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THE SOLUTIONS OF BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS WITH NON-LIPSCHITZ COEFFICIENTS

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ABSTRACT. In this paper, we shall establish a new theorem on the existence and uniqueness of the solution to a backward doubly stochastic differential equations under a weaker condition than the Lipschitz coefficient. We also show a comparison theorem for this kind of equations.

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1. Introduction

Nonlinear backward stochastic differential equations (BSDEs in short) were published by Pardoux and Peng [1] in 1990, BSDEs have attracted great interest from both mathematical community and financial community (cf. [2], [3] and the references therein). One of the main reasons is that the theory of BSDEs has been an important and fundamental tool for mathematical economics and for mathematical finance in particular. Another main reason is due to their enormous range of applications in such diverse fields as partial differential equations, stochastic partial differential equations, stochastic control, stochastic differential games, nonlinear mathematical expectations and so on. Amongst these researches of BSDEs, much effort was devoted to loosen the uniform Lipschitz conditions on coefficients of BSDEs (e.g., [4-6]).

After they introduced the theory of BSDEs, Pardoux and Peng [7] in 1994 brought forward a new kind of BSDEs, that is a class of backward doubly stochastic differential equations (BDSDEs in short) with two different directions of

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stochastic integrals, i.e., the equations involve both a standard (forward) stochastic integral dW_t and a backward stochastic integral dB_t. They have proved the existence and uniqueness of solution to BDSDEs under uniformly Lipschitz conditions on coefficients. That is, for a given terminal time $T > 0$, under the uniformly Lipschitz assumptions on coefficients f and g, for any square integrable terminal value ξ, the following BDSDE has a unique solution pair (y_t, z_t) in the interval $[0, T]$:

$$y_t = \xi + \int_t^T f(s, y_s, z_s)ds + \int_t^T g(s, y_s, z_s)dB_s - \int_t^T z_s dW_s. \tag{1}$$

Pardoux and Peng [7] showed that BDSDEs can produce a probabilistic representation for certain quasilinear stochastic partial differential equations (SPDEs). In order to study more general SPDEs, recently Peng and Shi [8] introduce a class of forward-backward doubly stochastic differential equations, under Lipschitz condition and monotonicity assumption. However, it is somehow too strong to require the uniform Lipschitz continuity in applications, e.g. in dealing with quasilinear parabolic SPDEs. So it is important to find some weaker conditions than the Lipschitz one under which BDSDE has a unique solution.

In [9] Shi et al. weaken the uniform Lipschitz assumption to linear growth and continuous conditions by virtue of the comparison theorem that is introduced by themselves. They obtained the existence of solutions to BDSDE but without uniqueness. The aim of the present paper is to obtain the existence and uniqueness of solution to BDSDE without uniform Lipschitz assumptions. We also give a comparison theorem of this kind equations, generalize the results of [9].

This paper is organized as follows: in Section 2 we present the setting of problems and the main assumptions; in Section 3 we prove the existence and uniqueness theorem of BDSDE under non-Lipschitz condition; at the end we discuss a comparison theorem in Section 4.

2. Setting of Backward Doubly Stochastic Differential Equations

The Euclidean norm of a vector $x \in R^k$ will be denoted by $|x|$, and for a $d \times k$ matrix A, we define $\|A\| = \sqrt{TrAA^*}$, where A^* is the transpose of A.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and T be an arbitrarily fixed positive constant throughout this paper. Let $\{W_t; 0 \le t \le T\}$ and $\{B_t; 0 \le t \le T\}$ be two mutually independent standard Brownian Motions with values in R^d and R^l, respectively, defined on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t \doteq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B,$$

where for any process $\{\eta_t\}$, $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s; s \le r \le t\} \vee \mathcal{N}, \mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$.

We note that the collection $\{\mathcal{F}_t; t \in [0, T]\}$ is neither increasing nor decreasing, so it does not constitute a classical filtration.

For any $n \in N$, let $M^2(0,T;R^n)$ denote the set of (classes of $dP\otimes dt$ a.e. equal) n-dimensional jointly measurable stochastic processes $\{\varphi_t; t \in [0,T]\}$ which satisfy:

(i) $\|\varphi\|^2_{M^2} := E\int_0^T |\varphi_t|^2 dt < \infty$;

(ii) φ_t is $\mathcal{F}_t$-measurable, for any $t \in [0,T]$.

Similarly, we denote by $S^2([0,T];R^n)$ the set of n-dimensional continuous stochastic processes $\{\varphi_t; t \in [0,T]\}$ which satisfy:

(iii) $\|\varphi\|^2_{S^2} := E(\sup_{0\le t\le T} |\varphi_t|^2) < \infty$;

(iv) φ_t is $\mathcal{F}_t$-measurable, for any $t \in [0,T]$.

Obviously, $M^2(0,T;R^n)$ and $S^2([0,T];R^n)$ are Hilbert Space.

Given $\xi \in L^2(\Omega, \mathcal{F}_T, P; R^k)$, we consider the following BDSDE:

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s,Z_s)dB_s - \int_t^T Z_s dW_s, 0 \le t \le T. \quad (2)$$

We note that the integral with respect to $\{B_t\}$ is a "backward Itô integral" and the integral with respect to $\{W_t\}$ is a standard forward Itô integral, these two types of integrals are particular cases of the Itô-Skorohod integral (see [10]).

Definition 1. A pair of processes $(Y,Z) : \Omega \times [0,T] \to R^k \times R^{k\times d}$ is called a solution of BDSDE (2), if $(Y,Z) \in S^2([0,T];R^k) \times M^2(0,T;R^{k\times d})$ and satisfy BDSDE (2).

Let $f : \Omega \times [0,T] \times R^k \times R^{k\times d} \to R^k$, $\quad g : \Omega \times [0,T] \times R^k \times R^{k\times d} \to R^{k\times l}$ be jointly measurable and satisfy the following assumption:

(H1) $f(\cdot,0,0) \in M^2(0,T;R^k)$, $\qquad g(\cdot,0,0) \in M^2(0,T;R^{k\times l})$.

(H2) For all $(\omega,t) \in \Omega \times [0,T]$, $(y_1,z_1),(y_2,z_2) \in R^k \times R^{k\times d}$ and $t \in [0,T]$,

$$|f(t,y_1,z_1) - f(t,y_2,z_2)|^2 \le \rho(t,|y_1-y_2|^2) + c||z_1-z_2||^2,$$
$$\|g(t,y_1,z_1) - g(t,y_2,z_2)\|^2 \le \rho(t,|y_1-y_2|^2) + \alpha||z_1-z_2||^2,$$

where $c > 0$ and $0 < \alpha < 1$ are two constants, $\rho : [0,T] \times R^+ \to R^+$ is a continuous non-random function, for fixed t, with respect to u is a concave nondecreasing function, such that $\rho(t,0) = 0$, $\forall t \in [0,T]$ and the following ODE

$$\begin{cases} u' = -\rho(t,u) \\ u(T) = 0 \end{cases}$$

has a unique solution $u(t) \equiv 0, \quad t \in [0,T]$.

3. Existence and uniqueness theorem

Under assumptions (H1) and (H2), we can construct an approximate sequence using a Picard-type iteration . Let $y_t^0 \equiv 0$, and let $\{y_t^n, z_t^n\}_{n\ge1}$ be a sequence in

$S^2\left([0,T];R^k\right)\times M^2\left(0,T;R^{k\times d}\right)$ defined recursively by

$$\begin{aligned} y_t^n = \xi &+ \int_t^T f(s,y_s^{n-1},z_s^n)ds + \int_t^T g(s,y_s^{n-1},z_s^n)dB_s \\ &- \int_t^T z_s^n dW_s,\ 0\le t\le T. \end{aligned} \tag{3}$$

For each given $\left(y_s^{n-1}\right)\in S^2\left([0,T];R^k\right)$, According to the results of Pardoux and Peng (1994), there exists a unique pair $(y_s^n,z_s^n)\in S^2\left([0,T];R^k\right)\times M^2\left(0,T;R^{k\times d}\right)$ satisfying (3).

We first give some lemmas.

Lemma 1. If $\xi\in L^2(\Omega,\mathcal{F}_T,P;R^k)$, f and g satisfy $(H1)$ and $(H2)$, $\forall\ 0\le t\le T$, it holds that

$$E|y_t^1|^2\le e^{\frac{(2c+\alpha)T}{1-\alpha}}\left[E|\xi|^2+\frac{2(1-\alpha)}{2c+\alpha}E\int_t^T|f(s,0,0)|^2ds+\frac{2c+1}{1-\alpha}E\int_t^T||g(s,0,0)||^2ds\right].$$

Proof. Applying the extension of Itô formula (see [4]) to $|y_t^1|^2$, one can derive that

$$\begin{aligned} &E|y_t^1|^2+E\int_t^T||z_s^1||^2ds \\ &= E|\xi|^2+2E\int_t^T(y_s^1,f(s,0,z_s^1))ds+E\int_t^T||g(s,0,z_s^1)||^2ds \\ &\le E|\xi|^2+\frac{1}{\theta}E\int_t^T|y_s^1|^2ds+\theta E\int_t^T|f(s,0,z_s^1)|^2ds+E\int_t^T||g(s,0,z_s^1)||^2ds. \end{aligned}$$

From (H2) we have

$$\begin{aligned} &|f(s,0,z_s^1)|^2\le 2|f(s,0,0)|^2+2c||z_s^1||^2, \\ &||g(s,0,z_s^1)||^2\le\frac{1+\theta}{\theta}||g(s,0,0)||^2+(1+\theta)\alpha||z_s^1||^2. \end{aligned}$$

So

$$\begin{aligned} E|y_t^1|^2+E\int_t^T||z_s^1||^2ds &\le E|\xi|^2+\frac{1}{\theta}E\int_t^T|y_s^1|^2ds \\ &\quad+E\int_t^T\left[2\theta|f(s,0,0)|^2+\frac{1+\theta}{\theta}||g(s,0,0)||^2\right]ds \\ &\quad+(2\theta c+(1+\theta)\alpha)E\int_t^T||z_s^1||^2ds. \end{aligned}$$

We let $\theta=\frac{1-\alpha}{2c+\alpha}>0$ is a constant, then

$$E|y_t^1|^2\le E|\xi|^2+\frac{1}{\theta}E\int_t^T|y_s^1|^2ds+E\int_t^T[2\theta|f(s,0,0)|^2+\frac{1+\theta}{\theta}||g(s,0,0)||^2]ds$$

Now fix $r \in [0,T]$ arbitrarily. If $t \leq r \leq T$, then

$$E|y_r^1|^2 \leq E|\xi|^2 + \frac{1}{\theta}E\int_r^T |y_s^1|^2 ds + E\int_t^T [2\theta|f(s,0,0)|^2 + \frac{1+\theta}{\theta}||g(s,0,0)||^2]ds$$

From Gronwall inequality, we see that

$$E|y_r^1|^2 \leq e^{\frac{(2c+\alpha)(T-r)}{1-\alpha}}\left[E|\xi|^2 + \frac{2(1-\alpha)}{2c+\alpha}E\int_t^T |f(s,0,0)|^2 ds + \frac{2c+1}{1-\alpha}E\int_t^T ||g(s,0,0)||^2 ds\right].$$

Since r is arbitrary, so

$$E|y_t^1|^2 \leq e^{\frac{(2c+\alpha)T}{1-\alpha}}\left[E|\xi|^2 + \frac{2(1-\alpha)}{2c+\alpha}E\int_t^T |f(s,0,0)|^2 ds + \frac{2c+1}{1-\alpha}E\int_t^T ||g(s,0,0)||^2 ds\right].$$

□

Lemma 2. Under the assumptions of Lemma 1, for $\forall 0 \leq t \leq T, n, m \geq 1$, such that

$$E|y_t^{n+m} - y_t^n|^2 \leq e^{\frac{cT}{1-\alpha}}\left(\frac{1-\alpha}{c}+1\right)\int_t^T \rho(s, E|y_s^{n+m-1} - y_s^{n-1}|^2)ds.$$

Proof. Applying the extension of Itô formula to $|y_t^{n+m} - y_t^n|^2$ we have

$$\begin{aligned}
&E|y_t^{n+m} - y_t^n|^2 + E\int_t^T ||z_s^{n+m} - z_s^n||^2 ds \\
&= 2E\int_t^T (y_s^{n+m} - y_s^n, f(s, y_s^{n+m-1}, z_s^{n+m}) - f(s, y_s^{n-1}, z_s^n))ds \\
&\quad + E\int_t^T ||g(s, y_s^{n+m-1}, z_s^{n+m}) - g(s, y_s^{n-1}, z_s^n)||^2 ds \\
&\leq \frac{1}{\theta}E\int_t^T |y_s^{n+m} - y_s^n|^2 ds + \theta E\int_t^T |f(s, y_s^{n+m-1}, z_s^{n+m}) - f(s, y_s^{n-1}, z_s^n)|^2 ds \\
&\quad + E\int_t^T ||g(s, y_s^{n+m-1}, z_s^{n+m}) - g(s, y_s^{n-1}, z_s^n)||^2 ds \\
&\leq \frac{1}{\theta}E\int_t^T |y_s^{n+m} - y_s^n|^2 ds + (\theta+1)E\int_t^T \rho(s, |y_s^{n+m-1} - y_s^{n-1}|^2)ds \\
&\quad + (\theta c + \alpha)E\int_t^T ||z_s^{n+m} - z_s^n||^2 ds \\
&\leq \frac{1}{\theta}E\int_t^T |y_s^{n+m} - y_s^n|^2 ds + (\theta+1)\int_t^T \rho(s, E|y_s^{n+m-1} - y_s^{n-1}|^2)ds \\
&\quad + (\theta c + \alpha)E\int_t^T ||z_s^{n+m} - z_s^n||^2 ds.
\end{aligned}$$

The last inequality is due to Jensen's inequality, let $\theta = \frac{1-\alpha}{c} > 0$ is a constant, in the same way as the Lemma 1 we have

$$E|y_t^{n+m} - y_t^n|^2 \leq e^{\frac{cT}{1-\alpha}} \left(\frac{1-\alpha}{c} + 1\right) \int_t^T \rho(s, E|y_s^{n+m-1} - y_s^{n-1}|^2) ds.$$

□

Theorem 1. *Under the assumptions of Lemma 1, let* $M = \max_{(t,u)\in[0,T]\times[0,b]} \rho(t,u)$. *If*

$$e^{\frac{(2c+\alpha)T}{1-\alpha}} [E|\xi|^2 + \frac{2(1-\alpha)}{2c+\alpha} E \int_t^T |f(s,0,0)|^2 ds + \frac{2c+1}{1-\alpha} E \int_t^T ||g(s,0,0)||^2 ds] \leq b, \forall t \in [0,T].$$

Then there exists a unique solution $(y_t, z_t) \in S^2([T_1,T]; R^k) \times M^2(T_1,T; R^{k\times d})$ *satisfy BDSDE (2), where* $T - T_1 = min\left\{T, \frac{b}{M(\frac{1-\alpha}{c}+1)e^{\frac{cT}{1-\alpha}}}\right\}$.

Proof. **(Existence):** For $\forall t \in [T_1, T]$, we let

$$\phi_0(t) = (\frac{1-\alpha}{c} + 1) M e^{\frac{cT}{1-\alpha}} (T-t) \leq b,$$
$$\phi_{n+1}(t) = (\frac{1-\alpha}{c} + 1) e^{\frac{cT}{1-\alpha}} \int_t^T \rho(s, \phi_n(s)) ds.$$

Obviously

$$\phi_1(t) = (\frac{1-\alpha}{c} + 1) e^{\frac{cT}{1-\alpha}} \int_t^T \rho(s, \phi_0(s)) ds \leq (\frac{1-\alpha}{c} + 1) M e^{\frac{cT}{1-\alpha}} (T-t) = \phi_0(t) \leq b,$$
$$\phi_2(t) = (\frac{1-\alpha}{c} + 1) e^{\frac{cT}{1-\alpha}} \int_t^T \rho(s, \phi_1(s)) ds \leq (\frac{1-\alpha}{c} + 1) e^{\frac{cT}{1-\alpha}} \int_t^T \rho(s, \phi_0(s)) ds = \phi_1(t) \leq b$$

$\cdots$

By induction, for all $n = 0, 1, 2, \cdots, \phi_n(t)$ satisfies

$$0 \leq \phi_{n+1}(t) \leq \phi_n(t) \leq \cdots \leq \phi_1(t) \leq \phi_0(t) \leq b,$$

and

$$|\phi_{n+1}^{'}(t)| = (\frac{1-\alpha}{c} + 1) e^{\frac{cT}{1-\alpha}} |\rho(s, \phi_n(t))| \leq (\frac{1-\alpha}{c} + 1) e^{\frac{cT}{1-\alpha}} M.$$

So $\{\phi_n(t)\}_{n\geq 0}$ is continuous on $[T_1, T]$ and nonincreasing monotonically as $n \to \infty$. Therefore we can define the function $\phi(t)$ by the limit of $\phi_n(t)$. Then $\phi(t)$ is continuous on $[T_1, T]$ and $\phi(T) = 0$. From the assumptions of the theorem, $\phi(t) = 0,\ t \in [T_1, T]$. From Lemma 1 we get

$$E|y_t^1|^2 \leq e^{\frac{(2c+\alpha)T}{1-\alpha}} [E|\xi|^2 + \frac{2(1-\alpha)}{2c+\alpha} E \int_t^T |f(s,0,0)|^2 ds + \frac{2c+1}{1-\alpha} E \int_t^T ||g(s,0,0)||^2 ds] \leq b.$$

From Lemma 2 we have

$$E|y_t^{n+1} - y_t^n|^2 \leq e^{\frac{cT}{1-\alpha}} (\frac{1-\alpha}{c} + 1) \int_t^T \rho(s, E|y_s^n - y_s^{n-1}|^2) ds.$$

So

$$E|y_t^2 - y_t^1|^2 \le e^{\frac{cT}{1-\alpha}}(\frac{1-\alpha}{c}+1)\int_t^T \rho(s,b)ds \le (\frac{1-\alpha}{c}+1)Me^{\frac{cT}{1-\alpha}}(T-t) = \phi_0(t)$$

$$E|y_t^3 - y_t^2|^2 \le e^{\frac{cT}{1-\alpha}}(\frac{1-\alpha}{c}+1)\int_t^T \rho(s,E|y_s^2 - y_s^1|^2)ds$$

$$\le e^{\frac{cT}{1-\alpha}}(\frac{1-\alpha}{c}+1)\int_t^T \rho(s,\phi_0(s))ds = \phi_1(t)$$

$\dots$

So for all $n \in N$, when $t \in [T_1, T]$, we obtain

$$E|y_t^{n+1} - y_t^n|^2 \le \phi_{n-1}(t) \le b.$$

Let $n, m \in N, m > n$, for all $t \in [T_1, T]$,

$$E|y_t^m - y_t^n|^2 \le 3[E|y_t^{n+1} - y_t^n|^2 + E|y_t^{m+1} - y_t^m|^2 + E|y_t^{m+1} - y_t^{n+1}|^2]$$

$$\le 3\phi_{n-1}(t) + 3\phi_{m-1}(t) + 3e^{\frac{cT}{1-\alpha}}(\frac{1-\alpha}{c}+1)\int_t^T \rho(s,E|y_s^m - y_s^n|^2)ds$$

$$\le 6\phi_{n-1}(t) + 3e^{\frac{cT}{1-\alpha}}(\frac{1-\alpha}{c}+1)\int_t^T \rho(s,E|y_s^m - y_s^n|^2)ds.$$

Since $\phi_n(t) \to 0$, when $n \to \infty$. So $\exists N_0$, such that $\phi_{n-1}(t) \le \frac{\varepsilon}{6}$ whenever $n \ge N_0$.
Therefore when $m > n \ge N_0$,

$$E|y_t^m - y_t^n|^2 \le \varepsilon + 3e^{\frac{cT}{1-\alpha}}(\frac{1-\alpha}{c}+1)\int_t^T \rho(s,E|y_s^m - y_s^n|^2)ds.$$

From the comparison of ODE we have

$$E|y_t^m - y_t^n|^2 \le r(t,\varepsilon), m > n \ge N_0, \quad t \in [T_1, T].$$

where $r(t,\varepsilon)$ is the maximum solution of the left column of the following equation:

$$\begin{cases} u' = -3e^{\frac{cT}{1-\alpha}}(\frac{1-\alpha}{c}+1)\rho(t,u) \\ u(T) = \varepsilon \end{cases}$$

When $\varepsilon \to 0$, $r(t,\varepsilon)$ uniformly convergence the maximum solution of the left column of the following problems,

$$\begin{cases} u' = -3e^{\frac{cT}{1-\alpha}}(\frac{1-\alpha}{c}+1)\rho(t,u) \\ u(T) = 0 \end{cases}$$

From (H2), we have $u(t) \equiv 0$. So

$$E|y_t^m - y_t^n|^2 \to 0, n, m \to \infty.$$

We immediately see that $\{y_t^n, n = 1, 2\cdots\}$ is a Cauchy sequence in $S^2\left([T_1,T];R^k\right)$ and $\{z_t^n, n = 1, 2\cdots\}$ is also a Cauchy sequence in $M^2\left(T_1, T; R^{k\times d}\right)$. Define their limits by $y(\cdot)$ and $z(\cdot)$ respectively, then $(y(\cdot), z(\cdot)) \in S^2\left([T_1,T];R^k\right) \times M^2\left(T_1, T; R^{k\times d}\right)$ and satisfy BDSDE (2). The existence has been proved.

(Uniqueness): Let both $(Y_\cdot^i, Z_\cdot^i)$ be the two solutions of (2), $(i = 1, 2)$. Applying the extension of the Itô formula to $\left|Y_t^1 - Y_t^2\right|^2$, Jensen's inequality and the assumption (H2) it follows that

$$\begin{aligned} & E|Y_t^1 - Y_t^2|^2 + E\int_t^T ||Z_s^1 - Z_s^2||^2 ds \\ \le & \frac{1}{\theta}E\int_t^T |Y_s^1 - Y_s^2|^2 ds + (\theta+1)\int_t^T \rho(s, E|Y_s^1 - Y_s^2|^2)ds \\ & + (\theta c + \alpha)E\int_t^T ||Z_s^1 - Z_s^2||^2 ds. \end{aligned}$$

Let $\theta = \frac{1-\alpha}{2c}$ we have

$$\begin{aligned} & E|Y_t^1 - Y_t^2|^2 + \frac{1-\alpha}{2}E\int_t^T ||Z_s^1 - Z_s^2||^2 ds \\ \le & \frac{2c}{1-\alpha}E\int_t^T |Y_s^1 - Y_s^2|^2 ds + (\frac{1-\alpha}{2c}+1)\int_t^T \rho(s, E|Y_s^1 - Y_s^2|^2)ds. \end{aligned} \tag{4}$$

From the Gronwall inequality

$$E|Y_t^1 - Y_t^2|^2 \le e^{\frac{2c}{1-\alpha}}(\frac{1-\alpha}{2c}+1)\int_t^T \rho(s, E|Y_s^1 - Y_s^2|^2)ds.$$

From the comparison of the ODE we have

$$E|Y_t^1 - Y_t^2|^2 \le r(t), \quad \forall t \in [T_1, T].$$

Where $r(t)$ is the maximum solution of the left column of the following equation:

$$\begin{cases} u' = -e^{\frac{2c}{1-\alpha}}(\frac{1-\alpha}{2c}+1)\rho(t,u) \\ u(T) = 0 \end{cases}$$

From the assumption $r(t) = 0,\ t \in [T_1, T]$. So $E|Y_t^1 - Y_t^2|^2 = 0,\ t \in [T_1, T]$, this means $Y_t^1 = Y_t^2,\ a.s.$. It then follows from (4) that $Z_t^1 = Z_t^2,\ a.s., \forall T_1 \le t \le T$. The uniqueness has been proved. □

If function $\rho(t, u)$ also satisfy the following assumption:

(H3): $\rho(t, u) \le a(t) + b(t)u,\ t \ge 0$, where $a(t) \ge 0, \quad b(t) \ge 0$ and such that

$$\int_0^T a(t)dt < +\infty, \int_0^T b(t)dt < +\infty.$$

Then we also can assert the following existence and uniqueness theorem.

Theorem 2. *If $\xi \in L^2(\Omega, \mathcal{F}_T, P; R^k)$. f and g satisfy $(H1), (H2)$ and $(H3)$. Then BDSDE (2)has a unique solution $(y_t, z_t) \in S^2([0,T]; R^k) \times M^2(0,T; R^{k\times d})$.*

Proof. From Lemma 1,

$$E|y_t^1|^2 \le e^{\frac{(2c+\alpha)T}{1-\alpha}}[E|\xi|^2 + \frac{2(1-\alpha)}{2c+\alpha}E\int_t^T |f(s,0,0)|^2 ds + \frac{2c+1}{1-\alpha}E\int_t^T ||g(s,0,0)||^2 ds].$$

From (H3) we have

$$E|y_t^1|^2 \le e^{\frac{(2c+\alpha)T}{1-\alpha}}[E|\xi|^2 + \frac{2(1-\alpha)}{2c+\alpha}E\int_t^T |f(s,0,0)|^2 ds + \frac{2c+1}{1-\alpha}E\int_t^T ||g(s,0,0)||^2 ds$$
$$+ (\frac{1-\alpha}{c}+1)\int_t^T a(s)ds] \le \bar{b} < +\infty. \quad (5)$$

We choose $T_1 \in [0,T]$, such that

$$(\frac{1-\alpha}{c}+1)e^{\frac{cT}{1-\alpha}}\int_t^T \rho(s,\bar{b})ds \le \bar{b}, \quad t \in [T_1,T]. \quad (6)$$

We can show T_1 is exist and its value does not depend on the final value ξ. Since (H3), we note that (6) holds if

$$(\frac{1-\alpha}{c}+1)e^{\frac{cT}{1-\alpha}}\int_t^T a(s)ds + (\frac{1-\alpha}{c}+1)e^{\frac{cT}{1-\alpha}}\bar{b}\int_t^T b(s)ds \le \bar{b}$$

but by (5),

$$(\frac{1-\alpha}{c}+1)e^{\frac{cT}{1-\alpha}}\int_t^T a(s)ds = (\frac{1-\alpha}{c}+1)e^{\frac{cT}{1-\alpha}}e^{-\frac{(2c+\alpha)T}{1-\alpha}}e^{\frac{(2c+\alpha)T}{1-\alpha}}\int_t^T a(s)ds \le \bar{b}e^{-\frac{(c+\alpha)T}{1-\alpha}},$$

this holds if

$$\bar{b}e^{-\frac{(c+\alpha)T}{1-\alpha}} + (\frac{1-\alpha}{c}+1)e^{\frac{cT}{1-\alpha}}\bar{b}\int_t^T b(s)ds \le \bar{b}$$

and so if

$$\int_t^T b(s)ds \le (1-e^{-\frac{(c+\alpha)T}{1-\alpha}})e^{\frac{-cT}{1-\alpha}}\frac{c}{1-\alpha+c}.$$

So we choose T_1, such that

$$\int_{T_1}^T b(s)ds = (1-e^{-\frac{(c+\alpha)T}{1-\alpha}})e^{\frac{-cT}{1-\alpha}}\frac{c}{1-\alpha+c}.$$

In fact, if

$$\int_0^T b(s)ds < (1-e^{-\frac{(c+\alpha)T}{1-\alpha}})e^{\frac{-cT}{1-\alpha}}\frac{c}{1-\alpha+c},$$

we choose $T_1 = 0$; otherwise we choose T_1, such that

$$\int_{T_1}^T b(s)ds = (1-e^{-\frac{(c+\alpha)T}{1-\alpha}})e^{\frac{-cT}{1-\alpha}}\frac{c}{1-\alpha+c},$$

so the value of T_1 does not depends on the final value ξ.
For all $t \in [T_1,T]$, let

$$\psi_1(t) = (\frac{1-\alpha}{c}+1)e^{\frac{cT}{1-\alpha}}\int_t^T \rho(s,\bar{b})ds$$
$$\psi_{n+1}(t) = (\frac{1-\alpha}{c}+1)e^{\frac{cT}{1-\alpha}}\int_t^T \rho(s,\psi_n(s))ds.$$

Obviously,

$$\begin{aligned}\psi_1(t) &\le \bar{b},\\ \psi_2(t) &= (\frac{1-\alpha}{c}+1)e^{\frac{cT}{1-\alpha}}\int_t^T \rho(s,\psi_1(s))ds\\ &\le (\frac{1-\alpha}{c}+1)e^{\frac{cT}{1-\alpha}}\int_t^T \rho(s,\bar{b})ds=\psi_1(t)\le \bar{b},\\ \psi_3(t) &= (\frac{1-\alpha}{c}+1)e^{\frac{cT}{1-\alpha}}\int_t^T \rho(s,\psi_2(s))ds\\ &\le (\frac{1-\alpha}{c}+1)e^{\frac{cT}{1-\alpha}}\int_t^T \rho(s,\psi_1(s))ds=\psi_2(t)\le \bar{b}\cdots\end{aligned}$$

So for all $n=1,2,\cdots,\ \psi_n(t)$ is continuous and

$$0\le \psi_{n+1}(t)\le \psi_n(t)\le\cdots\le\psi_1(t)\le\bar{b}.$$

In the same way of the Theorem 1, we have $\{\psi_n(t),t\in[T_1,T]\}_{n\ge1}$ convergence $\psi(t)=0$.
From Lemma 2,

$$E|y_t^{n+1}-y_t^n|^2\le e^{\frac{cT}{1-\alpha}}(\frac{1-\alpha}{c}+1)\int_t^T\rho(s,E|y_s^n-y_s^{n-1}|^2)ds.$$

So when $t\in[T_1,T]$, we have

$$\begin{aligned}E|y_t^2-y_t^1|^2 &\le e^{\frac{cT}{1-\alpha}}(\frac{1-\alpha}{c}+1)\int_t^T\rho(s,E|y_s^1|^2)ds\\ &\le (\frac{1-\alpha}{c}+1)e^{\frac{cT}{1-\alpha}}\int_t^T\rho(s,\bar{b})ds=\psi_1(t),\\ E|y_t^3-y_t^2|^2 &\le e^{\frac{cT}{1-\alpha}}(\frac{1-\alpha}{c}+1)\int_t^T\rho(s,E|y_s^2-y_s^1|^2)ds\\ &\le e^{\frac{cT}{1-\alpha}}(\frac{1-\alpha}{c}+1)\int_t^T\rho(s,\psi_1(s))ds=\psi_2(t)\cdots\end{aligned}$$

So by induction, for all $n\in N$

$$E|y_t^{n+1}-y_t^n|^2\le\psi_n(t),\quad t\in[T_1,T].$$

In the same way of Theorem 1, there exist a unique $(y_t,z_t)\in S^2([T_1,T];R^k)\times M^2(T_1,T;R^{k\times d})$ satisfies BDSDE (2). In other words, we have showed the existence of the solution on $[T_1,T]$. Since the value of T_1 does not depend on the final value ξ. If $T_1=0$, then we have showed the existence of the solution on $[0,T]$; otherwise we set the final value be y_{T_1}, in the same way, we get $(y_t,z_t)\in S^2\left([T_2,T_1];R^k\right)\times M^2(T_2,T_1;R^{k\times d})$ such that

$$y_t=y_{T_1}+\int_t^{T_1}f(s,y_s,z_s)ds+\int_t^{T_1}g(s,y_s,z_s)dB_s-\int_t^{T_1}z_sdW_s,$$

where if $\int_0^{T_1} b(s)ds \leq (1-e^{-\frac{(c+\alpha)T}{1-\alpha}})e^{\frac{-cT}{1-\alpha}}\frac{c}{1-\alpha+c}$, we let $T_2=0$, otherwise we let $\int_{T_2}^{T_1} b(s)ds = (1-e^{-\frac{(c+\alpha)T}{1-\alpha}})e^{\frac{-cT}{1-\alpha}}\frac{c}{1-\alpha+c}\ldots$, hence one can deduce by iteration the existence of the solution on $[T_n,T]$, i.e., $(y_t,z_t)\in S^2([T_n,T];R^k)\times M^2(T_n,T;R^{k\times d})$ such that BDSDE (2) holds on $[T_n,T]$, and T,T_n $(n=1,2,\ldots)$ satisfy

$$\int_{T_1}^{T} b(s)ds = \int_{T_2}^{T_1} b(s)ds = \cdots = \int_{T_n}^{T_{n-1}} b(s)ds = (1-e^{-\frac{(c+\alpha)T}{1-\alpha}})e^{\frac{-cT}{1-\alpha}}\frac{c}{1-\alpha+c},$$

$T\geq T_1\geq T_2\geq\cdots\geq T_n$. So there is an enough large n such that $\int_{T_n}^{T} b(s)ds = n(1-e^{-\frac{(c+\alpha)T}{1-\alpha}})e^{\frac{-cT}{1-\alpha}}\frac{c}{1-\alpha+c} > T$. Therefore we have $(y_t,z_t)\in S^2([0,T];R^k)\times M^2(0,T;R^{k\times d})$ satisfies (2). □

Remark: When $\rho(t,u)$ is independent of t, i.e.

(A1): For all $(y_1,z_1),(y_2,z_2)\in R^k\times R^{k\times d}$ and $t\in[0,T]$,

$$|f(t,y_1,z_1)-f(t,y_2,z_2)|^2 \leq \rho(|y_1-y_2|^2)+c||z_1-z_2||^2;$$
$$\|g(t,y_1,z_1)-g(t,y_2,z_2)\|^2 \leq \rho(|y_1-y_2|^2)+\alpha||z_1-z_2||^2.$$

where $c>0$ and $0<\alpha<1$ are two constants and $\rho: R^+\to R^+$ is a concave nondecreasing function, such that $\rho(0)=0,\ \rho(u)>0,\ \forall u>0$, and $\int_{0+}\frac{du}{\rho(u)}=\infty$,

If (A1) holds, then (H2) also holds. In fact, we consider the following ODE

$$\begin{cases} u' = -\rho(u) \\ u(T)=0 \end{cases}$$

Then

$$u(t)=\int_t^T \rho(u(s))ds, \quad t\in[0,T].$$

Since $\int_{0+}\frac{du}{\rho(u)}=\infty$, we can prove $u(t)=0,\ t\in[0,T]$, i.e. the ODE has a unique solution $u(t)\equiv 0,\quad t\in[0,T]$. That is, if the assumption (A1) holds, then (H2) also holds, while (H3) obviously holds. Thus, the conditions of Theorem 2 all hold, from the Theorem 2 BDSDE (2) has a unique solution.

Inference If $\xi\in L^2(\Omega,\mathcal{F}_T,P;R^k)$. f and g satisfy $(H1)$ and $(A1)$. Then BDSDE (2) has a unique solution $(y_t,z_t)\in S^2([0,T];R^k)\times M^2(0,T;R^{k\times d})$.

4. Comparison theorem

In this section, we only consider one-dimensional BDSDEs, i.e., $k=1$. We consider the following BDSDEs:

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s)ds+\int_t^T g(s,Y_s,Z_s)dB_s-\int_t^T Z_s dW_s, \quad 0\leq t\leq T, \tag{7}$$

$$\bar{Y}_t=\bar{\xi}+\int_t^T \bar{f}(s,\bar{Y}_s,\bar{Z}_s)ds+\int_t^T g(s,\bar{Y}_s,\bar{Z}_s)dB_s-\int_t^T \bar{Z}_s dW_s, \quad 0\leq t\leq T, \tag{8}$$

where $\xi,\ \bar{\xi} \in L^2(\Omega, \mathcal{F}_T, P; R)$, $f,\ \bar{f},\ g$ satisfy the conditions of theorem 2. Then there exist two pairs of processes (Y_t, Z_t) and $(\bar{Y}_t, \bar{Z}_t)$ satisfying BDSDEs (7) and (8), respectively.

We can assert the following comparison theorem.

Theorem 3. *Assume BDSDE* (7) *and* (8) *satisfy the conditions of Theorem 2, let* (Y_t, Z_t) *and* $(\bar{Y}_t, \bar{Z}_t)$ *be solutions of* (7) *and* (8), *respectively. If* $(i)\xi \leq \bar{\xi}, (ii) f(t, \bar{Y}_t, \bar{Z}_t) \leq \bar{f}(t, \bar{Y}_t, \bar{Z}_t)$, *then*

$$Y_t \leq \bar{Y}_t,\ a.s.,\ \ \forall t \in [0, T].$$

Proof. Let

$$\hat{Y}_t = Y_t - \bar{Y}_t, \quad \hat{Z}_t = Z_t - \bar{Z}_t, \quad \hat{\xi} = \xi - \bar{\xi},$$

$$\hat{f}_t = f(t, Y_t, Z_t) - f(t, \bar{Y}_t, \bar{Z}_t), \hat{g}_t = g(t, Y_t, Z_t) - g(t, \bar{Y}_t, \bar{Z}_t).$$

Apply the extension of the Itô formula to $\left|\hat{Y}_t^+\right|^2$ we have

$$\begin{aligned}
&E|\hat{Y}_t^+|^2 + E\int_t^T \mathbf{I}_{(Y_s > \bar{Y}_s)} ||\hat{Z}_s||^2 ds \\
&= 2E\int_t^T (\hat{Y}_s^+, f(s, Y_s, Z_s) - \bar{f}(s, \bar{Y}_s, \bar{Z}_s))ds + E\int_t^T \mathbf{I}_{(Y_s > \bar{Y}_s)} ||\hat{g}_s||^2 ds \\
&\leq 2E\int_t^T (\hat{Y}_s^+, f(s, Y_s, Z_s) - f(s, \bar{Y}_s, \bar{Z}_s))ds + E\int_t^T \mathbf{I}_{(Y_s > \bar{Y}_s)} ||\hat{g}_s||^2 ds \\
&\leq \frac{1}{\theta} E\int_t^T |\hat{Y_s}^+|^2 ds + \theta E\int_t^T \mathbf{I}_{(Y_s > \bar{Y}_s)} (\rho(s, |\hat{Y}_s|^2) + c||\hat{Z}_s||^2) ds \\
&\quad + E\int_t^T \mathbf{I}_{(Y_s > \bar{Y}_s)} (\rho(s, |\hat{Y}_s|^2) + \alpha||\hat{Z}_s||^2) ds \\
&\leq \frac{1}{\theta} E\int_t^T |\hat{Y_s}^+|^2 ds + (\theta + 1) E\int_t^T \mathbf{I}_{(Y_s > \bar{Y}_s)} \rho(s, |\hat{Y}_s|^2) ds \\
&\quad + (\theta c + \alpha) E\int_t^T \mathbf{I}_{(Y_s > \bar{Y}_s)} ||\hat{Z}_s||^2 ds \\
&\leq \frac{1}{\theta} E\int_t^T |\hat{Y_s}^+|^2 ds + (\theta + 1)\int_t^T \rho(s, E|\hat{Y}_s^+|^2) ds + (\theta c + \alpha) E\int_t^T \mathbf{I}_{(Y_s > \bar{Y}_s)} ||\hat{Z}_s||^2 ds
\end{aligned}$$

The last inequality is due to Jensen's inequality. Where $0 < \theta < \frac{1-\alpha}{c}$ is a constant. It implies that

$$E|\hat{Y}_t^+|^2 \leq \frac{1}{\theta} E\int_t^T |\hat{Y_s}^+|^2 ds + (\theta + 1)\int_t^T \rho(s, E|\hat{Y}_s^+|^2) ds.$$

By the same argument in the proof of uniqueness of Theorem 1 we have

$$E|(Y_t - \bar{Y}_t)^+|^2 = 0, \quad \forall t \in [0, T].$$

which implies that $Y_t \leq \bar{Y}_t, \quad a.s, \quad \forall 0 \leq t \leq T.$ □

Remark: If $\rho(t, u) = cu$, then the Theorem 3 is the Theorem 3.1 of [9], i.e. we generalize the results of [9].

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