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LARGE TIME-STEPPING METHOD BASED ON THE FINITE ELEMENT DISCRETIZATION FOR THE CAHN-HILLIARD EQUATION†

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Abstract. In this paper, a class of large time-stepping method based on the finite element discretization for the Cahn-Hilliard equation with the Neumann boundary conditions is developed. The equation is discretized by finite element method in space and semi-implicit schemes in time. For the first order fully discrete scheme, convergence property is investigated by using finite element analysis. Numerical experiment is presented, which demonstrates the effectiveness of the large time-stepping approaches.

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1. Introduction

In this work, we use a class of large time-stepping methods to solve numerically the Cahn-Hilliard equation with Neumann boundary conditions. We consider the fourth-order nonlinear evolutionary equation: for real u,

$$\begin{cases} u_t + \Delta\left(\nu\Delta u - \Phi(u)\right) = 0, \quad x \in \Omega, \quad 0 < t \leq T, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial\left(\nu\Delta u - \Phi(u)\right)}{\partial n} = 0, \quad x \in \partial\Omega, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where $u_t = \dfrac{\partial u}{\partial t}$, n is the outward normal, $\Phi(u)$ is usually of the form $\Phi(u) = u^3 - u$ and Ω is a bounded domain.

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The Cahn-Hilliard equation was originally introduced by Cahn and Hilliard [2] to describe the complicated phase separation and coarsening phenomena. It has been widely accepted as a good model to describe the phase separation and coarsening phenomena in a melted alloy. For the background, derivation and discussion of the Cahn-Hilliard equation, we refer to [1, 3, 6, 12, 11, 16, 17] and the references therein. There has been a large body of work dealing with the numerical approximation of the Cahn-Hilliard equation. Zhang [23] obtained optimal order error estimates in L^∞ and $H^{1,\infty}$ norms and the superconvergence property in derivative approximation. In [7, 8] Elliot et al. investigated the nonconforming finite element method for multidimensional problem and obtained in [9] optimal order error bounds by applying a second order splitting method. A multigrid finite element solver has been developed by Kay and Welford in [15]. In [18] Xia et al. studied the local discontinuous Galerkin methods. Ye et al. developed the Fourier collocation method and the Fourier spectral method for Cahn-Hilliard equation with periodic boundary conditions, respectively (see [20], [22]); Ye also presented the Legendre collocation method (see [21]). In [19], Xu and Tang analyzed the large time-stepping methods for epitaxial growth models. Feng and Prohl in [10] proposed and analyzed a semi-discrete and a fully-discrete finite element method for a class of Cahn-Hilliard equation involving a small parameter. They proved that the stability constant increases to infinity algebraically instead of exponentially (if one uses the Gronwall inequality) as the small parameter goes to zero.

Our main task in this work is to investigate the time-stepping methods for the problem (1). The classical first order semi-implicit scheme reads:

$$\frac{u^n - u^{n-1}}{\Delta t} + \nu\Delta^2 u^n = \Delta\Phi(u^{n-1}),\ n \geq 1, \tag{2}$$

where Δt is the time-step and $t_n = n\Delta t$, u^n is an approximation to $u(x, t_n)$. In practice, it is known that the semi-implicit scheme in time allows a consistently larger time-step size. To improve the stability of the semi-implicit method (2), the term $\mathcal{O}(\Delta t u_t)$ is added to the scheme (2):

$$\frac{u^n - u^{n-1}}{\Delta t} + \nu\Delta^2 u^n = A\Delta(u^n - u^{n-1}) + \Delta\Phi(u^{n-1}),\ n \geq 1, \tag{3}$$

where A is a positive constant. A second order backward difference (BDF) for u_t and a second order Adams-Bashforth (AB) for the explicit treatment of the nonlinear term for (1) is the following second-order BDF/AB scheme:

$$\frac{3u^n - 4u^{n-1} + u^{n-2}}{2\Delta t} + \nu\Delta^2 u^n = \Delta(2\Phi(u^{n-1}) - \Phi(u^{n-2})),\ n \geq 2. \tag{4}$$

Similarly, to improve stability, an $\mathcal{O}(\Delta t^2 u_{tt})$ term is added in the scheme (4), from which we get the following second-order time discretization scheme:

$$\frac{3u^n - 4u^{n-1} + u^{n-2}}{2\Delta t} + \nu\Delta^2 u^n = A\Delta\delta_{tt}u^{n-1} + \Delta(2\Phi(u^{n-1}) - \Phi(u^{n-2})),\ n \geq 2, \tag{5}$$

where $\delta_{tt}u^{n-1} = u^n - 2u^{n-1} + u^{n-2}$.

In [13], these methods together with the Fourier discretization in space were applied to solve the Cahn-Hilliard equation with periodic boundary condition. Our main contribution is to use these methods combined with finite element discretization to solve Cahn-Hilliard equation with Neumann boundary conditions and hopefully to get the similar results.

The remainder of the paper is organized as follows. In section 2, we give semi-discrete form for the Cahn-Hilliard equation; stability and convergence property of the first-order method is investigated in section 3; numerical results are presented in section 4.

2. Semi-discrete scheme with finite element method

Let $L^2(\Omega)$ denote the set of all square integrable functions with the inner product $(u,v)=\int_\Omega u(x)v(x)dx$ and the norm $\|u\|^2=(u,u)$. Let $L^\infty(\Omega)$ denote the Lebesgue space with the norm $\|u\|_\infty = \text{ess}\sup_{x\in\Omega}|u(x)|$ and $H^m(\Omega)$ denote the usual Sobolev space with the norm $\|u\|_m = (\sum_{|\alpha|\le m}\|D^\alpha u\|^2)^{\frac{1}{2}}$. Denote

$$L^2(0,T;H^m(\Omega)) = \left\{ u(x,t)\in H^m(\Omega); \int_0^T \|u\|_m^2 < \infty \right\},$$

namely $u(x,t)\in H^m(\Omega)$ for all $0\le t\le T$.

Let τ_h be a subdivision of domain Ω, N is the dimension of the approximation space, h_i is the mesh size, $h=\max_{1\le i\le N} h_i$. Let S_h^3 denote the finite element space consisting of piecewise third-order polynomials, and

$$S_h^3 \subset H_E^2(\Omega) = \{u\in H^2(\Omega): \frac{\partial u}{\partial n}=0 \text{ on } \partial\Omega\}. \tag{6}$$

τ_h is assumed to be regular in the usual sense [4, 5]. Then the following inverse inequality holds

$$\|\nabla v_h\| \le ch^{-1}\|v_h\|,\ \forall v_h\in S_h^3,$$

where $c>0$ is independent of h.

Denote $P_N: L^2(\Omega)\to H_E^2(\Omega)$ the $L^2(\Omega)-$projector onto $H_E^2(\Omega)$, which is defined by

$$(P_N u - u, v) = 0, \forall\ v\in H_E^2(\Omega).$$

The variational form of (1) is to seek $u(t,x):(0,T]\times\Omega\to H_E^2(\Omega), u(0,x)=u_0(x)$ such that

$$(u_t,v)+\nu(\Delta u,\Delta v) = (\Phi(u),\Delta v),\ \forall v\in H_E^2(\Omega). \tag{7}$$

Similarly, we define the semi-discrete approximation of problem (1): find $u_h(t)$: $(0,T]\to S_h^3$ such that

$$\begin{cases} (\partial_t u_h, v_h) + \nu(\Delta u_h, \Delta v_h) = (\Phi(u_h), \Delta v_h), \ \forall v_h \in S_h^3, \\ u_h(0) \in S_h^3, \end{cases} \tag{8}$$

for all $t > 0$ with $u_h(0) = P_N u_0$. Further, we define the energy functional of the solution u

$$E(u) = \frac{\nu}{2}\|\nabla u\|^2 + \frac{1}{4}\|u^2 - 1\|^2. \tag{9}$$

The following lemma of Gronwall type will be used in the later section (see, for instance, [14] for a proof).

Lemma 2.1 (Discrete Gronwall Lemma)**.** *Let C_0, Δt be nonnegative numbers and a_k, b_k, c_k, d_k be nonnegative sequences satisfying*

$$a_n + \Delta t \sum_{k=0}^{n} b_k \le \Delta t \sum_{k=0}^{n-1} d_k a_k + \Delta t \sum_{k=0}^{n-1} c_k + C_0, \ \forall \ n \ge 1$$

then

$$a_n + \Delta t \sum_{k=0}^{n} b_k \le \exp\left(\Delta t \sum_{k=0}^{n-1} d_k\right)\left(\Delta t \sum_{k=0}^{n-1} c_k + C_0\right), \ \forall \ n \ge 1.$$

3. Stability and error analysis for semi-implicit scheme

Discretize the time interval $[0,T] : 0 = t_0 < t_1 < \cdots < t_M = T, t_n - t_{n-1} = \Delta t = T/M, M > 0$ is an integer. Introduce the Euler backward formula

$$u_t(t_n) = \frac{u(t_n) - u(t_{n-1})}{\Delta t} + \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} (\tau - t_{n-1}) u_{tt}(\tau) d\tau = \delta_t u^n + \varepsilon^n, \tag{10}$$

here, if $u(t)$ is continuous, then $u^n = u(t_n)$.

Next we define the fully discrete scheme to approximate problem (7) by the first-order scheme (3). Find $U^n \in \mathbf{S}_h^3 (n = 1, 2, \cdots, M)$ satisfying

$$(\delta_t U^n, v_h) + \nu(\Delta U^n, \Delta v_h) = (\Phi(U^{n-1}), \Delta v_h) + A((U^n - U^{n-1}), \Delta v_h), \tag{11}$$

when U^{n-1} is known and Δt is sufficiently small, we can get U^n by solving a definite linear equation system which is equivalent to equation (11). Similarly, the second-order fully discrete formula by (5) is

$$\begin{aligned} &\left(\frac{3U^n - 4U^{n-1} + U^{n-2}}{2\Delta t}, v_h\right) + \nu(\Delta U^n, \Delta v_h) \\ &= (A\delta_{tt} U^{n-1}, \Delta v_h) + (2\Phi(U^{n-1}) - \Phi(U^{n-2}), \Delta v_h). \end{aligned} \tag{12}$$

3.1. Stability analysis for the first order semi-implicit scheme.

Theorem 1. *If A in (11) satisfies*

$$A \geq \max_{x\in\Omega}\left\{\frac{1}{2}|U^{n-1}(x)|^2 + \frac{1}{4}|U^n(x)+U^{n-1}(x)|^2\right\} - \frac{1}{2}, \forall n \geq 1, \tag{13}$$

then for all $m \geq 0$ there holds

$$E(U^m) + \Delta t\sum_{n=1}^{m}\|\nabla(A(U^n - U^{n-1}) + (|U^{n-1}|^2-1)U^{n-1} - \nu\Delta U^n)\|^2 \leq E(U^0) \tag{14}$$

where $E(u)$ is defined by (9).

Proof. Taking $v = (A(U^n - U^{n-1}) + (|U^{n-1}|^2 - 1)U^{n-1} - \nu\Delta U^n)\Delta t$ in (11) and using the equalities

$$2a(a-b) = a^2 - b^2 + (a-b)^2,\ 2ab = a^2 + b^2 - (a-b)^2,$$

we obtain

$$\begin{aligned}0 = &\|\nabla(A(U^n - U^{n-1}) + (|U^{n-1}|^2-1)U^{n-1} - \nu\Delta U^n)\|^2\Delta t\\ &+ \frac{\nu}{2}(\|\nabla U^n\|^2 - \|\nabla U^{n-1}\|^2 + \|\nabla(U^n - U^{n-1})\|^2) + A\|U^n - U^{n-1}\|^2 + I_n,\end{aligned} \tag{15}$$

where

$$\begin{aligned}I_n &= \big((|U^{n-1}|^2-1)U^{n-1}, U^n - U^{n-1}\big)\\ &= \frac{1}{2}\big((|U^{n-1}|^2-1), |U^n|^2 - |U^{n-1}| - |U^n - U^{n-1}|^2\big)\\ &= \frac{1}{2}\big(1-|U^{n-1}|^2, |U^n - U^{n-1}|^2\big) - \frac{1}{2}\|U^n\|^2 + \frac{1}{2}\|U^{n-1}\|^2\\ &\quad - \frac{1}{4}\||U^n|^2 - |U^{n-1}|^2\|^2 + \frac{1}{4}\|U^n\|_{L^4}^4 - \frac{1}{4}\|U^{n-1}\|_{L^4}^4\\ &= \frac{1}{2}(1 - |U^{n-1}|^2 - \frac{1}{2}|U^n + U^{n-1}|^2, |U^n - U^{n-1}|^2)\\ &\quad + \frac{1}{4}\||U^n|^2-1\|^2 - \frac{1}{4}\||U^{n-1}|^2 - 1\|^2.\end{aligned} \tag{16}$$

Combining (15) and (16) yields

$$\begin{aligned}0 \geq &\|\nabla(A(U^n - U^{n-1}) + (|U^{n-1}|^2-1)U^{n-1} - \nu\Delta U^n)\|^2\Delta t + E(U^n)\\ &- E(U^{n-1}) + \left(A + \frac{1}{2} - \frac{1}{2}|U^{n-1}|^2 - \frac{1}{4}|U^n + U^{n-1}|^2, |U^n - U^{n-1}|^2\right),\end{aligned} \tag{17}$$

which gives the desired result (14) provided the assumption (13) is satisfied. □

Note that the condition (13) for A is not a satisfactory one from the pointview that the right-hand side of (13) is also dependent of A. Nevertheless, it will not bring us critical difficulty when choosing proper A for computation. In fact, we can roughly take as

$$A \geq \frac{3}{2}|U^{n-1}|^2 - \frac{1}{2},\ \text{a.e. in } \Omega.$$

On the other hand, from numerical experiments we find out that for some larger values of A, scheme (11) leads to divergent solutions.

3.2. Error analysis for fully discrete finite element approximation. In this section, we present the fully discrete scheme and derive error estimates for the first-order scheme. In order to carry out error analysis, we first introduce a projector for a steady state problem. Set $a(u,v) = \nu(\Delta u, \Delta v) + (u,v)$ and assume $u \in H_E^2(\Omega)$ and define biharmonic projector R_h of u satisfying

$$a(u - R_h u, v_h) = 0, \ \forall v_h \in S_h^3, \tag{18}$$

following

$$\|\nabla u\|^2 = |(\nabla u, \nabla u)| = |(u, \Delta u)| \le \|u\|\|\Delta u\| \le \frac{1}{2}\|u\|^2 + \frac{1}{2}\|\Delta u\|^2, \ \forall u \in H_E^2(\Omega),$$

we get

$$\nu^\star \|u\|_2^2 \le a(u,u), \ \forall u \in H_E^2(\Omega), \nu^\star = \frac{1}{2}\min(\nu, 1).$$

Therefore, $a(u,v)$ is a symmetric positive-define bilinear form on S_h^3. Furthermore, we can see that the solution to problem (18) is existent and unique. Using the standard finite element analysis for biharmonic equation, we get

$$\|u - R_h u\| + h\|u - R_h u\|_1 + h^2\|u - R_h u\|_2 \le Ch^{r+1}\|u\|_{r+1}, \ 2 \le r \le 3. \tag{19}$$

Denote

$$\eta^n = u^n - R_h u^n, \quad \theta^n = R_h u^n - U^n,$$

then

$$u^n - U^n = \eta^n + \theta^n, \ \theta^n \in S_h^3.$$

Combining equations (7) and (18), we get

$$(u_t, v_h) - (u - R_h u, v_h) + \nu(\Delta R_h u, \Delta v_h) = (\Phi(u), \Delta v_h). \tag{20}$$

Substracting (11) from (20) at $t = t_n$, we obtain

$$\begin{aligned} &(u_t(t_n) - \delta_t U^n, v_h) - (\eta^n, v_h) + \nu(\Delta\theta^n, \Delta v_h) \\ &= (\Phi(u^n) - \Phi(U^{n-1}) - A(U^n - U^{n-1}), \Delta v_h). \end{aligned} \tag{21}$$

Combining the following equality with (21)

$$\begin{aligned} u_t(t_n) - \delta_t U^n &= u_t(t_n) - \delta_t u^n + \delta_t u^n - \delta_t R_h u^n + \delta_t R_h u^n - \delta_t U^n \\ &= \varepsilon^n + \delta_t \eta^n + \delta_t \theta^n, \end{aligned}$$

we have

$$\begin{aligned} &(\delta_t \theta^n, v_h) + \nu(\Delta\theta^n, \Delta v_h) \\ &= (\eta^n - \varepsilon^n - \delta_t \eta^n, v_h) + (\Phi(u^n) - \Phi(U^{n-1}) - A(U^n - U^{n-1}), \Delta v_h). \end{aligned} \tag{22}$$

Theorem 2. *Denote $u(t)$ and U^n solutions to problems (7) and (11) respectively, if $u(0) \in H^4(\Omega), u_t \in L^2(0,T;H^4(\Omega)), u_{tt} \in L^2(0,T;L^2(\Omega))$, U^0 satisfying $\|u(0)-U^0\| \le Ch^4\|u(0)\|_4$, mesh ratio satisfying $\Delta t/h^2 \le c$, then if h is small enough, there exists $C = C(u)$ independent of $h, \Delta t, n$, satisfying*

$$\|u(t_n) - U^n\| \le C(\Delta t + h^4), \ n = 0, 1, \cdots, M. \tag{23}$$

Proof. First, we make a posteriori assumption whose correctness will be verified later: if $0 < h \le h_0$, there exists h_0 satisfying

$$\|u(t_m) - U^m\|_{0,\infty} \le 1, \ m = 1, 2, \cdots, M. \tag{24}$$

Setting $v_h = \theta^n$ in (22), using the Cauchy inequality and the Young inequality, we get

$$\begin{aligned}
(\delta_t\theta^n, \theta^n) + \nu\|\Delta\theta^n\|^2 \le\, & \|\eta^n - \varepsilon^n - \delta_t\eta^n\|\|\theta^n\| + \|\theta^n + \eta^n\|\|\Delta\theta^n\| \\
& + \left(\|(u^n)^3 - (U^{n-1})^3\| + (1+A)\|U^n - U^{n-1}\|\right)\|\Delta\theta^n\| \\
\le\, & \frac{1}{2}\|\eta^n - \varepsilon^n - \delta_t\eta^n\|^2 + \frac{1}{2}\|\theta^n\|^2 + \frac{1}{\nu}(\|\eta^n\|^2 + \|\theta^n\|^2) \\
& + \frac{1}{\nu}(\|(u^n)^3 - (U^{n-1})^3\|^2 + (1+A)^2\|U^n - U^{n-1}\|^2) + \nu\|\Delta\theta^n\|^2.
\end{aligned}$$

Applying the definition of $\delta_t\theta^n$ and the Cauchy inequality, we obtain

$$\begin{aligned}
\|\theta^n\|^2 \le \|\theta^{n-1}\|^2 + 2\Delta t\{ & \frac{1}{2}\|\eta^n - \varepsilon^n - \delta_t\eta^n\|^2 + \frac{1}{2}\|\theta^n\|^2 + \frac{1}{\nu}(\|\eta^n\|^2 + \|\theta^n\|^2) \\
& + \frac{1}{\nu}(\|(u^n)^3 - (U^{n-1})^3\|^2 + (1+A)^2\|U^n - U^{n-1}\|^2)\}.
\end{aligned} \tag{25}$$

Combining (19), (10) and (24), we have the following estimates:

$$\|\eta^n\| = \|u^n - R_h u^n\| \le Ch^4\|u(t_n)\|_4,$$

$$\|\delta_t\eta^n\| = \|\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} \eta_t(\tau)d\tau\| \le \frac{Ch^4}{\Delta t}\int_{t_{n-1}}^{t_n} \|u_t(\tau)\|_4 d\tau \le \frac{Ch^4}{(\Delta t)^{\frac{1}{2}}}\left(\int_{t_{n-1}}^{t_n} \|u_t(\tau)\|_4^2 d\tau\right)^{\frac{1}{2}},$$

$$\|\varepsilon^n\| \le \int_{t_{n-1}}^{t_n} \|u_{tt}(\tau)\|d\tau \le (\Delta t)^{\frac{1}{2}}\left(\int_{t_{n-1}}^{t_n} \|u_{tt}(\tau)\|^2 d\tau\right)^{\frac{1}{2}},$$

$$\begin{aligned}
\|U^n - U^{n-1}\| =& \|U^n - u^n + u^n - u^{n-1} + u^{n-1} - U^{n-1}\| \\
=& \|\theta^n + \eta^n + u^n - u^{n-1} + \theta^{n-1} + \eta^{n-1}\| \\
\le& C\left\{(\Delta t)^{\frac{1}{2}}(\int_{t_{n-1}}^{t_n} \|u_t(\tau)\|^2 d\tau)^{\frac{1}{2}} + h^4\|u(t_n)\|_4 + \|\theta^n\| + \|\theta^{n-1}\|\right\},
\end{aligned}$$

$$\begin{aligned}
\left\|(u^n)^3 - (U^{n-1})^3\right\| =& \|\left((u^n)^2 - (u^{n-1})^2\right)u^n + \left((u^{n-1})^2 - (U^{n-1})^2\right)u^n \\
& + (U^{n-1})^2(u^n - U^{n-1})\| \\
\le& |u^n + u^{n-1}|_\infty |u^n|_\infty \|u^n - u^{n-1}\| \\
& + |u^{n-1} + U^{n-1}|_\infty |u^n|_\infty \|u^{n-1} - U^{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&+|U^{n-1}|_\infty^2(\int_{t_{n-1}}^{t_n}\|u_t(\tau)\|d\tau+\|\eta^{n-1}\|+\|\theta^{n-1}\|)\\
&\leq C\left\{(\Delta t)^{\frac{1}{2}}\left(\int_{t_{n-1}}^{t_n}\|u_t(\tau)\|^2d\tau\right)^{\frac{1}{2}}+h^4\|u(t_n)\|_4+\|\theta^{n-1}\|\right\},
\end{aligned}
$$

here we have used the following inequalities:

$$
|u^n|_\infty\leq C\|u(t_n)\|_4\leq C(\|u(0)\|_4+\int_0^{t_n}\|u_t(\tau)\|_4d\tau),
$$

$$
|U^{n-1}|_\infty\leq|U^{n-1}-u^{n-1}|_\infty+|u^{n-1}|_\infty\leq 1+|u^{n-1}|_\infty.
$$

Applying the above inequalities into (25), we get

$$
\begin{aligned}
\|\theta^n\|^2-\|\theta^{n-1}\|^2\leq& C\Delta t(\|\theta^n\|^2+\|\theta^{n-1}\|^2+h^8\|u(t_n)\|_4^2)\\
&+C(h^8+(\Delta t)^2)\left(\int_{t_{n-1}}^{t_n}(\|u_t(\tau)\|_4^2+\|u_t(\tau)\|^2+\|u_{tt}(\tau)\|^2)\right).
\end{aligned}
$$

Summing for $n=1$ to M and noting that $\|\theta^0\|\leq Ch^4\|u(0)\|_4, n\Delta t=t_n\leq T$, we obtain

$$
\|\theta^n\|^2\leq C\Delta t\sum_{i=1}^{n}\|\theta^i\|^2+C\left((\Delta t)^2+h^8\right)\left(\|u(0)\|_4^2+\int_0^{t_n}(\|u_t(\tau)\|_4^2+\|u_{tt}(\tau)\|^2)d\tau\right).
$$

When Δt is small enough that $C\Delta t\leq 1/2$, we have

$$
\|\theta^n\|^2\leq C\Delta t\sum_{i=1}^{n-1}\|\theta^i\|^2+C\left((\Delta t)^2+h^8\right).
$$

From the discrete Gronwall Lemma 2.1, we get

$$
\|\theta^n\|\leq C(\Delta t+h^4).
$$

Therefore, combining (19) and the triangular inequality, we come to

$$
\|u^n-U^n\|\leq\|\theta^n\|+\|\eta^n\|\leq C(\Delta t+h^4).
$$

To complete the proof, next we verify the correctness of the assumption (24) by the method of induction. When $m=0$, with initial approximation condition and the inverse inequality, we obtain that when $h\leq h_0,\ h_0$ is sufficiently small, then (24) holds. Assume (24) holds for $m=n-1$, from the proof above, we have (23), where C is independent of $n,\Delta t,\ h$ (note that $t_n\leq T$). In the case of $m=n$, we introduce u_I, the interpolation approximation of u. Applying the triangular inequality, the inverse inequality, interpolation approximation properties, the mesh ratio condition and (24), we come to the conclusion that when $h\leq h_0, h_0$ is sufficiently small, then

$$
\begin{aligned}
\|u^n-U^n\|_{0,\infty}\leq&\|u^n-u_I^n\|_\infty+\|u_I^n-U^n\|_\infty\\
\leq&\|u^n-u_I^n\|_1+Ch^{-1}\|u_I^n-U^n\|\\
\leq&C(h^3+h)\leq Ch_0\leq 1,
\end{aligned}
$$

namely, (24) holds when $m = n$. According to the method of induction, the correctness of assumption (24) is verified, and the proof of the theorem is thus completed. □

4. Numerical experiments

We present an example for the Cahn-Hilliard equation using our schemes (11) and (12).

We investigate the Cahn-Hilliard equation (1) in $[0, 2\pi]$ with the initial condition $u_0(x) = 0.5\cos(x)$, $\nu = 0.03$ and $T = 5$. Since no exact solution to (1) is known, we take numerical results of the second-order scheme (12) with $\Delta t = 0.0001$ and $N = 128$ as the "exact" solution, which will be used in the computation of the L^2- error.

Figure 1 and Figure 2 give numerical solutions of schemes (11) and (12) for $\nu = 0.03$ with different values of A and Δt respectively. Figure 1 with $(A, \Delta t)$=(0,0.01), (0.5,0.1), (1,0.1) and Figure 2 with $(A, \Delta t)$=(0,0.001), (0.5,0.01), (1,0.01). We can see that there is a good agreement between the numerical results obtained by using the standard semi-implicit time-stepping method (A=0) with small Δt and the modified methods (11) and (12) with larger Δt.

In Figure 3, the patterns hardly change after $t = 3$. Figure 4 presents the time dependency of energy functional (9) of numerical solutions of scheme (11). We can see that the energy functional of numerical solutions decreases as time passes. In [6], the similar results were given.

In Table 1, we list the values of Δt_c with different ν and different choices of A, where Δt_c denotes the largest possible time which allows stable numerical computation. In other words, the numerical solution will blow up if the time-step is larger than Δt_c. It is observed that the time-steps can be increased a few times by adding a non-zero A term in both first and second order semi-implicit schemes.

Table 2 presents the L^2-errors obtained by both schemes (11) and (12). It is obvious that the numerical errors are almost the same for computations with and without using the A terms, and larger time-steps can be applied by adding an A term. Furthermore, we can see that the L^2-error for scheme (11) is of first-order and the L^2-error for scheme (12) is of second-order which are correspondent with our theoretical results.

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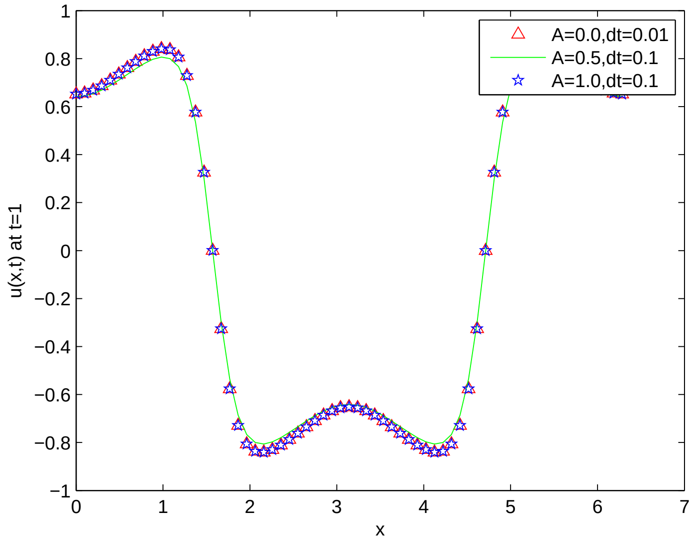

FIGURE 1. Numerical results obtained by using the first-order method with $\nu = 0.03, N = 64$

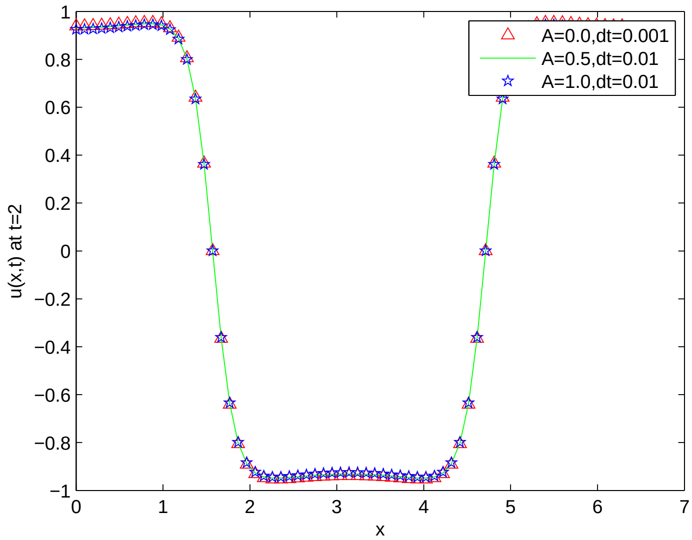

FIGURE 2. Numerical results obtained by using the second-order method with $\nu = 0.03, N = 64$

TABLE 1. $\Delta t_{\max}$ with different ν and A

ν	A	Δt_c for scheme (11)	Δt_c for scheme (12)
ν=0.3	A=0	$\Delta t_c \approx 0.1$	$\Delta t_c \approx 0.01$
	A=0.5	$\Delta t_c \approx 1$	$\Delta t_c \approx 0.4$
	A=1	$\Delta t_c \approx 1$	$\Delta t_c \approx 0.8$
ν=0.03	A=0	$\Delta t_c \approx 0.01$	$\Delta t_c \approx 0.002$
	A=0.5	$\Delta t_c \approx 0.1$	$\Delta t_c \approx 0.025$
	A=1	$\Delta t_c \approx 0.1$	$\Delta t_c \approx 0.04$
ν=0.003	A=0	$\Delta t_c \approx 0.002$	$\Delta t_c \approx 0.0001$
	A=0.5	$\Delta t_c \approx 0.015$	$\Delta t_c \approx 0.003$
	A=1	$\Delta t_c \approx 0.02$	$\Delta t_c \approx 0.006$

TABLE 2. Numerical errors of schemes (11) and (12) with $\nu = 0.03, N = 32$.

	Δt	L^2-error of scheme(11)	L^2-error of scheme (12)
A=0	0.01	4.33334562e-3	unstable
	0.005	2.34241035e-3	5.02824385e-5
	0.0025	1.23188030e-3	1.30128099e-5
A=0.5	0.01	4.33172538e-3	1.91138811e-4
	0.005	2.33153557e-3	5.03125295e-5
	0.0025	1.23142862e-3	1.30129302e-5
A=1	0.01	4.32942604e-3	1.91128550e-4
	0.005	2.33153557e-3	5.03130222e-5
	0.0025	1.23093536e-3	1.30130501e-5

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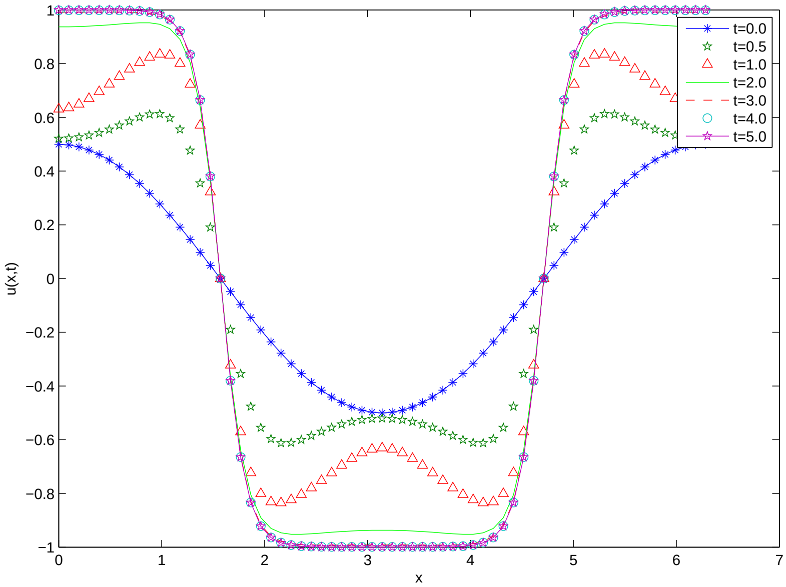

FIGURE 3. The evolution from $t = 0$ to 5, $\Delta t = 0.01, N = 64$

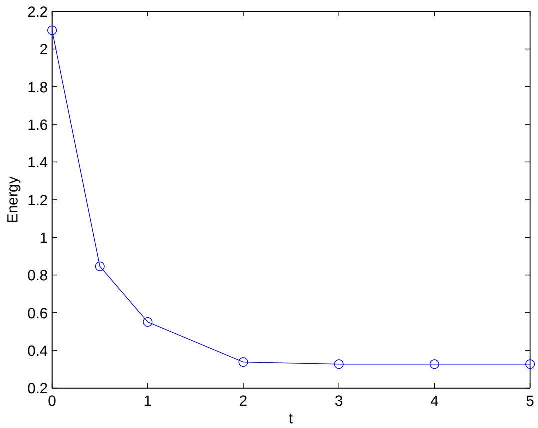

FIGURE 4. The energy from $t = 0$ to 5, $\Delta t = 0.01, N = 64$

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