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APPROXIMATE J^*-DERIVATIONS ON J^*-ALGEBRAS

Hark-Mahn Kim* and Sanghoon Lee**

Abstract. We establish alternative stability and superstability of J^*-derivations in J^*-algebras for a generalized Jensen type functional equation by using the direct method and the fixed point alternative method.

1. Introduction

Let $\mathcal{H}$, $\mathcal{K}$ be two Hilbert spaces and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ be the space of all bounded operators from $\mathcal{H}$ into $\mathcal{K}$. By a J^*-algebra we mean a closed subspace $\mathcal{A}$ of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $xx^*x \in \mathcal{A}$ whenever $x \in \mathcal{A}$. In general, by a J^*-algebra we mean a closed subspace $\mathcal{A}$ of a C^*-algebra such that $xx^*x \in \mathcal{A}$ whenever $x \in \mathcal{A}$ [10]. Many familiar spaces are J^*-algebras [9, 10], for example: (i) every Cartan factor of type I, i.e., the space of all bounded operators $\mathcal{B}(\mathcal{H}, \mathcal{K})$ between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$; (ii) every Cartan factor of type IV, i.e., a closed $*$-subspace $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ in which the square of each operator in $\mathcal{A}$ is a scalar multiple of the identity operator on $\mathcal{H}$; (iii) every JC^*-algebra; (iv) every ternary algebra of operators [11]. Of course J^*-algebras are not algebras in the ordinary sense. However from the point of view they may be considered a generalization of C^*-algebras; see [8, 10, 11]. In particular any Hilbert space may be thought as a J^*-algebra identified with $\mathcal{L}(\mathcal{H}, \mathbb{C})$. Also any C^*-algebra in $\mathcal{B}(\mathcal{H})$ is a J^*-algebra. A J^*-derivation on a J^*-algebra $\mathcal{A}$ is defined to be a $\mathbb{C}$-linear mapping $d : \mathcal{A} \to \mathcal{A}$ such that

$$d(aa^*a) = d(a)a^*a + ad(a)^*a + aa^*d(a)$$

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Correspondence should be addressed to Sanghoon Lee, sahunee@hanmail.net.
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for all $a \in \mathcal{A}$ [9]. In particular, every $*$-derivation on a C^*-algebra is a J^*-derivation.

The stability of functional equations was first introduced by Ulam [25] in 1940. More precisely, he proposed the following problem: Given a group G_1, a metric group (G_2, d) and a positive number ϵ, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $h : G_1 \to G_2$ such that $d(f(x), h(x)) < \epsilon$ for all $x \in G_1$? As mentioned above, when this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, Hyers [12] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Hyers' theorem was generalized by T. Aoki [1] and D.G. Bourgin [2] for additive mappings by considering an unbounded Cauchy difference. In 1978, Th.M. Rassias [21] also provided a generalization of Hyers' theorem for linear mappings which allows the Cauchy difference to be unbounded like this $\|x\|^p + \|y\|^p$.

During the last decades, several stability problems of functional equations have been investigated by a lot of mathematicians. A large list of references concerning the stability of functional equations can be found in [3, 13, 14, 15, 22]. Recently, Cădariu and Radu [4, 5, 6, 7] applied the fixed point method to the investigation of stability problem for the functional equations(see also [18, 23, 24]). In [19], Park establish the stability of homomorphisms between C^*-algebras(see also [17, 18, 20]). In the paper [9], authors established the stability and superstability of J^*-derivations in J^*-algebras for the generalized Jensen type functional equation

$$rf(\frac{x+y}{r}) + rf(\frac{x-y}{r}) = 2f(x). \tag{1.1}$$

Let $r \in (1, \infty)$ and let $f : \mathcal{A} \to \mathcal{A}$ be a mapping with $f(0) = 0$ for which there exists a function $\phi : \mathcal{A}^3 \to [0, \infty)$ such that

$$\begin{aligned}
&\Phi(a, b, c) := \sum_{n=0}^{\infty} r^{-n} \phi(r^n a, r^n b, r^n c) < \infty, \\
&\Big\| r\mu f(\frac{a+b}{r}) + r\mu f(\frac{a-b}{r}) - 2f(\mu a) \\
&\qquad + f(cc^*c) - f(c)c^*c - cf(c)^*c - cc^*f(c) \Big\| \leq \phi(a, b, c)
\end{aligned}$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}$. Then there exists a unique J^*-derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(a) - D(a)\| \le \frac{1}{2}\Phi(a, 0, 0)$$

for all $a \in \mathcal{A}$. Furthermore, they investigated the stability of J^*-derivations by using the fixed point alternative method.

In Section 2 of this paper, we establish alternative stability and superstability of J^*-derivations in J^*-algebras for the generalized Jensen type functional equation by using the direct method. In Section 3, we will use the fixed point alternative method of Cădariu and Radu to prove alternative stability and superstability of J^*-derivations on J^*-algebras for the generalized Jensen type functional equation (1.1).

2. Stability of the equation (1.1) by direct method

Throughout this paper, assume that $\mathcal{A}$ is a J^*-algebra. We present here a useful lemma to prove our stability theorems.

Lemma 2.1. ([16]) *If an element u of a C^*-algebra $\mathfrak{U}$ has the property that $\|u\| < 1 - \frac{2}{n}$ for some integer n greater than 2, then there are n unitary elements $u_1, \cdots, u_n$ in $\mathfrak{U}$ such that $u = \frac{u_1 + u_2 + \cdots + u_n}{n}$.*

We start our investigation with superstability of J^*-derivations.

Theorem 2.2. *Let $r, s \in (1, \infty)$, and let $D : \mathcal{A} \to \mathcal{A}$ a mapping for which $D(sa) = sD(a)$ for all $a \in \mathcal{A}$. Suppose there exists a function $\phi : \mathcal{A}^3 \to [0, \infty)$ such that*

$$\lim_{n\to\infty} s^{3n}\phi(\frac{a}{s^n}, \frac{b}{s^n}, \frac{c}{s^n}) = 0,$$

$$\begin{aligned}(2.1)\qquad &\Big\|r\mu D(\frac{a+b}{r}) + r\mu D(\frac{a-b}{r}) - 2D(\mu a)\\ &\quad + D(cc^*c) - D(c)c^*c - cD(c)^*c - cc^*D(c)\Big\| \le \phi(a, b, c)\end{aligned}$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}$, where $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Then D is a J^*-derivation.

Proof. It follows that $\phi(0, 0, 0) = 0$ and so $D(0) = 0$ by setting $a, b, c := 0$ in the inequality (2.1). Replacing a,b and c by $\frac{a}{s^n}$, $\frac{b}{s^n}$ and 0, respectively, in (2.1), we obtain

$$\Big\|r\mu D(\frac{a+b}{r}) + r\mu D(\frac{a-b}{r}) - 2D(\mu a)\Big\| \le s^n\phi(\frac{a}{s^n}, \frac{b}{s^n}, 0)$$

for all $a, b \in \mathcal{A}$ and $n \in \mathbb{N}$. Taking $n \to \infty$, one gets the equation

$$r\mu D(\frac{a+b}{r}) + r\mu D(\frac{a-b}{r}) = 2D(\mu a) \tag{2.2}$$

for all $a, b \in \mathcal{A}$ and all $\mu \in \mathbb{T}$. This means that D satisfies the generalized Jensen type functional equation (1.1), and so it is additive. If we replace b by a in the above equation, then we obtain $\mu D(a) = D(\mu a)$ for all $a \in \mathcal{A}$ and all $\mu \in \mathbb{T}$. Now, let $\lambda \in \mathbb{C}(\lambda \neq 0)$ and $M \in \mathbb{N}$ such that $M > 4|\lambda|$ and $\left|\frac{\lambda}{M}\right| < \frac{1}{4} < \frac{1}{3} = 1 - \frac{2}{3}$. Then by Lemma 2.1, there exist three elements μ_1, μ_2 and μ_3 in $\mathbb{T}$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. Therefore, we see that

$$\begin{aligned} D(\lambda a) &= D(\frac{M}{3} 3\frac{\lambda}{M} a) = \frac{M}{3} D(\mu_1 a + \mu_2 a + \mu_3 a) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3) D(a) = \lambda D(a) \end{aligned} \tag{2.3}$$

for all $a \in \mathcal{A}$ and all $\lambda \in \mathbb{C}(\lambda \neq 0)$. This equation is trivial for $\lambda = 0$. Hence D is $\mathbb{C}$-linear.

Put $a = b := 0$, $c := \frac{c}{s^n}$ in (2.1). Then

$$\begin{aligned} &\left\| D(\frac{c}{s^n}\frac{c^*}{s^n}\frac{c}{s^n}) - D(\frac{c}{s^n})\frac{c^*}{s^n}\frac{c}{s^n} - \frac{c}{s^n} D(\frac{c}{s^n})^* \frac{c}{s^n} - \frac{c}{s^n}\frac{c^*}{s^n} D(\frac{c}{s^n}) \right\| \\ &\le \phi(0, 0, \frac{c}{s^n}) \end{aligned} \tag{2.4}$$

for all $c \in \mathcal{A}$ and all $n \in \mathbb{N}$. Multiplying s^{3n} on both sides, then we get

$$\|D(cc^*c) - D(c)c^*c - cD(c)^*c - cc^*D(c)\| \le s^{3n}\phi(0, 0, \frac{c}{s^n})$$

for all $c \in \mathcal{A}$ and all $n \in \mathbb{N}$. Since the right-hand side tends to zero as $n \to \infty$, the mapping D is a J^*-derivation. □

Now, we investigate stability of J^*-derivations on J^*-algebras for the generalized Jensen type functional equation by using the direct method.

THEOREM 2.3. *Let $r \in (1, \infty)$, and let $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there exists a function $\phi : \mathcal{A}^3 \to [0, \infty)$ such that*

$$\sum_{n=0}^{\infty} r^n \phi(\frac{a}{r^n}, \frac{b}{r^n}, \frac{c}{r^n}) < \infty, \quad \lim_{n\to\infty} r^{3n}\phi(\frac{a}{r^n}, \frac{b}{r^n}, \frac{c}{r^n}) = 0,$$

$$\begin{aligned} &\left\| r\mu f(\frac{a+b}{r}) + r\mu f(\frac{a-b}{r}) - 2f(\mu a) \right. \\ &\quad \left. + f(cc^*c) - f(c)c^*c - cf(c)^*c - cc^*f(c) \right\| \le \phi(a, b, c) \end{aligned} \tag{2.5}$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}$. Then there exists a unique J^*-derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(a) - D(a)\| \leq \frac{1}{2}\sum_{n=0}^{\infty} r^n \phi(\frac{a}{r^n}, 0, 0) \tag{2.6}$$

for all $a \in \mathcal{A}$.

Proof. It follows that $\phi(0, 0, 0) = 0$ and so $f(0) = 0$ by setting $a, b, c := 0$ in the inequality (2.5). Put $\mu := 1$ and $b = c := 0$ in (2.5). Then it follows that

$$\left\| rf(\frac{a}{r}) - f(a) \right\| \leq \frac{1}{2}\phi(a, 0, 0) \tag{2.7}$$

for all $a \in \mathcal{A}$. By induction, it is easy to see that

$$\left\| r^n f(\frac{a}{r^n}) - f(a) \right\| \leq \frac{1}{2}\sum_{k=0}^{n-1} r^k \phi(\frac{a}{r^k}, 0, 0) \tag{2.8}$$

for all $a \in \mathcal{A}$ and all $n \in \mathbb{N}$. Replacing a by $\frac{a}{r^m}$ in (2.8) and then multiplying r^m, we obtain

$$\begin{aligned} \left\| r^{m+n} f(\frac{a}{r^{m+n}}) - r^m f(\frac{a}{r^m}) \right\| &\leq \frac{1}{2}\sum_{k=0}^{n-1} r^{m+k} \phi(\frac{a}{r^{m+k}}, 0, 0) \\ &= \frac{1}{2}\sum_{k=m}^{m+n-1} r^k \phi(\frac{a}{r^k}, 0, 0) \end{aligned}$$

for all $a \in \mathcal{A}$ and all $n, m \geq 0$. Hence $\{r^n f(\frac{a}{r^n})\}$ is a Cauchy sequence in $\mathcal{A}$ and $D(a) := \lim_{n\to\infty} r^n f(\frac{a}{r^n})$ exists for all $a \in \mathcal{A}$. In particular, taking $n \to \infty$ in (2.8), we can have the desired estimation (2.6).

The proof for $\mathbb{C}$-linearity and J^*-derivation of D follows similarly from the corresponding parts (2.2)–(2.4) of Theorem 2.2.

To prove the uniqueness of the J^*-derivation D, let us assume that there exists another J^*-derivation $D' : \mathcal{A} \to \mathcal{A}$ satisfying (2.6). Since D and D' are $\mathbb{C}$-linear, we get $D(a) = r^n D(\frac{a}{r^n})$ and $D'(a) = r^n D'(\frac{a}{r^n})$. Thus for all $a \in \mathcal{A}$ and all $n \in \mathbb{N}$, we have

$$\begin{aligned} \|D(a) - D'(a)\| &= \left\| r^n D(\frac{a}{r^n}) - r^n D'(\frac{a}{r^n}) \right\| \\ &\leq \left\| r^n D(\frac{a}{r^n}) - r^n f(\frac{a}{r^n}) \right\| + \left\| r^n f(\frac{a}{r^n}) - r^n D'(\frac{a}{r^n}) \right\| \\ &\leq \sum_{i=0}^{\infty} r^{n+i} \phi(\frac{a}{r^{i+n}}, 0, 0) = \sum_{i=n}^{\infty} r^i \phi(\frac{a}{r^i}, 0, 0), \end{aligned}$$

which implies the uniqueness of D since the series tends to zero as $n \to \infty$. This completes the proof. $\square$

In the paper [9], the authors proved Hyers-Ulam-Rassias stability problem for J^*-derivations on J^*-algebras for $0 < p < 1$. On the other hand, we prove the following Hyers-Ulam-Rassias stability problem for J^*-derivations on J^*-algebras for $p > 3$.

Corollary 2.4. *Let $p > 3$, $\theta \in [0, \infty)$ and $r \in (1, \infty)$ be real numbers. Suppose that a mapping $f : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\Big\| r\mu f(\frac{a+b}{r}) + r\mu f(\frac{a-b}{r}) - 2f(\mu a) + f(cc^*c) \\ - f(c)c^*c - cf(c)^*c - cc^*f(c)\Big\| \leq \theta(\|a\|^p + \|b\|^p + \|c\|^p)$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}$. Then there exists a unique J^*-derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(a) - D(a)\| \leq \frac{r^p\,\theta}{2(r^p - r)}\|a\|^p$$

for all $a \in \mathcal{A}$.

3. Stability of the equation (1.1) by fixed point method

Before proceeding to the main results of this section, we will state the following theorem.

Theorem 3.1. *(The Alternative of Fixed Point). Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either $d(T^m x, T^{m+1} x) = \infty$ for all $m \geq 0$ or there exists a natural number m_0 such that*
- *$d(T^m x, T^{m+1} x) < \infty$ for all $m \geq m_0$,*
- *the sequence $\{T^m x\}$is convergent to a fixed point y^* of T,*
- *y^* is the unique fixed point of T in the set $\Lambda = \{y \in \Omega : d(T^{m_0} x, y) < \infty\}$ and*
- *$d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.*

We prove alternative stability and superstability of J^*-derivations on J^*-algebras for the generalized Jensen type functional equation by using the fixed point alternative method.

THEOREM 3.2. *Let $r \in (1, \infty)$ be a real number and $f : \mathcal{A} \to \mathcal{A}$ be a mapping for which there exists a function $\phi : \mathcal{A}^3 \to [0, \infty)$ such that*

$$(3.1) \qquad \Big\| r\mu f(\frac{a+b}{r}) + r\mu f(\frac{a-b}{r}) - 2f(\mu a) + f(cc^*c) - f(c)c^*c - cf(c)^*c - cc^*f(c) \Big\| \le \phi(a, b, c)$$

for all $\mu \in \mathbb{T}$ and all $a, b, c \in \mathcal{A}$. If there exists a real $0 < L < 1$ such that

$$(3.2) \qquad \phi(\frac{a}{r}, \frac{b}{r}, \frac{c}{r}) \le \frac{L}{r^3}\phi(a, b, c)$$

for all $a, b, c \in \mathcal{A}$, then there exists a unique J^*-derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$(3.3) \qquad \|f(a) - D(a)\| \le \frac{1}{2(1-L)}\phi(a, 0, 0)$$

for all $a \in \mathcal{A}$.

Proof. It follows from (3.2) that $\phi(0, 0, 0) = 0$, and thus $f(0) = 0$ by setting $a, b, c := 0$ in the inequality (3.1). Consider a set $X := \{g | g : \mathcal{A} \to \mathcal{A},\ g(0) = 0\}$ and we introduce a generalized metric d on X:

$$d(g, h) := \inf\{C \in \mathbb{R}^+ : \|g(a) - h(a)\| \le C\phi(a, 0, 0), \forall a \in \mathcal{A}\}.$$

It is easy to see that (X, d) is complete. Now we define a linear mapping $J : X \to X$ by $J(h)(a) = rh(\frac{a}{r})$ for all $a \in \mathcal{A}$. Then we see that $d(J(g), J(h)) \le Ld(g, h)$, $\forall g, h \in X$, and so J is a strictly contractive self-mapping on X with Lipschitz constant L. It follows from (2.7) that $d(J(f), f) \le \frac{1}{2}$. By Theorem 3.1, J has a unique fixed point D in the set $X_1 = \{h \in X : \ d(f, h) < \infty\}$. Thus D is the unique mapping such that $D(\frac{a}{r}) = \frac{1}{r}D(a)$ and there exists $C \in (0, \infty)$ satisfying $\|D(a) - f(a)\| \le C\phi(a, 0, 0)$ for all $a \in \mathcal{A}$. On the other hand, we have $d(J^n f(a), D(a)) \to 0$ as $n \to \infty$ for all $a \in \mathcal{A}$ by Theorem 3.1. It means that $\lim_{n\to\infty} r^n f(\frac{a}{r^n}) = D(a)$ for all $a \in \mathcal{A}$. Moreover, it follows from $d(f, D) \le \frac{1}{1-L}d(f, J(f))$ that $\|f(a) - D(a)\| \le \frac{1}{2(1-L)}\phi(a, 0, 0)$ for all $a \in \mathcal{A}$.

Now, we claim that $D : \mathcal{A} \to \mathcal{A}$ is a J^*-derivation. Replacing a, b and c by $\frac{a}{r^n}, \frac{b}{r^n}$ and 0, respectively, in (3.1) and then multiplying r^n, we have the following inequality

$$\begin{aligned}&\Big\|rD(\frac{a+b}{r})+rD(\frac{a-b}{r})-2D(a)\Big\|\\&\quad=\lim_{n\to\infty}r^n\Big\|rf(\frac{a+b}{r^{n+1}})+rf(\frac{a-b}{r^{n+1}})-2f(\frac{a}{r^n})\Big\|\\&\quad\le\lim_{n\to\infty}r^n\phi(\frac{a}{r^n},\frac{b}{r^n},0)=0\end{aligned}$$

for all $a,b\in\mathcal{A}$. This means that D is a Jensen type function and so D is additive because $D(0)=0$. Replacing a,b and c by $\frac{a}{r^n}$, $\frac{a}{r^n}$ and 0, respectively, in (3.1) and then multiplying r^n, we have an inequality

$$\Big\|\mu\, r^{n+1}f(\frac{2a}{r^{n+1}})-2r^nf(\frac{\mu a}{r^n})\Big\|\le r^n\phi(\frac{a}{r^n},\frac{a}{r^n},0).$$

for all $a\in\mathcal{A}$ and all $n\in\mathbb{N}$. Taking $n\to\infty$, we obtain the inequality

$$\begin{aligned}\|\mu D(2a)-2D(\mu a)\| &= \lim_{n\to\infty}\Big\|\mu r^{n+1}f(\frac{2a}{r^{n+1}})-2r^nf(\frac{\mu a}{r^n})\Big\|\\&\le \lim_{n\to\infty}r^n\phi(\frac{a}{r^n},\frac{a}{r^n},0)=0\end{aligned}$$

for all $\mu\in\mathbb{T}$ and all $a\in\mathcal{A}$. Thus $\mu D(a)=D(\mu a)$ for all $\mu\in\mathbb{T}$ and all $a\in\mathcal{A}$, so D is $\mathbb{C}$-linear by the same argument as in the proof of Theorem 2.2. It follows from (3.2) that

$$\lim_{n\to\infty}r^{3n}\phi(\frac{a}{r^n},\frac{b}{r^n},\frac{c}{r^n})=0$$

for all $a,b,c\in\mathcal{A}$. Thus we have that

$$\begin{aligned}&\|D(cc^*c)-D(c)c^*c-cD(c)^*c-cc^*D(c)\|\\&=\lim_{n\to\infty}\Big\|r^{3n}f(\frac{c}{r^n}\frac{c^*}{r^n}\frac{c}{r^n})+r^nf(\frac{c}{r^n})c^*c-cr^nf(\frac{c}{r^n})^*c-cc^*r^nf(\frac{c}{r^n})\Big\|\\&\le\lim_{n\to\infty}r^{3n}\phi(0,0,\frac{c}{r^n})=0\end{aligned}$$

for all $c\in\mathcal{A}$. Thus $D:\mathcal{A}\to\mathcal{A}$ is a J^*-derivation.

To prove the uniqueness of the J^*-derivation D, let us assume that there exists another J^*-derivation $D':\mathcal{A}\to\mathcal{A}$ satisfying (3.3). Since D and D' are $\mathbb{C}$-linear, $D(a)=r^nD(\frac{a}{r^n})$ and $D'(a)=r^nD'(\frac{a}{r^n})$. Thus we have

$$\begin{aligned}\|D(a)-D'(a)\| &= \Big\|r^nD(\frac{a}{r^n})-r^nD'(\frac{a}{r^n})\Big\|\\&\le \Big\|r^nD(\frac{a}{r^n})-r^nf(\frac{a}{r^n})\Big\|+\Big\|r^nf(\frac{a}{r^n})-r^nD'(\frac{a}{r^n})\Big\|\\&\le \frac{1}{1-L}r^n\phi(\frac{a}{r^n},0,0)\le\frac{1}{1-L}\frac{L^n}{r^{2n}}\phi(a,0,0)\end{aligned}$$

for all $a \in \mathcal{A}$ and all $n \in \mathbb{N}$. Taking $n \to \infty$ in the last inequality, we leads to the uniqueness of the mapping D. This completes the proof. $\square$

We prove the following Hyers-Ulam-Rassias stability problem for J^*-derivations on J^*-algebras for $p > 3$.

COROLLARY 3.3. *Let $p > 3$, $r \in (1, \infty)$, and $\theta \in [0, \infty)$ be real numbers. Suppose that $f : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\begin{aligned}(3.4)\qquad &\Big\| r\mu f(\frac{a+b}{r}) + r\mu f(\frac{a-b}{r}) - 2f(\mu a) + f(cc^*c) \\ &- f(c)c^*c - cf(c)^*c - cc^*f(c)\Big\| \leq \theta(\|a\|^p + \|b\|^p + \|c\|^p)\end{aligned}$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}$. Then there exists a unique J^*-derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(a) - D(a)\| \leq \frac{r^p\ \theta}{2(r^p - r^3)}\|a\|^p$$

for all $a \in \mathcal{A}$.

COROLLARY 3.4. *Let $p := p_1 + p_2 + p_3 > 3$, $p_i > 0$ $(i = 1, 2, 3)$, $r \in (1, \infty)$, and $\theta \in [0, \infty)$ be real numbers. Suppose $f : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\begin{aligned}(3.5)\qquad &\Big\| r\mu f(\frac{a+b}{r}) + r\mu f(\frac{a-b}{r}) - 2f(\mu a) + f(cc^*c) \\ &- f(c)c^*c - cf(c)^*c - cc^*f(c)\Big\| \leq \theta(\|a\|^{p_1}\|b\|^{p_2}\|c\|^{p_3})\end{aligned}$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}$. Then f is in fact a J^*-derivation.

COROLLARY 3.5. *Let $p > 3$, $r \in (1, \infty)$, and $\theta_1, \theta_2 \in [0, \infty)$ be real numbers. Suppose $f : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\begin{aligned}(3.6)\qquad &\Big\| r\mu f(\frac{a+b}{r}) + r\mu f(\frac{a-b}{r}) - 2f(\mu a) + f(cc^*c) - f(c)c^*c \\ &- cf(c)^*c - cc^*f(c)\Big\| \leq \theta_1(\|a\|^p + \|b\|^p) + \theta_2(\|a\|\|b\|\|c\|)^{\frac{p}{3}}\end{aligned}$$

for all $a, b, c \in \mathcal{A}$ and all $\mu \in \mathbb{T}$. Then there exists a unique J^*-derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(a) - D(a)\| \leq \frac{r^p\ \theta_1}{2(r^p - r^3)}\|a\|^p$$

for all $a \in \mathcal{A}$.

Theorem 3.6. *Let $r, s \in (1, \infty)$ be real numbers and let $f : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying $f(sa) = sf(a)$ for all $a \in \mathcal{A}$. Let $\phi : \mathcal{A}^3 \to [0, \infty)$ be a mapping satisfying (3.1). If there exists an $0 < L < 1$ such that*

$$\phi(\frac{a}{s}, \frac{b}{s}, \frac{c}{s}) \leq \frac{L}{s^3}\phi(a, b, c) \tag{3.7}$$

for all $a, b, c \in \mathcal{A}$, then f is a J^*-derivation.

Proof. By iterative process of (3.7), we get

$$\lim_{n\to\infty} s^{3n}\phi(\frac{a}{s^n}, \frac{b}{s^n}, \frac{c}{s^n}) \leq \lim_{n\to\infty} L^n\phi(a, b, c) = 0.$$

for all $a, b, c \in \mathcal{A}$. Thus, by Theorem 2.2, f is a J^*-derivation. □

Corollary 3.7. *Let $r, s \in (1, \infty)$, and $\theta \in [0, \infty)$ be real numbers. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a function satisfying $f(sa) = sf(a)$ for all $a \in \mathcal{A}$ and one of approximate inequalities (3.4), (3.5) or (3.6) with $p > 3$. Then f is a J^*-derivation.*

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2 (1950), 64–66.

[2] D. G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. 57 (1951), 223-237.

[3] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.

[4] L. Cădariu, V. Radu, *Fixed points and the stability of quadratic functional equations*, An. Univ. Vest Timis. **41** (2003), 25–48.

[5] L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4** (2003)(Article ID 4).

[6] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: A fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.

[7] L. Cădariu, V. Radu, *The fixed points method for the stability of some functional equations*, Carpathian J. Math. **23** (2007), 63–72.

[8] M. Elin, L. Harris, S. Reich, D. Shoikhet, *Evolution equations and geometric function theory in J^*-algebras* , J. Nonlinear Convex Anal. **3** (2002), 81–121.

[9] M. E. Gordji, M.B. Ghaemi, S.K. Gharetapeh, S. Shames, A. Ebadian, *On the stability of J^*-derivations*, J. Geo. Phy. **60** (2010), 454–459.

[10] L. A. Harris, *Bounded symmetric homogeneous domains in infinite-dimensional spaces, in: Lecture Notes in Mathematics*, vol. 364, Springer, Berlin (1974).

[11] L. A. Harris, *Operator siegel domains* , Proc. Roy. Soc. Edinburgh Sect. A. **79** (1977), 137–156.

[12] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA. **27** (1941), 222–224.

[13] D. H. Hyers, Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44**(2-3) (1992), 125–153.

[14] D. H. Hyers, G. Isac, Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
[15] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.
[16] R. V. Kadison, G. Pedersen, *Mean and convex combinations of unitary operators*, Math. Scan. **57** (1985), 249–266.
[17] M. S. Moslehian, C. Park, *On the stability of J^*-homomorphisms*, Nonlinear Anal.-TMA **63** (2005), 42–48.
[18] C. Park, J.M. Rassias, *Stability of the Jensen-type functional equation in C^*-algebras: A fixed point approach*, Abstact Appl. Anal. (2009) (Article ID 360432, 17 pages).
[19] C. Park, *Isomorphisms between C^*-ternary algebras*, J. Math. Anal. Appl. **327** (2007), 101–115.
[20] C. Park, *Homomorphisms between Poisson JC^*-algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97.(MR2132832(2005m:39047)).
[21] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[22] Th. M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62**(1), (2000) 23–130.
[23] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory 4 (2003), 91–96.
[24] I. A. Rus, *Principles and Applications of Fixed Point Theory*, Ed. Dacia, Cluj-Napoca, 1979 (in Romanian).
[25] S. M. Ulam, *Problems in Modern Mathematics*, Science Ed. Wiley, New York, 1940 (Chapter VI).

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Department of Mathematics
Chungnam National University
Daejeon 305–764, Republic of Korea
E-mail: hmkim@cnu.ac.kr

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Department of Mathematics
Chungnam National University
Daejeon 305–764, Republic of Korea
E-mail: sahunee@hanmail.net