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CONVERGENCE THEOREMS FOR THE M_α-INTEGRAL

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Abstract. In this paper, we investigate some properties of the M_α-integral and prove convergence theorems for the M_α-integral.

1. Introduction and preliminaries

It is well-known [8] that the Monotone Convergence Theorem and the Dominated Convergence Theorem are valid for the Lebesgue, Perron, Denjoy, Henstock and C-integrals. In this paper, we prove convergence theorems for the M_α-integral.

Throughout this paper, $I_0 = [a, b]$ is a compact interval in R. Let D be a finite collection of interval-point pairs $\{(I_i, \xi_i)\}_{i=1}^n$, where $\{I_i\}_{i=1}^n$ are non-overlapping subintervals of I_0 and let δ be a positive function on I_0, i.e. $\delta : I_0 \to R^+$. We say that $D = \{(I_i, \xi_i)\}_{i=1}^n$ is

(1) a partial tagged partition of I_0 if $\cup_{i=1}^n I_i \subset I_0$,

(2) a tagged partition of I_0 if $\cup_{i=1}^n I_i = I_0$,

(3) a δ-fine McShane partition of I_0 if $I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in I_o$ for all $i = 1, 2, ..., n$,

(4) a δ-fine M_α-partition of I_0 for a constant $\alpha > 0$ if it is a δ-fine McShane partition of I_0 that satisfies the condition

$$\sum_{i=1}^n dist(\xi_i, I_i) < \alpha,$$

where $\text{dist}(\xi_i, I_i) = inf\{|t - \xi_i| : t \in I_i\}$,

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2. Properties of the M_α-integral

We present the definition of the M_α-integral.

DEFINITION 2.1. *A function $f : I_0 \to R$ is M_α-integrable if there exists a real number A such that for each $\epsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that*

$$|S(f, D) - A| < \epsilon$$

for each δ-fine M_α-partition $D = \{(I_i, \xi_i)\}$ of I_0. The real number A is called the M_α-integral of f on I_0 and we write $A = \int_{I_0} f$ or $A = (M_\alpha)\int_{I_0} f$.

The function f is M_α-integrable on the set $E \subset I_0$ if the function $f\chi_E$ is M_α-integrable on I_0. We write $\int_E f = \int_{I_0} f\chi_E$.

We can easily get the following theorems.

THEOREM 2.2. *A function $f : I_0 \to R$ is M_α-integrable if and only if for each $\epsilon > 0$ there is a positive function $\delta(\xi) : I_0 \to R^+$ such that*

$$|S(f, D_1) - S(f, D_2)| < \epsilon$$

for any δ-fine M_α-partitions D_1 and D_2 of I_0.

THEOREM 2.3. *Let $f : I_0 \to R$.*

(1) If f is M_α-integrable on I_0, then f is M_α-integrable on every subinterval of I_0.

(2) If f is M_α-integrable on each of the intervals I_1 and I_2, where I_1 and I_2 are non-overlapping and $I_1 \cup I_2 = I_0$, then f is M_α-integrable on I_0 and $\int_{I_1} f + \int_{I_2} f = \int_{I_0} f$.

The following theorem shows that the M_α-integral is linear.

THEOREM 2.4. *Let f and g be M_α-integrable functions on I_0. Then*
(1) αf is M_α-integrable on I_0 and $\int_{I_0} \alpha f = \alpha \int_{I_0} f$ for each $\alpha \in R$,
(2) $f + g$ is M_α-integrable on I_0 and $\int_{I_0}(f + g) = \int_{I_0} f + \int_{I_0} g$.

DEFINITION 2.5. *Let $F : I_0 \to R$ and let E be a subset of I_0.*

(a) F is said to be AC_α on E if for each $\epsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta : I_0 \to R^+$ such that $|\sum_i F(I_i)| < \epsilon$ for each δ-fine partial M_α-partition $D = \{(I_i, \xi_i)\}$ of I_0 satisfying $\xi_i \in E$ and $\sum_i |I_i| < \eta$.

(b) F is said to be ACG_α on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is AC_α.

THEOREM 2.6. *([12]) If a function $f : I_0 \to R$ is M_α-integrable on I_0 if and only if there is an ACG_α function F on I_0 such that $F' = f$ almost everywhere on I_0.*

THEOREM 2.7. *([12]) Let $f : I_0 \to R$ be a function.*
(a) If f is McShane integrable on I_0, then f is M_α-integrable on I_0.
(b) If f is M_α-integrable on I_0, then f is Henstock integrable on I_0.

3. Convergence Theorems for the M_α-integral

We will prove the convergence theorems for the M_α-integral.

THEOREM 3.1. *(Uniform Convergence Theorem)*
Let $\{f_n\}$ be a sequence of M_α-integral functions defined on $[a, b]$ and suppose that $\{f_n\}$ converges to f uniformly on $[a, b]$. Then f is M_α-integral on $[a, b]$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n \ .$$

Proof. Given $\epsilon > 0$, there exists N such that if $n \geq N$, then

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in [a, b]$. Consequently, if $m, n \geq N$, then

$$-2\epsilon < f_n(x) - f_m(x) < 2\epsilon \ \text{ for } \ x \in [a, b]$$

Hence, $-2\epsilon(b - a) < \int_a^b f_n - \int_a^b f_m < 2\epsilon(b - a)$, where $\left|\int_a^b f_n - \int_a^b f_m\right| < 2\epsilon(b - a)$. Since $\epsilon > 0$ is arbitrary, the sequence $\{\int_a^b f_n\}$ is a Cauchy sequence. Let $\lim_{n\to\infty} \int_a^b f_n = L$. If $D = \{(I_i, \xi_i)\}_{i=1}^p$ is any M_α-partition of $[a, b]$ and $n \geq N$, then

$$\begin{aligned} |S(f_n, D) - S(f, D)| &= \Big|\sum_{i=1}^p [f_n(\xi_i) - f(\xi_i)]|I_i|\Big| \\ &\leq \sum_{i=1}^p |f_n(\xi_i) - f(\xi_i)||I_i| \\ &\leq \sum_{i=1}^p \epsilon|I_i| = \epsilon(b - a) \ . \end{aligned}$$

Choose a fixed number $n_0 \geq N$ such that $\left|\int_a^b f_{n_0} - L\right| < \epsilon$. Let δ be a positive function on $[a, b]$ such that $\left|\int_a^b f_{n_0} - S(f_{n_0}, D)\right| < \epsilon$ whenever

D is a δ-fine M_α-partition of $[a,b]$. Then

$$\begin{aligned} |S(f,D)-L| &\leq |S(f,D)-S(f_{n_0},D)| \\ &\quad +|S(f_{n_0},D)-\int_a^b f_{n_0}| + |\int_a^b f_{n_0} - L| \\ &< \epsilon(b-a)+\epsilon+\epsilon = \epsilon(b-a+2). \end{aligned}$$

Hence, f is M_α-integrable on $[a,b]$ and

$$\int_a^b f = L = \lim_{n\to\infty}\int_a^b f_n.$$

□

THEOREM 3.2. *(Monotone Convergence Theorem)*
Let $\{f_n\}$ be a monotone increasing sequence of M_α-integrable functions defined on $[a,b]$ and suppose that $\{f_n\}$ converges pointwise to a measurable function f on $[a,b]$. If $\lim_{n\to\infty}\int_a^b f_n$ is finite, then f is M_α-integrable on $[a,b]$ and

$$\int_a^b f = \lim_{n\to\infty}\int_a^b f_n.$$

Proof. Since the sequence $\{f_n\}$ is increasing, $\{f_n - f_1\}$ is an increasing sequence of nonnegative M_α-integrable functions on $[a,b]$. Since $f_n - f_1$ is nonnegative for each n, it follows that each $f_n - f_1$ is Lebesgue integrable on $[a,b]$ and $\lim_{n\to\infty}(f_n - f_1) = f - f_1$.

By the Monotone Convergence Theorem for the Lebesgue integral, the function $f - f_1$ is Lebesgue integrable on $[a,b]$ and

$$\begin{aligned} (L)\int_a^b (f-f_1) &= \lim_{n\to\infty}(L)\int_a^b (f_n - f_1) \\ &= \lim_{n\to\infty}\int_a^b (f_n - f_1) \\ &= \lim_{n\to\infty}(\int_a^b f_n - \int_a^b f_1) \\ &= \lim_{n\to\infty}\int_a^b f_n - \int_a^b f_1. \end{aligned}$$

Since $f - f_1$ and f_1 are M_α-integrable on $[a,b]$, the function $f = (f - f_1) + f_1$ is M_α-integrable on $[a,b]$. Hence,

$$\int_a^b f = \int_a^b (f-f_1) + \int_a^b f_1$$

$$\begin{aligned} &= (L)\int_a^b (f-f_1) + \int_a^b f_1 \\ &= \lim_{n\to\infty}\int_a^b f_n - \int_a^b f_1 + \int_a^b f_1 \\ &= \lim_{n\to\infty}\int_a^b f_n. \end{aligned}$$

□

THEOREM 3.3. *(Dominated Convergence Theorem)*

Let $\{f_n\}$ be a sequence of M_α-integrable functions defined on $[a,b]$ and suppose that $\{f_n\}$ converges to a measurable function f almost everywhere on $[a,b]$. If there exist M_α-integrable functions g and h on $[a,b]$ such that $g \le f_n \le h$ almost everywhere on $[a,b]$ for all n, then the function f is M_α-integrable on $[a,b]$ and

$$\int_a^b f = \lim_{n\to\infty}\int_a^b f_n.$$

Proof. Since $0 \le f_n - g \le h - g$ and $h-g$ is a nonnegative M_α-integrable function on $[a,b]$, $h-g$ is Lebesgue integrable. Since $\{f_n\}$ converges pointwise to f almost everywhere on $[a,b]$, $0 \le f-g \le h-g$ almost everywhere on $[a,b]$. Hence, $f-g$ is Lebesgue integrable. Since $f_n - g$ converges pointwise to $f-g$ almost everywhere on $[a,b]$ and f_n-g is Lebesgue integrable on $[a,b]$, by the Dominated Convergence Theorem for the Lebesgue integral we have

$$(L)\int_a^b (f-g) = \lim_{n\to\infty}(L)\int_a^b (f_n-g).$$

Since $f-g$ and g are M_α-integrable, $f=(f-g)+g$ is M_α-integrable and

$$\begin{aligned} \int_a^b f &= \int_a^b (f-g) + \int_a^b g \\ &= (L)\int_a^b (f-g) + \int_a^b g \\ &= \lim_{n\to\infty}(L)\int_a^b (f_n-g) + \int_a^b g \\ &= \lim_{n\to\infty}\int_a^b (f_n-g) + \int_a^b g \end{aligned}$$

$$= \lim_{n\to\infty}\int_a^b f_n - \int_a^b g + \int_a^b g = \lim_{n\to\infty}\int_a^b f_n.$$

□

COROLLARY 3.4. *Let $\{f_n\}$ be a sequence of M_α-integrable functions defined on $[a,b]$ and suppose that $\{f_n\}$ converges to a measurable function f almost everywhere on $[a,b]$. If there exist an M_α-integrable function g and a Henstock integrable function h such that $g \le f_n \le h$ almost everywhere on $[a,b]$ for all n, then f is M_α-integrable on $[a,b]$ and*

$$\int_a^b f = \lim_{n\to\infty}\int_a^b f_n.$$

Proof. Since $0 \le f_n - g \le h - g$ and $h - g$ is a nonnegative Henstock integrable function on $[a,b]$, $h-g$ is Lebesgue integrable on $[a,b]$. Hence, $h = (h-g)+g$ is M_α-integrable on $[a,b]$. By Theorem 3.3, f is M_α-integrable on $[a,b]$ and

$$\int_a^b f = \lim_{n\to\infty}\int_a^b f_n.$$

□

We begin with the concept of uniform M_α-integrability. The idea behind this concept is that there exists a single positive function δ that works for all of the functions.

DEFINITION 3.5. *Let $\{f_n\}$ be a sequence of M_α-integrable functions defined on I_0. The sequence $\{f_n\}$ is uniformly M_α-integrable on I_0 if for each $\epsilon > 0$ there exists a positive function $\delta : I_0 \to R^+$ such that*

$$|S(f_n, D) - \int_{I_0} f_n| < \epsilon$$

for all n, whenever $D = \{(I_i, \xi_i)\}_{i=1}^n$ is a δ-fine M_α-partition of I_0

THEOREM 3.6. *Assume that $\{f_n\}$ is uniformly M_α-integrable on I_0 such that*

$$\lim_{n\to\infty} f_n(\xi) = f(\xi).$$

Then the function $f : I_0 \to R$ is M_α-integrable on I_0 and we have

$$\lim_{n\to\infty}\int_{I_0} f_n = \int_{I_0} f.$$

Proof. Since $\{f_n\}$ is uniformly M_α-integrable on I_0, for each $\epsilon > 0$ there is a positive function $\delta : I_0 \to R^+$ such that

$$|S(f_n, D) - \int_{I_0} f_n| < \frac{\epsilon}{3}$$

for all n, whenever D is a δ-fine M_α-partition of I_0. Let D be a δ-fine M_α-partition of I_0. Since $\lim_{n\to\infty} f_n(\xi) = f(\xi)$, there exists an $N \in \mathbb{N}$ such that

$$|S(f_n, D) - S(f, D)| < \epsilon$$

for all $n > N$. Then we have

$$\begin{aligned}
&|\int_{I_0} f_n - \int_{I_0} f_m| \\
&\le |S(f, D) - \int_{I_0} f_n| + |S(f, D) - \int_{I_0} f_m| \\
&\le |S(f_n, D) - S(f, D)| + |S(f_n, D) - \int_{I_0} f_n| \\
&\quad + |S(f_m, D) - S(f, D)| + |S(f_m, D) - \int_{I_0} f_m| \\
&< \frac{8}{3}\epsilon
\end{aligned}$$

for all $m, n > N$. Hence $\{\int_{I_0} f_n\}$ is a Cauchy sequence. Let

$$\lim_{n\to\infty} \int_{I_0} f_n = A.$$

Then there exists an $M \in \mathbb{N}$ such that $| \int_{I_0} f_n - A| < \frac{\epsilon}{3}$ for all $n > M$. Take any δ-fine M_α-partition $D = \{(I, \xi)\}$ of I_0. Since $\lim_{n\to\infty} f_n(\xi) = f(\xi)$, there exists a $k > M$ such that $|S(f_k, D) - S(f, D)| < \frac{\epsilon}{3}$.
Then we have

$$\begin{aligned}
&|S(f, D) - A| \\
&\le |S(f, D) - S(f_k, D)| + |S(f_k, D) - \int_{I_0} f_k| + | \int_{I_0} f_k - A| \\
&< \epsilon
\end{aligned}$$

Hence f is M_α-integrable on I_0 and $\lim_{n\to\infty} \int_{I_0} f_n = \int_{I_0} f$. □

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