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A MIXED-TYPE SPLITTING ITERATIVE METHOD†

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Abstract. In this paper, a preconditioned mixed-type splitting iterative method for solving the linear systems $Ax = b$ is presented, where A is a Z-matrix. Then we also obtain some results to show that the rate of convergence of our method is faster than that of the preconditioned AOR (PAOR) iterative method and preconditioned SOR (PSOR) iterative method. Finally, we give one numerical example to illustrate our results.

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1. Introduction

For solving linear system

$$Ax = b, \tag{1}$$

where A is an $n \times n$ square matrix, and x and b are n-dimensional vectors, the basic iterative method is

$$Mx^{k+1} = Nx^k + b,\ k = 0,\ 1,\ \ldots. \tag{2}$$

where $A = M - N$ and M is nonsingular. Thus (2) can be written as

$$x^{k+1} = Tx^k + c,\ k = 0,\ 1,\ \ldots,$$

where $T = M^{-1}N$, $c = M^{-1}b$.

Let $A = D - L - U$, where D is a diagonal matrix, $-L$ and $-U$ are strictly lower and strictly upper triangular parts of A, respectively.

Transform the original system (1) into the preconditioned form $PAx = Pb$.

Then, we can define the basic iterative scheme:

$$M_p x^{k+1} = N_p x^k + Pb, k = 0, 1, \ldots,$$

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where $PA = M_p - N_p$ and M_p is nonsingular. Thus the equation above can also be written as

$$x^{k+1} = Tx^k + c, k = 0, 1, \ldots,$$

where $T = M_p^{-1} N_p, c = M_p^{-1} Pb$.

In paper [1], Guang-Hui Cheng et al. presented the mixed-type splitting iterative method

$$(D + D_1 + L_1 - L)x^{k+1} = (D_1 + L_1 + U)x^k + b, \quad k = 0, 1, 2 \cdots$$

whose iteration matrix is

$$T = (D + D_1 + L_1 - L)^{-1}(D_1 + L_1 + U), \tag{3}$$

where D_1 is an auxiliary nonnegative diagonal matrix, L_1 is an auxiliary strictly lower triangular matrix and $0 \leq L_1 \leq L$.

In this paper, we establish the preconditioned mixed-type splitting iterative method with the preconditioner P_α for solving linear systems, where

$$P_\alpha = I + S_\alpha = \begin{pmatrix} 1 & -\alpha a_{12} & & & \\ & 1 & -\alpha a_{23} & & \\ & & \ddots & \ddots & \\ & & & \ddots & \\ & & & \ddots & -\alpha a_{n-1,n} \\ & & & & 1 \end{pmatrix} \tag{4}$$

and $0 \leq \alpha \leq 1$, a_{12},a_{23},$\cdots$, $a_{n-1,n}$ are the entries of matrix A. And we obtain some comparison results which show that the rate of convergence of the preconditioned mixed-type splitting iterative method is faster than that of the mixed-type splitting iterative method.

2. Preconditioned mixed-type splitting iterative method

For the linear system (1) we consider its preconditioned form

$$P_\alpha Ax = P_\alpha b$$

with the preconditioner $P_\alpha = I + S_\alpha$, i.e.,

$$A_\alpha x = b_\alpha.$$

We apply the mixed-type splitting iterative method to it and have the corresponding preconditioned mixed-type splitting iterative method

$$(D_\alpha + D_1 + L_1 - L_\alpha)x^{k+1} = (D_1 + L_1 + U_\alpha)x^k + b_\alpha, \quad k = 0, 1, 2 \cdots$$

with the iteration matrix

$$\tilde{T} = (D_\alpha + D_1 + L_1 - L_\alpha)^{-1}(D_1 + L_1 + U_\alpha), \tag{5}$$

where D_α,$-L_\alpha$,$-U_\alpha$ are the diagonal, strictly lower and strictly upper triangular matrices obtained from A_α and D_1 is an auxiliary nonnegative diagonal matrix, L_1 is an auxiliary strictly lower triangular matrix and $0 \leq L_1 \leq L_\alpha$.

If we choose certain auxiliary matrices, we can get the classical iterative methods:

(1) The PSOR method

$$D_1 = \frac{1}{r}(1-r)D, L_1 = 0,$$

$$\tilde{L}_r = (D_\alpha - rL_\alpha)^{-1}[(1-r)D_\alpha + rU_\alpha]. \quad (6)$$

(2) The PAOR method

$$D_1 = \frac{1}{w}(1-w)D, L_1 = \frac{1}{w}(w-r)L,$$

$$\tilde{L}_{r,w} = (D_\alpha - rL_\alpha)^{-1}[(1-w)D_\alpha + (w-r)L_\alpha + wU_\alpha]. \quad (7)$$

We need the following definitions and results.

Definition 2.1 (Young [3])**.** A matrix A is a Z-matrix if $a_{ij} \leq 0$, for all $i, j = 1, 2, \ldots n$, such that $i \neq j$.

Definition 2.2 (Young [3])**.** A matrix A is a M-matrix if A is a nonsingular Z-matrix, and $A^{-1} \geq 0$.

Definition 2.3 ([7])**.** Let $M, N \in R^{n,n}$. Then $A = M - N$ is called a regular splitting if $M^{-1} \geq 0$ and $N \geq 0$.

Lemma 2.1. *Let $A = M - N$ be a regular splitting of A. Then the splitting is convergent if and only if $A^{-1} \geq 0$.*

Lemma 2.2 (Young [3])**.** *Let $A \geq 0$ be an irreducible nonnegative matrix. Then*

(1) A has a positive real eigenvalue equals to its spectral radius;

(2) For $\rho(A)$, there corresponds an eigenvector x >0;

(3) $\rho(A)$ is a simple eigenvalue of A.

Lemma 2.3 (Varga [4])**.** *Let A be a nonnegative matrix. Then*

(1) If $\alpha x \leq Ax$ for some nonnegative vector x, $x \neq 0$, then $\alpha \leq \rho(A)$;

(2) If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then

$$\alpha \leq \rho(A) \leq \beta$$

and x is a positive vector.

Lemma 2.4 ([5])**.** *Let $A = M - N$ be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is a nonsingular M-matrix.*

Lemma 2.5 ([6])**.** *Let A be a Z-matrix. Then A is a nonsingular M-matrix if and only if there is a positive vector x such that $Ax \geq 0$.*

3. Convergence analysis and comparison results

Now we give main results as follows.

Theorem 3.1. *Let $A = D - L - U$ be an M-matrix, $D_1 \geq 0$ and $0 \leq L_1 \leq L_\alpha$, where $-L$, $-U$ are the strictly lower and strictly upper triangular parts of A, respectively. Then the preconditioned mixed-type splitting iterative method is convergent.*

Proof. Let us first denote

$$D_\alpha = D - S_1, L_\alpha = L + S_2, U_\alpha = U - S_\alpha + S_\alpha U$$

and

$$M = D_\alpha + D_1 + L_1 - L_\alpha, N = D_1 + L_1 + U_\alpha.$$

Since A is an M-matrix and $0 \leq L_1 \leq L_\alpha$, we have

$$M^{-1} = (D_\alpha + D_1 + L_1 - L_\alpha)^{-1} = [(D_\alpha + D_1) - (L_\alpha - L_1)]^{-1} \geq 0,$$

$$A^{-1} \geq 0, N = D_1 + L_1 + U_\alpha \geq 0.$$

According to the Lemma 2.2, Lemma 2.2 and Definition 2.3, we know that the preconditioned mixed-type splitting method for M-matrix is convergent. □

Corollary 3.2. *If the coefficient matrix A is an M-matrix and $0 < r < 1$, then the PSOR iterative method is convergent.*

Corollary 3.3. *If the coefficient matrix A is an M-matrix and $0 < r < w < 1$, then the PAOR iterative method is convergent.*

Theorem 3.4. *Let $A = D - L - U$ be an Z-matrix, where $-L$, $-U$ are the strictly lower and strictly upper triangular parts of A, respectively. Assume that $0 \leq D_1 \leq (\frac{1}{w} - 1)D_\alpha$, $0 \leq L_1 \leq (1 - \frac{r}{w})L_\alpha$, $1 - a_{ii+1}a_{i+1i} > 0$, $i = 1, 2 \cdots n-1$, $0 \leq r < w \leq 1$, and $\tilde{T}$, $\tilde{L}_{r,w}$ are the iteration matrix given by (5) and (7), respectively. If $\tilde{T}$ and $\tilde{L}_{r,w}$ are irreducible, then*

$$\rho(\tilde{T}) > \rho(\tilde{L}_{r,w}) \quad if\ \rho(\tilde{L}_{r,w}) > 1;$$
$$\rho(\tilde{T}) = \rho(\tilde{L}_{r,w}) \quad if\ \rho(\tilde{L}_{r,w}) = 1;$$
$$\rho(\tilde{T}) < \rho(\tilde{L}_{r,w}) \quad if\ \rho(\tilde{L}_{r,w}) < 1.$$

Proof. Firstly, from the splitting of A and the definition of $\tilde{T}$ and $\tilde{L}_{r,w}$, we can easily obtain that $\tilde{T}$ and $\tilde{L}_{r,w}$ are nonnegative. Thus, from Lemma 2.2, we know that there exists a positive vector $x = (x_1, x_2 \cdots x_n)^T$ such that $\tilde{L}_{r,w}x = \lambda x$ where we denote $\lambda = \rho(\tilde{L}_{r,w})$. And equivalently,

$$[(1 - w)D_\alpha + (w - r)L_\alpha + wU_\alpha]x = \lambda(D_\alpha - rL_\alpha)x.$$

Now we consider

$$\tilde{T}x - \tilde{L}_{r,w}x = (D_\alpha + D_1 + L_1 - L_\alpha)^{-1}(D_1 + L_1 + U_\alpha)x - \lambda x$$
$$= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1}[(D_1 + L_1 + U_\alpha) - \lambda(D_\alpha + D_1 + L_1 - L_\alpha)]x$$

$$= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1}[(D_1(1-\lambda) + L_1(1-\lambda) + U_\alpha + \lambda L_\alpha - \lambda D_\alpha)]x$$
$$= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1}\Big\{(D_1(1-\lambda) + L_1(1-\lambda) +$$
$$\frac{1}{w}[\lambda(1-w)D_\alpha + \lambda(w-r)L_\alpha - (1-w)D_\alpha - (w-r)L_\alpha]\Big\}$$
$$= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1}\Big\{(D_1(1-\lambda) + L_1(1-\lambda)$$
$$+ \frac{1}{w}[(1-w)(\lambda-1)D_\alpha + (w-r)(\lambda-1)L_\alpha]\Big\}$$
$$= (1-\lambda)(D_\alpha + D_1 + L_1 - L_\alpha)^{-1}\left[D_1 + L_1 + \left(\frac{w-1}{w}\right)D_\alpha + \left(\frac{r-w}{w}\right)L_\alpha\right]$$
$$= (1-\lambda)(D_\alpha + D_1 + L_1 - L_\alpha)^{-1}\left\{\left[D_1 - \left(\frac{1}{w}-1\right)D_\alpha\right] + \left[L_1 - \left(1-\frac{r}{w}\right)L_\alpha\right]\right\}.$$

Since $[D_1 - (\frac{1}{w}-1)D_\alpha] + [L_1 - (1-\frac{r}{w})L_\alpha] \le 0$, we have

$$(D_\alpha + D_1 + L_1 - L_\alpha)^{-1}\{[D_1 - (\frac{1}{w}-1)D_\alpha] + [L_1 - (1-\frac{r}{w})L_\alpha]\} \le 0.$$

(1) If $0 < \lambda < 1$, then $\tilde{T}x \le \lambda x$. By Lemma 2.3, we get $\rho(\tilde{T}) < \rho(\tilde{L}_{r,w})$;
(2) If $\lambda = 1$, then $\tilde{T}x = \lambda x$. By Lemma 2.3, we get $\rho(\tilde{T}) = \rho(\tilde{L}_{r,w})$;
(3) If $\lambda > 1$, then $\tilde{T}x \ge \lambda x$. By Lemma 2.3, we get $\rho(\tilde{T}) > \rho(\tilde{L}_{r,w})$. □

Theorem 3.5. *Let $A = D - L - U$ be an Z-matrix, where $-L$, $-U$ are the strictly lower and strictly upper triangular parts of A, respectively. Assume that $0 \le D_1 \le (\frac{1}{w}-1)D_\alpha$, $L_1 = 0$, $1 - a_{ii+1}a_{i+1i} > 0$, $i = 1, 2\cdots n-1$, $0 \le r < 1$, and $\tilde{T}$, $\tilde{L}_r$ are the iteration matrix given by (5) and (6), respectively. If $\tilde{T}$and$\tilde{L}_r$ are irreducible, then*

$$\rho(\tilde{T}) > \rho(\tilde{L}_r) \quad if\ \rho(\tilde{L}_r) > 1;$$
$$\rho(\tilde{T}) = \rho(\tilde{L}_r) \quad if\ \rho(\tilde{L}_r) = 1;$$
$$\rho(\tilde{T}) < \rho(\tilde{L}_r) \quad if\ \rho(\tilde{L}_r) < 1.$$

Proof. The proof is similar to the proof of the Theorem 3.4, if we let $r = w$, we can easily obtain Theorem 3.5, so we omit it. □

Next, we will illustrate the rate of convergence of the preconditioned mixed-type splitting iterative method is faster than that of the mixed-type splitting iterative method.

Theorem 3.6. *Let $A = D - L - U$ be an Z-matrix, where $-L$, $-U$ are the strictly lower and strictly upper triangular parts of A, respectively. Assume that $D_1 \ge 0$, $0 \le L_1 \le L_\alpha$, $1 - a_{ii+1}a_{i+1i} > 0$, $i = 1, 2\cdots n-1$, and $\tilde{T}$, T are the iteration matrices given by (5) and (6), respectively. If $\tilde{T}$ and T are irreducible, then*

$$\rho(\tilde{T}) > \rho(T) \quad if\ \rho(T) > 1;$$
$$\rho(\tilde{T}) = \rho(T) \quad if\ \rho(T) = 1;$$
$$\rho(\tilde{T}) < \rho(T) \quad if\ \rho(T) < 1.$$

Proof. First, from the splitting of A, and the definition of $\tilde{T}$ and T, we can easy obtain that $\tilde{T}$ and T are nonnegative. Thus, from Lemma 2.2, we know that there exists a positive vector $x = (x_1, x_2 \cdots x_n)^T$ such that $Tx = \lambda x$ where we denote $\lambda = \rho(T)$, which is equivalent to

$$(D_1 + L_1 + U)x = \lambda(I + D_1 + L_1 - L)x,$$
$$(U - \lambda D + \lambda L)x = [(\lambda - 1)D_1 + (\lambda - 1)L_1]x.$$

Now we consider

$$\begin{aligned}
\tilde{T}x - Tx &= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1}(D_1 + L_1 + U_\alpha)x - \lambda x \\
&= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1}[(D_1 + L_1 + U_\alpha) - \lambda(D_\alpha + D_1 + L_1 - L_\alpha)]x \\
&= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1}[(D_1 + L_1 + U - S_\alpha + S_\alpha U) \\
&\quad - \lambda(I - S_1 + D_1 + L_1 - L - S_2)]x \\
&= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1}\{[(D_1 + L_1 + U) - \lambda(I + D_1 + L_1 - L)]x + \\
&\quad [S_\alpha U - S_\alpha D + \lambda(S_1 + S_2)]x\} \\
&= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1}[S_\alpha U - S_\alpha D + \lambda(S_1 + S_2)]x \\
&= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1}[(\lambda - 1)D_1 + (\lambda - 1)L_1 + (\lambda - 1)D]x \\
&= (\lambda - 1)(D_\alpha + D_1 + L_1 - L_\alpha)^{-1}[D_1 + L_1 + D]x.
\end{aligned}$$

Since $S_\alpha \geq 0$, and $D_1 + L_1 + D \geq 0$, we have

(1) if $\lambda > 1$, then $\tilde{T}x \leq Tx$. By Lemma 2.3, we get $\rho(\tilde{T}) < \rho(T)$;
(2) if $\lambda = 1$, then $\tilde{T}x = Tx$. By Lemma 2.3, we get $\rho(\tilde{T}) = \rho(T)$;
(3) if $\lambda < 1$, then $\tilde{T}x \geq Tx$. By Lemma 2.3, we get $\rho(\tilde{T}) > \rho(T)$. □

Corollary 3.7. *Let $A = D - L - U$ be an Z-matrix, where $-L$, $-U$ are the strictly lower and strictly upper triangular parts of A, respectively. Assume that $D_1 \geq 0$, $0 \leq L_1 \leq L_\alpha$, $1 - a_{ii+1}a_{i+1i} > 0$, $i = 1, 2 \cdots n-1$, $0 \leq r < w \leq 1$, and $\tilde{T}_{rw}$, T_{rw} are the iteration matrices of PAOR and AOR, respectively. If $\tilde{T}_{rw}$ and T_{rw} are irreducible, then*

$$\begin{aligned}
\rho(\tilde{T}_{rw}) &> \rho(T_{rw}) \quad if\ \rho(T_{rw}) > 1; \\
\rho(\tilde{T}_{rw}) &= \rho(T_{rw}) \quad if\ \rho(T_{rw}) = 1; \\
\rho(\tilde{T}_{rw}) &< \rho(T_{rw}) \quad if\ \rho(T_{rw}) < 1.
\end{aligned}$$

Corollary 3.8. *Let $A = D - L - U$ be an Z-matrix, where $-L$, $-U$ are the strictly lower and strictly upper triangular parts of A, respectively. Assume that $D_1 \geq 0$, $0 \leq L_1 \leq L_\alpha$, $1 - a_{ii+1}a_{i+1i} > 0$, $i = 1, 2 \cdots n-1$, $0 \leq r < 1$, and $\tilde{T}_r$, T_r are the iteration matrices of PSOR and SOR, respectively. If $\tilde{T}_r$ and T_r are irreducible, then*

$$\begin{aligned}
\rho(\tilde{T}_r) &> \rho(T_r) \quad if\ \rho(T_r) > 1; \\
\rho(\tilde{T}_r) &= \rho(T_r) \quad if\ \rho(T_r) = 1; \\
\rho(\tilde{T}_r) &< \rho(T_r) \quad if\ \rho(T_r) < 1.
\end{aligned}$$

TABLE 1. Spectral radius for different α

$\rho(T)$	$\rho(\tilde{L}_r)$	$\rho(\tilde{L}_{r,w})$	$\rho(\tilde{T})$	α
0.9132	0.8521	0.8310	0.7763	0
0.9132	0.8492	0.8275	0.7714	0.05
0.9132	0.8462	0.8240	0.7664	0.10
0.9132	0.8432	0.8205	0.7615	0.15
0.9132	0.8402	0.8171	0.7565	0.20
0.9132	0.8373	0.8136	0.7516	0.25
0.9132	0.8343	0.8101	0.7466	0.30
0.9132	0.8313	0.8066	0.7416	0.35
0.9132	0.8284	0.8031	0.7366	0.40
0.9132	0.8254	0.7996	0.7316	0.45
0.9132	0.8225	0.7962	0.7265	0.50
0.9132	0.8195	0.7927	0.7215	0.55
0.9132	0.8166	0.7892	0.7164	0.60
0.9132	0.8137	0.7858	0.7114	0.65
0.9132	0.8107	0.7823	0.7063	0.70
0.9132	0.8078	0.7789	0.7013	0.75
0.9132	0.8049	0.7754	0.6962	0.80
0.9132	0.8020	0.7720	0.6911	0.85
0.9132	0.7991	0.7686	0.6861	0.90
0.9132	0.7962	0.7651	0.6810	0.95
0.9132	0.7934	0.7617	0.6759	1.00

4. Numerical examples

We consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} 1 & -0.1 & -0.1 & 0 & -0.2 & -0.4 \\ -0.3 & 1 & -0.2 & 0 & -0.3 & -0.2 \\ 0 & -0.2 & 1 & -0.5 & -0.1 & 0 \\ -0.1 & -0.3 & -0.1 & 1 & -0.2 & -0.1 \\ -0.2 & -0.3 & -0.2 & -0.1 & 1 & -0.1 \\ -0.3 & -0.1 & -0.1 & -0.2 & -0.1 & 1 \end{pmatrix}.$$

We choose $D_1 = 0.8D$,$L_1 = 0.7L$, $r = 0.7$and $w = 0.8$, then we can obtain the following results by Theorem 3.4-3.6.

In Figure 1, '–'denotes$\rho(T)$, '**' denotes $\rho(\tilde{L}_r)$, '–' denotes $\rho(\tilde{L}_{r,w})$ and 'ooo' denotes $\rho(\tilde{T})$.

From the above Table and Figure, we can conclude that the rate of convergence of the preconditioned mixed-type splitting method is faster than that of the mixed-type splitting method. And the preconditioned mixed-type splitting method convergences faster than the PAOR method and PSOR method.

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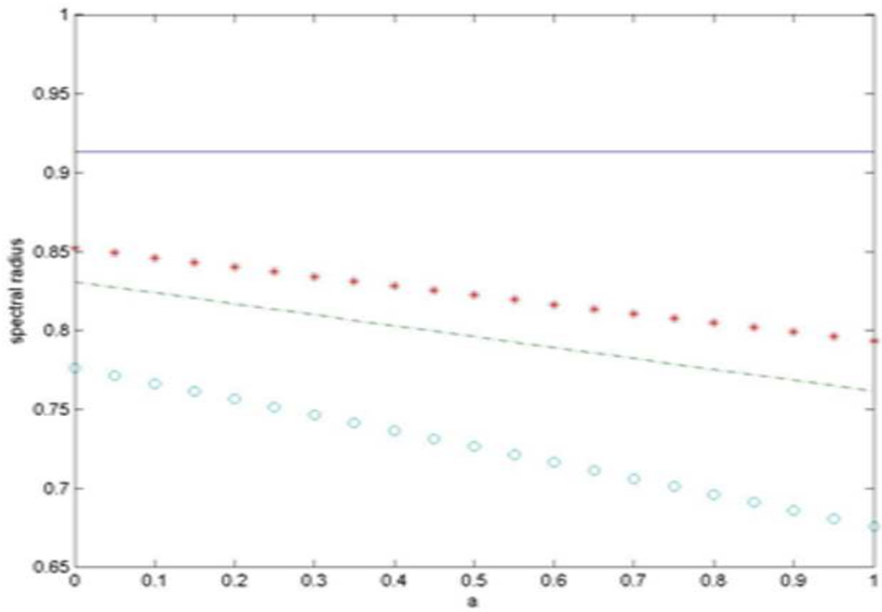

FIGURE 1. The relationship between $\rho(T)$, $\rho(\tilde{L}_r)$, $\rho(\tilde{L}_{r,w})$ and $\rho(\tilde{T})$.

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