $v = 0$, and the system (5) is unstable around the interior equilibrium point in the case of positive economic profit. In this section, when we have a positive economic profit, a feedback controller is designed to eliminate the SIB of the system (5) and stabilize the system around the interior equilibrium. Furthermore, a numerical simulation is given to illustrate the effectiveness of the controller.

4.1 Design of the feedback control According to the leading matrix A in the model (5) and J_{P_3}, it can be calculated that the $rank\,(J_{P_3}\ AJ_{P_3}\ A^2J_{P_3}\ A^3J_{P_3}) =$

4. By using theorem 2-2.1 in Dai [4], it is easy to show the system (5) is locally controllable at P_3. Consequently, a feedback controller can be applied to stabilize the system (5) at P_3.
By using the theorem 3-1.2 in Dai [4], a feedback controller $u(t) = K(E(t) - E^*)$ where K is the feedback gain and E^* is a component of P_3 i.e.
$E^* = \frac{(a-b-ce)pq-c_1(c+d)}{cpq^2}$, can be applied to stabilize the system (5) around P_3.
Applying the controller $u(t) = K(E(t) - E^*)$ we get a controlled differential-algebraic model system as follows:

$$\begin{aligned} \dot{y_1} &= ay_2 - by_1 - cy_1^2 - dy_1y_3, \\ \dot{y_2} &= y_1 - y_2, \\ \dot{y_3} &= y_3(-e + y_1 - y_3 - qE), \\ 0 &= (pqy_3 - c_1)E - v + K(E(t) - E^*), \end{aligned} \tag{10}$$

where $y_1, y_2, y_3, E, a, b, c, d, e, c_1, v, q$ have the same biological interpretations as mentioned in the system (5). The feedback controller $u(t) = K(E(t) - E^*)$ will be designed in the following theorem 2.

Theorem 2. *If the feedback gain K satisfies the following inequality,*

$$K > max\{P, Q, R\}$$

where

$$P = \frac{pq^2E^*y_3}{a + cy_1^* + y_3^* + 1}, \ Q = \frac{pq^2E^*c}{c+d}, \ R = \frac{B_1 + \sqrt{B_1^2 - 4A_1C_2}}{2A_1}$$

and

$$\begin{aligned} A_1 &= (a + cy_1^* + y_3^*)\{cy_1^* + y_3^*(a + cy_1^* + dy_1^* + 1)\} + (cy_1^* + ay_3^* + y_3^*), \\ B_1 &= pq^2E^*y_3^*\left\{(a + cy_1^* + 1)^2 + 2y_3^*(a + cy_1^* + 1) + dy_1^*y_3^*\right\}, \\ C_2 &= (pq^2E^*y_3^*)^2(a + cy_1^* + 1). \end{aligned}$$

then the system (10) is stable around P_3.

Proof. The Jacobian of model (10) evaluated around P_3 takes the form

$$\tilde{J}_{P_3} = \begin{pmatrix} -a - cy_1^* & a & -dy_1^* & 0 \\ 1 & -1 & 0 & 0 \\ y_3^* & 0 & -y_3^* & -qy_3^* \\ 0 & 0 & pqE^* & K \end{pmatrix} \tag{11}$$

According to the leading matrix A in the model (5) and $\tilde{J}_{P_3}$, the characteristic equation of the system (10) around P_3 is $det(\lambda A - \tilde{J}_{P_3}) = 0$, which can be expressed as follows:

$$\lambda^3 + A_2\lambda^2 + B_2\lambda + C_3 = 0$$

where

$$\begin{aligned} A_2 &= (a + cy_1^* + y_3^* + 1) - \frac{pq^2E^*y_3^*}{K}, \\ B_2 &= cy_1^* + y_3^*\,(a + cy_1^* + dy_1^* + 1) - \frac{(a + cy_1^* + 1)(pq^2E^*y_3^*)}{K}, \\ C_3 &= y_1^*y_3^*(c + d) - \frac{pq^2E^*y_3^*cy_1^*}{K}. \end{aligned}$$

By using Routh-Hurwitz criteria [6], the necessary and sufficient condition for the stability of the system (10) around P_3 is that the feedback gain K satisfies $K > max\{P, Q, R\}$ where

$$P = \frac{pq^2E^*y_3}{a + cy_1^* + y_3^* + 1},\ Q = \frac{pq^2E^*c}{c + d},\ R = \frac{B_1 + \sqrt{B_1^2 - 4A_1C_2}}{2A_1}$$

and

$$\begin{aligned} A_1 &= (a + cy_1^* + y_3^*)\,\{cy_1^* + y_3^*(a + cy_1^* + dy_1^* + 1)\} + (cy_1^* + ay_3^* + y_3^*)\,, \\ B_1 &= pq^2E^*y_3^*\left\{(a + cy_1^* + 1)^2 + 2y_3^*(a + cy_1^* + 1) + dy_1^*y_3^*\right\}, \\ C_2 &= (pq^2E^*y_3^*)^2(a + cy_1^* + 1). \end{aligned}$$

Consequently, if the feedback gain K satisfies the above inequality, then system (10) is stable around P_3 and this completes the proof. □

5. The model with positive economic profit

When economic profit v is positive then we can get two interior equilibrium points of the system (5) as $\hat{P}_1(\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{E})$ and $\hat{P}_2(\hat{y}_1, \hat{y}_2, \bar{\bar{y}}_3, \hat{E})$ where

$$\begin{aligned} \hat{y}_1 &= \hat{y}_2 = \frac{a - b - d\hat{y}_3}{c}, \\ \hat{y}_3 &= \frac{c_1(c + d) + (a - b - ce)pq + \sqrt{\{c_1(c + d) - (a - b - ce)pq\}^2 - 4pq^2cv(c + d)}}{2(c + d)pq}, \\ \bar{\bar{y}}_3 &= \frac{c_1(c + d) + (a - b - ce)pq - \sqrt{\{c_1(c + d) - (a - b - ce)pq\}^2 - 4pq^2cv(c + d)}}{2(c + d)pq}, \\ \hat{E} &= \frac{v}{pq\hat{y}_3 - c_1}. \end{aligned}$$

These two interior equilibrium points exist if

$$b - a + ce \ < \ min\{\tfrac{c_1(c+d)}{pq}, \tfrac{cqv}{c_1}\}$$

We consider the stability of an interior equilibrium point when economic profit is positive numerically in the subsequent article.

6. Numerical Simulation

Numerical simulations are given to illustrate the results obtained earlier. For this we take the parameters value as:
$a = 150, b = 15, c = 5, d = 3, e = 1.2, p = 2.5, q = 1, c_1 = 30$

Case-I: (i) When $v = 0$ we find that the interior equilibrium point of the system (5) is $(19.8, 19.8, 12, 6.6)$.

(ii) When v is taken as positive, i.e. $v = 0.0001$, then the model (5) becomes:

$$\begin{aligned} \dot{y_1} &= 150y_2 - 15y_1 - 5y_1^2 - 3y_1y_3, \\ \dot{y_2} &= y_1 - y_2, \\ \dot{y_3} &= y_3(-1.2 + y_1 - y_3 - E), \\ 0 &= (2.5y_3 - 30)E - 0.0001. \end{aligned}$$

In this case we find the eigen values of the system from the Jacobian matrix J_1 given above as $1.3068 \times 10^7, -249.603, -0.396629$. As one of the eigen values is positive, the system is unstable around the equilibrium point $(19.8, 19.8, 12, 6.6)$.

(iii) When v is taken as negative, i.e. $v = -0.0001$, then the model (5) becomes:

$$\begin{aligned} \dot{y_1} &= 150y_2 - 15y_1 - 5y_1^2 - 3y_1y_3, \\ \dot{y_2} &= y_1 - y_2, \\ \dot{y_3} &= y_3(-1.2 + y_1 - y_3 - E), \\ 0 &= (2.5y_3 - 30)E + 0.0001. \end{aligned}$$

In this case we find the eigen values of the system from the Jacobian matrix J_1 given above as $-1.3068 \times 10^7, -249.603, -0.396629$. As all the eigen values are negative, the system is stable around the equilibrium point $(19.8, 19.8, 12, 6.6)$.

Remark. In case-I, we see that the system (5) is stable at $(19.8, 19.8, 12, 6.6)$ as $v < 0$ and it is unstable as $v > 0$.Thus the system (5) has singularity induced bifurcation (SIB) at the interior equilibrium point $(19.8, 19.8, 12, 6.6)$, $v = 0$ is a bifurcation value and a stability switch occurs as v increases through 0.

(iv) Based on the analysis in section 4, a feedback controller $u(t) = K(E(t) - 6.6)$ can be applied to stabilize the system (5) at $(19.8, 19.8, 12, 6.6)$ and then the system (5) with the state feedback controller takes the form as follows:

$$\begin{aligned} \dot{y_1} &= 150y_2 - 15y_1 - 5y_1^2 - 3y_1y_3, \\ \dot{y_2} &= y_1 - y_2, \\ \dot{y_3} &= y_3(-1.2 + y_1 - y_3 - E), \end{aligned} \tag{12}$$

$$0 = (2.5y_3 - 30)E + K(E(t) - 6.6).$$

We find that if $K > max\{P, Q, R\}$, i.e. if $K > max\{0.755725, 10.3125, 12.9914\}$, i.e. if $K > 12.991$, then the singular induced bifurcation is eliminated and the system (5) is stable around $(19.8, 19.8, 12, 6.6)$. If we take $K = 15$, three eigen values of the system (12) are $-246.735, -1.03226+1.15839i, -1.03226-1.15839i$. Thus all the three eigen values have negative real parts and the system is stable around $(19.8, 19.8, 12, 6.6)$.

Case-II: (i) For positive economic profit, we take $v = 10$ and all other parameter values remain same, i.e. $a = 150, b = 15, c = 5, d = 3, e = 1.2, p = 2.5, q = 1, c_1 = 30$ and we find that the interior equilibrium point of the system (5) is (19.3571, 19.3571, 12.7381, 5.41896). In this case the eigen values of the system are $-245.131, 78.4889, -0.367894$. As one of the eigen values is positive, the system (5) is unstable around $(19.3571, 19.3571, 12.7381, 5.41896)$ for the positive economic profit.

(ii) When a feedback controller $u(t) = K(E(t) - 5.41896)$ is applied then the system (5) with $v = 10$ takes the form:

$$\begin{aligned} \dot{y_1} &= 150y_2 - 15y_1 - 5y_1^2 - 3y_1y_3, \\ \dot{y_2} &= y_1 - y_2, \\ \dot{y_3} &= y_3(-1.2 + y_1 - y_3 - E), \\ 0 &= (2.5y_3 - 30)E - 10 + K(E(t) - 5.41896). \end{aligned} \tag{13}$$

We find $K = 15$ and the eigen values are $-244.344, -5.15755, -0.778514$ and therefore the system becomes stable.

Remark. In case-I we see that when $v = 0$ then a singularity induced bifurcation occurs at the interior equilibrium point $(19.8, 19.8, 12, 6.6)$ and this singularity is eliminated by applying a feedback controller and also the system becomes stable.

In case-II we see that when $v > 0$ (here we take $v = 10$), then the system is unstable at the interior equilibrium point $(19.3571, 19.3571, 12.7381, 5.41896)$ and when we use a feedback controller then the system becomes stable.

7. Conclusion

This paper analyzes the dynamical behavior of a prey-predator system with stage structure for prey. It has been shown that the system has a singularity induced bifurcation around an interior equilibrium point for zero economic profit and this singularity brings impulse and as a result the ecosystem will collapse. Also the system becomes unstable around the interior equilibrium point in case of positive economic profit. After applying the feedback controller, the system can be stabilized around the interior equilibrium point and the impulse phenomenon is also eliminated. The elimination of singularity induced bifurcation implies that the ecological balance in the prey-predator ecosystem is restored. This provides us a bio-economic way of maintaining the sustainable development of the prey-predator ecosystem in the case of positive economic profit. Numerical simulations are given to show the consistency of the results with theoretical analysis.

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T. K. Kar is an Associate Professor at the Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah, India. His research interests include dynamical systems, stability and bifurcation theory, population dynamics, mathematical modelling in ecology and epidemiology, management and conservation of fisheries and bioeconomic modelling of renewable resources. He wrote around 70 academic papers on those topics. He also supervised several students of master and doctor degree.

e-mail: tkar1117@gmail.com

U. K. Pahari received his Ph.D. degree from Bengal Engineering and Science University, Shibpur, Howrah-711103, India. He is an Assistant professor of the Department of Mathematics, Sree Chaitanya College, Habra, 24 Pgs(N), India . His research interests include dynamical systems, stability and bifurcation theory, population dynamics, management and conservation of fisheries and bioeconomic modelling of renewable resources. He wrote around 10 academic papers on those topics.

e-mail: pahariu@yahoo.in