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NEW EXACT TRAVELLING WAVE SOLUTIONS FOR SOME NONLINEAR EVOLUTION EQUATIONS

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Abstract. In this work, we obtain new solitary wave solutions for some nonlinear partial differential equations. The Jacobi elliptic function rational expansion method is used to establish new solitary wave solutions for the combined KdV-mKdV and Klein-Gordon equations. The results reveal that Jacobi elliptic function rational expansion method is very effective and powerful tool for solving nonlinear evolution equations arising in mathematical physics.

1. Introduction

Nonlinear wave phenomena appears in various scientific and engineering fields such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and so on. In order to understand better the nonlinear phenomena as well as further application in the practical life, it is important to seek their more exact travelling wave solutions. Many methods are used to obtain travelling solitary wave solutions to nonlinear partial differential equations (PDEs) such as tanh method [4, 7], sine-cosine method [2], variational iteration method [6], exp-function method [1, 9] and so on.

However, practically there is no unified method that can be used to handle all types of nonlinear partial differential equations. Another important method used to obtain exact solutions of nonlinear partial differential equation is the Jacobi elliptic function rational expansion method. One of the most effective straightforward method to construct exact solutions of PDEs is the Jacobi elliptic function rational expansion method [10]. The extended Jacobi elliptic function rational expansion method is more powerful than the method in [5]. The different Jacobi

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function expansion may lead to new Jacobi doubly periodic wave solutions, triangular periodic solutions and soliton solutions.

The KdV and mKdV equations are most popular soliton equations and have been extensively investigated. Consider the combined KdV-mKdV equation

$$u_t + (\alpha + \beta u)uu_x + u_{xxx} = 0, \tag{1.1}$$

where α and β are some arbitrary constants. This equation may describe the wave propagation of a bound particle, a sound wave and a thermal pulse [8]. The improved sub-ODE method is developed to obtain some exact travelling wave solutions to the combined KdV and mKdV equation in [11].

Next we consider Klein-Gordon equation of the form

$$u_{tt} - u_{xx} - u + u^3 = 0 \tag{1.2}$$

The Klein-Gordon equation plays an important role in mathematical physics. The equation has attracted much attention in studying solitons in condensed matter physics,interaction of solitons in a collisionless plasma, and the recurrence of initial states [3]. In this paper, we use the Jacobi elliptic function rational expansion method with symbolic computation to special equations (1.1) and (1.2) for constructing their new Jacobi doubly periodic wave solutions. It is shown that soliton and triangular solutions can be established as the limits of the Jacobi doubly periodic wave solutions.

2. The Jacobi elliptic function rational expansion method and its algorithm

Step 1. Consider a given nonlinear PDE in two variables

$$P(u, u_t, u_x, u_{xx}, u_{tt}, ...) = 0. \tag{2.1}$$

we make the transformation

$$u(x,t) = u(\xi), \xi = kx - \omega t, \tag{2.2}$$

where k and w are the wave number and wave speed respectively, we can rewrite Eq.(2.1) in the following nonlinear ODE:

$$Q(u, -\omega u', ku', k^2u'', \omega^2u'', ...) = 0. \tag{2.3}$$

Step 2. By the Jacobi elliptic function rational expansion method, $u(\xi)$ can be expressed as a finite series of Jacobi elliptic function

$$u(\xi) = a_0 + \sum_{i=1}^{n} \frac{\mathrm{sn}^{(i-1)}(\xi)(a_i \mathrm{sn}(\xi) + b_i \mathrm{cn}(\xi))}{(\mu \mathrm{sn}(\xi) + 1)^i}, \tag{2.4}$$

where $a_i (i = 0, 1, \cdots, n), b_i (i = 1, 2, \cdots, n)$ are constants to be determined later. Here $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$ and $\mathrm{dn}(\xi)$ are Jacobi elliptic functions. They are double periodic and possess properties of triangular functions

$$\mathrm{cn}^2(\xi) + \mathrm{sn}^2(\xi) = \mathrm{dn}^2(\xi) + m^2 \,\mathrm{sn}^2(\xi) = 1, \tag{2.5}$$

$$\begin{aligned} \mathrm{sn}'(\xi) &= \mathrm{cn}(\xi)\, \mathrm{dn}(\xi) \\ \mathrm{cn}'(\xi) &= -\,\mathrm{sn}(\xi)\, \mathrm{dn}(\xi) \\ \mathrm{dn}'(\xi) &= -m^2 \mathrm{sn}(\xi) \mathrm{cn}(\xi), \end{aligned} \tag{2.6}$$

where m $(0 < m < 1)$ is the modulus. When $m \to 1$, Jacobi elliptic functions degenerate to the hyperbolic functions, ie,

$$\lim_{m\to 1} \mathrm{sn}(\xi) = \tanh(\xi), \lim_{m\to 1} \mathrm{cn}(\xi) = \mathrm{sech}(\xi), \lim_{m\to 1} \mathrm{dn}(\xi) = \mathrm{sech}(\xi).$$

When $m \to 0$, Jacobi elliptic functions degenerate to the triangular functions, ie,

$$\lim_{m\to 0} \mathrm{sn}(\xi) = \sin(\xi), \lim_{m\to 0} \mathrm{cn}(\xi) = \cos(\xi), \lim_{m\to 0} \mathrm{dn}(\xi) = 1.$$

Step 3. We define a polynomial degree function as $D(u(\xi)) = n$, thus we have

$$D\left(u^p(\xi)\right) = np,\ D\left(\left(\frac{d^s u(\xi)}{d\xi^s}\right)^q\right) = q(n+s),\ p, q = 0, 1, 2, \cdots .$$

Therefore, we can determine the parameter n by balancing the highest order linear term with the nonlinear term in Eq. (2.3). (If n is not a positive integer, then we first make the transformation $u = v^n$, and then perform the third step again.)

Step 4. Substitute (2.4) into (2.3) along with (2.5) and (2.6), and set all coefficients of $\mathrm{sn}^i(\xi)\ \mathrm{cn}^j(\xi)$ $(i = 0, 1, 2, \cdots, j = 1, 2)$ to zero. We get a set of algebraic equations with respect to the unknowns $k, \omega, \mu, a_i (i = 0, 1, 2 \cdots, n)$ and $b_i (i = 1, 2, \cdots, n)$.

Step 5. Solving the systems of algebraic equations using Maple we can obtain the explicit expressions for $k, \omega, \mu, a_i (i = 0, 1, 2 \cdots, n)$ and $b_i (i = 1, 2, \cdots, n)$ and substituting these values in (2.4) we can get double periodic solutions with Jacobi elliptic functions for equation (2.1).

3. Solutions to nonlinear PDEs

3.1. Wave solutions to KdV-mKdV equation

We now proceed to apply the method as outlined in previous section to formally derive distinct exact travelling wave solutions to the nonlinear KdV-mKdV equation (1.1).

The famous combined KdV-mKdV equation is

$$u_t + (\alpha + \beta u)uu_x + u_{xxx} = 0. \tag{3.1}$$

To look for the travelling wave solution of Eq.(3.1), we make transformation $u(x,t) = u(\xi), \xi = x - \omega t$, and change Eq.(3.1) into the form

$$-\omega u' + (\alpha + \beta u)uu' + u''' = 0. \tag{3.2}$$

According to Step 3, we have n=1 and assume that Eq.(3.2) has the solution

$$u(\xi) = a_0 + \frac{a_1 \mathrm{sn}(\xi) + b_1 \mathrm{cn}(\xi)}{\mu \mathrm{sn}(\xi) + 1}. \tag{3.3}$$

With the aid of Maple, substituting (3.3) into (3.2) along with Eqs. (2.5) and (2.6), yields a set of algebraic equations for $\mathrm{sn}^i(\xi)\ \mathrm{cn}^j(\xi)(i = 0, 1, 2, \cdots, j = 1, 2)$. Setting the coefficients of $\mathrm{sn}^i(\xi)\ \mathrm{cn}^j(\xi)$ to zero, we obtain a set algebraic equations with respect to the unknowns a_0, a_1, b_1, ω and μ. Solving the system of algebraic equations using Maple gives the following set of nontrivial solutions:

$$\begin{aligned}&\Big\{a_0 = \pm\frac{6m^2\mu + 6\mu \pm \alpha K_1 - 12\mu^3}{2\beta K_1}, a_1 = \pm K_1,\\ &b_1 = 0, \omega = K_2/4\beta(\mu^2 m^2 + \mu^2 - \mu^4 - m^2), \mu = \mu\Big\}\end{aligned} \tag{3.4}$$

where $K_1 = \sqrt{-\frac{6m^2 - 6\mu^2 m^2 - 6\mu^2 + 6\mu^4}{\beta}}$, $K_2 = 2\mu^2 m^4\beta - 20\mu^2 m^2\beta + 4\mu^4 m^2\beta + 2\mu^2\beta + 4\mu^4\beta - \mu^2 m^2\alpha^2 - \mu^2\alpha^2 + \mu^4\alpha^2 + 4m^4\beta + 4m^2\beta + m^2\alpha^2$

$$\begin{aligned}&\Big\{a_0 = -\frac{\alpha}{2\beta}, a_1 = 0, b_1 = \pm m\sqrt{\frac{6}{\beta}},\\ &\omega = (-\alpha^2 - 4\beta + 8m^2\beta)/4\beta, \mu = 0\Big\},\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\Big\{a_0 = -\frac{\alpha}{2\beta}, a_1 = \pm m\sqrt{-\frac{3}{2\beta}}, b_1 = \pm m\sqrt{\frac{3}{2\beta}},\\ &\omega = (-\alpha^2 - 4\beta + 2m^2\beta)/4\beta, \mu = 0\Big\},\end{aligned} \tag{3.6}$$

$$\left\{a_0 = -\frac{\alpha}{2\beta}, a_1 = 0, b_1 = \pm\sqrt{\frac{3m^2-3}{2\beta}},\right.$$
$$\left.\omega = (-\alpha^2 + 2\beta + 2m^2\beta)/4\beta, \mu = \pm 1\right\}, \tag{3.7}$$

$$\left\{a_0 = -(\pm K_3\alpha - 3\mu^3 + 3\mu)/2K_3\beta, a_1 = \pm K_3,\right.$$
$$\left.b_1 = \pm m\sqrt{-\frac{3\mu^2-3m^2}{2\beta}}, \omega = \frac{K_4}{4\beta(-\mu^2+m^2)}, \mu = \mu\right\} \tag{3.8}$$

where $K_3 = \sqrt{-\frac{3m^2-3\mu^2m^2-3\mu^2+3\mu^4}{2\beta}}$, $K_4 = 4\mu^2m^2\beta - 2\mu^2\beta + \mu^2\alpha^2 - 4m^2\beta - m^2\alpha^2 + 2m^4\beta$

$$\left\{a_0 = (\pm 2K_5\alpha \pm 15i)/\pm 4\beta K_5, a_1 = \pm K_5,\right.$$
$$\left.b_1 = \pm\frac{1}{2}\sqrt{\frac{3+12m^2}{2\beta}}, \omega = \frac{K_6}{4\beta(4m^2+1)}, \mu = \pm\frac{1}{2}i\right\} \tag{3.9}$$

where $K_5 = \frac{1}{4}\sqrt{-\frac{15+60m^2}{2\beta}}$, $K_6 = -20m^2\beta + 8m^4\beta + 2\beta - 4m^2\alpha^2 - \alpha^2$.

From Eq.(3.3) and (3.4),we obtain a Jacobi doubly periodic wave solution

$$u_1(x,t) = \pm\frac{6\mu(m^2+1) \pm \alpha K_1 - 12\mu^3}{2\beta K_1} \pm \frac{K_1 \cdot \mathrm{sn}(x-\omega t)}{\mu \cdot \mathrm{sn}(x-\omega t)+1} \tag{3.10}$$

where K_1 and ω are defined as in Eq. (3.4). As $m \to 1$, this solution degenerates to the following soliton solution

$$u_{11}(x,t) = \pm\frac{12\mu \pm \alpha K_{11} - 12\mu^3}{2\beta K_{11}} \pm \frac{K_{11} \cdot \tanh(x-\omega t)}{\mu \cdot \tanh(x-\omega t)+1} \tag{3.11}$$

where $K_{11} = \sqrt{-\frac{6(\mu^4-2\mu^2+1)}{\beta}}$, $\omega = \frac{K_{21}}{-4\beta(2\mu^2-\mu^4-1)}$ and $K_{21} = -16\mu^2\beta + 8\mu^4\beta - 2\mu^2\alpha^2 + \mu^4\alpha^2 + 8\beta + \alpha^2$.

When $m \to 0$, the solution (3.10) degenerates to a triangular periodic solution of the form

$$u_{10}(x,t) = \pm\frac{12\mu \pm \alpha K_{10} - 12\mu^3}{2\beta K_{10}} \pm \frac{K_{10} \cdot \sin(x-\omega t)}{\mu \cdot \sin(x-\omega t)+1} \tag{3.12}$$

where $K_{10} = \sqrt{-\frac{-6\mu^2+6\mu^4}{\beta}}$,$\omega_{10} = \frac{K_{20}}{4\beta(\mu^2-\mu^4)}$ and $K_{20} = 2\mu^2\beta + 4\mu^4\beta - \mu^2\alpha^2 + \mu^4\alpha^2$.

From Eq.(3.3) and Eqs.(3.5)-(3.9),we obtain the following $\mathrm{sn}(\xi)$ and $\mathrm{cn}(\xi)$ rational formal doubly periodic wave solutions

$$u_2(x,t) = -\frac{\alpha}{2\beta} \pm m\sqrt{\frac{6}{\beta}} \cdot \mathrm{cn}(x-\omega t), \tag{3.13}$$

where ω defined as in Eq. (3.5),

$$\begin{aligned} u_3(x,t) = -\frac{\alpha}{2\beta} &\pm m\sqrt{-\frac{3}{2\beta}} \cdot \mathrm{sn}(x-\omega t) \\ &\pm m\sqrt{\frac{3}{2\beta}} \cdot \mathrm{cn}(x-\omega t), \end{aligned} \tag{3.14}$$

where ω defined as in Eq. (3.6),

$$u_4(x,t) = -\frac{\alpha}{2\beta} \pm \frac{\sqrt{\frac{3m^2-3}{2\beta}} \cdot \mathrm{cn}(x-\omega t)}{\pm\mathrm{sn}(x-\omega t)+1}, \tag{3.15}$$

where ω defined as in Eq. (3.7)

$$\begin{aligned} u_5(x,t) = &-\frac{\pm K_3\alpha - 3\mu^3 + 3\mu}{2K_3\beta} \\ &+ \frac{\pm K_3 \cdot \mathrm{sn}(x-\omega t) \pm m\sqrt{-\frac{3\mu^2-3m^2}{2\beta}} \cdot \mathrm{cn}(x-\omega t)}{\mu \cdot \mathrm{sn}(x-\omega t)+1}, \end{aligned} \tag{3.16}$$

where ω defined as in Eq. (3.8),

$$\begin{aligned} u_6(x,t) = &\frac{\pm 2K_5\alpha \pm 15i}{4\beta K_5} \\ &+ \frac{\pm K_5 \cdot \mathrm{sn}(x-\omega t) \pm \frac{1}{2}\sqrt{\frac{3+12m^2}{2\beta}} \cdot \mathrm{cn}(x-\omega t)}{\pm\frac{1}{2}i \cdot \mathrm{sn}(x-\omega t)+1}, \end{aligned} \tag{3.17}$$

where ω defined as in Eq. (3.9).

As $m \to 1$, the solutions (3.13) and (3.14) degenerate to the following soliton solutions

$$u_{21}(x,t) = -\frac{\alpha}{2\beta} \pm \sqrt{-\frac{6}{\beta}} \cdot \mathrm{sech}\left(x - \frac{4\beta-\alpha^2}{4\beta}t\right). \tag{3.18}$$

and

$$u_{31}(x,t) = -\frac{\alpha}{2\beta} \pm \sqrt{-\frac{3}{2\beta}} \cdot \tanh\left(x - \frac{-2\beta-\alpha^2}{4\beta}t\right) \pm \sqrt{\frac{3}{2\beta}} \cdot \operatorname{sech}\left(x - \frac{-2\beta-\alpha^2}{4\beta}t\right) \tag{3.19}$$

respectively. When $m \to 0$ the solution (3.15) degenerates to a triangular periodic solution of the form

$$u_{40}(x,t) = -\frac{\alpha}{2\beta} \pm \frac{\sqrt{-\frac{3}{2\beta}} \cdot \cos(x-\omega t)}{\pm\sin(x-\omega t)+1}, \quad \text{where } \omega = \frac{-\alpha^2+\beta}{4\beta}.$$

As $m \to 1$, the solutions (3.16) and (3.17) degenerate to the following soliton solutions

$$u_{51}(x,t) = -\frac{\pm K_{31}\alpha - 3\mu^3 + 3\mu}{2K_{31}\beta} + \frac{\pm K_{31} \cdot \tanh(x-\omega t) \pm \sqrt{\frac{3-3\mu^2}{2\beta}} \cdot \operatorname{sech}(x-\omega t)}{\mu \cdot \tanh(x-\omega t)+1}$$

where $K_{31} = \sqrt{-\frac{3-6\mu^2+3\mu^4}{2\beta}}$,$\omega = \frac{K_{41}}{4\beta(-\mu^2+1)}$ and $K_{41} = 2\mu^2\beta + \mu^2\alpha^2 -2\beta - \alpha^2$.

$$u_{61}(x,t) = \frac{\pm 2K_{51}\alpha \pm 15i}{4\beta K_{51}} + \frac{\pm K_{51} \cdot \tanh(x-\omega t) \pm \frac{1}{2}\sqrt{\frac{15}{2\beta}} \cdot \operatorname{sech}(x-\omega t)}{\pm\frac{1}{2}i \cdot \tanh(x-\omega t)+1},$$

where $K_{51} = \frac{1}{4}\sqrt{-\frac{75}{2\beta}}, \omega = \frac{K_{61}}{20\beta}$ and $K_{61} = -10\beta - 5\alpha^2$.

Also when $m \to 0$ the solutions (3.16) and (3.17) degenerate to triangular periodic solutions of the form

$$u_{50}(x,t) = -\frac{K_{30}\alpha - 3\mu^3 + 3\mu}{2K_{30}\beta} + \frac{\pm K_{30} \cdot \sin(x-\omega t)}{\mu \cdot \sin(x-\omega t)+1},$$

where $K_{30} = \sqrt{\frac{-3\mu^2+3\mu^4}{2\beta}}$,$\omega = \frac{K_{40}}{-4\mu^2\beta}$ and $K_{40} = -2\mu^2\beta + \mu^2\alpha^2$, and

$$u_{60}(x,t) = \frac{\pm 2K_{50}\alpha \pm 15i}{4\beta K_{50}} + \frac{\pm K_{50} \cdot \sin(x-\omega t) \pm \frac{1}{2}\sqrt{\frac{3}{2\beta}} \cdot \cos(x-\omega t)}{\pm\frac{1}{2}i \cdot \sin(x-\omega t)+1},$$

where $K_{50} = \frac{1}{4}\sqrt{-\frac{15}{2\beta}}, \omega = \frac{K_{60}}{4\beta}$ and $K_{60} = 2\beta - \alpha^2$ respectively.

3.2. Wave solutions to Klein-Gordon equation

Consider the Klein-Gordon equation

$$u_{tt} - u_{xx} - u + u^3 = 0. \tag{3.20}$$

To find the travelling wave solutions of Eq.(3.20), we consider the transformation $u(x,t) = u(\xi), \xi = x - \omega t$, where ω is a constant to be determined later, and Eq.(3.20) reduces to

$$(\omega^2 - 1)u'' - u + u^3 = 0. \tag{3.21}$$

According to Step 3, by balancing the highest order derivative term u'' with the nonlinear term u^3 in (3.21), we obtain n = 1 and thus suppose that (3.21) has the solution in the form

$$u(\xi) = a_0 + \frac{a_1 \mathrm{sn}(\xi) + b_1 \mathrm{cn}(\xi)}{\mu \mathrm{sn}(\xi) + 1}. \tag{3.22}$$

With the aid of Maple, substituting (3.22) into (3.21) along with Eqs. (2.5) and (2.6), yields a set of algebraic equations for $\mathrm{sn}^i(\xi)\ \mathrm{cn}^j(\xi)(i = 0, 1, 2, \cdots, j = 1, 2)$. Setting the coefficients of $\mathrm{sn}^i(\xi)\ \mathrm{cn}^j(\xi)$ to zero, we obtain a set algebraic equations with respect to the unknowns a_0, a_1, b_1, ω and μ. Solving the system of algebraic equations using Maple gives the following set of nontrivial solutions:

$$\begin{aligned} &\Big\{a_0 = 0, a_1 = \pm m\sqrt{\frac{2}{1+m^2}}, \\ &b_1 = 0, \omega = \pm m\sqrt{\frac{1}{1+m^2}}, \mu = 0\Big\}, \end{aligned} \tag{3.23}$$

$$\begin{aligned} &\Big\{a_0 = \pm(m-1)\sqrt{\frac{1}{C_1}}, a_1 = \pm 2(m-1)\sqrt{\frac{m}{C_1}}, \\ &b_1 = 0, \omega = \pm\sqrt{\frac{C_1+2}{C_1}}, \mu = \pm C_1\sqrt{\frac{1}{C_1}}\sqrt{\frac{m}{C_1}}\Big\} \end{aligned} \tag{3.24}$$

where $C_1 = m^2 + 6m + 1$.

$$\begin{aligned} &\Big\{a_0 = \pm(m+1)\sqrt{\frac{1}{C_2}}, a_1 = \pm 2(m+1)\sqrt{-\frac{m}{C_2}}, \\ &b_1 = 0, \omega = \pm\sqrt{\frac{C_2+2}{C_2}}, \mu = \pm C_2\sqrt{\frac{1}{C_2}}\sqrt{-\frac{m}{C_2}}\Big\} \end{aligned} \tag{3.25}$$

where $C_2 = m^2 - 6m + 1$,

$$\begin{aligned}\Big\{a_0 = 0, a_1 = 0, b_1 = \pm m\sqrt{\frac{2}{2m^2-1}},\\ \omega = \pm m\sqrt{\frac{2}{2m^2-1}}, \mu = 0\Big\},\end{aligned} \tag{3.26}$$

$$\begin{aligned}\Big\{a_0 = 0, a_1 = \pm m\sqrt{-\frac{1}{m^2-2}},\\ b_1 = \pm m\sqrt{\frac{1}{m^2-2}}, \omega = \pm m\sqrt{\frac{1}{m^2-2}}, \mu = 0\Big\},\end{aligned} \tag{3.27}$$

$$\begin{aligned}\Big\{a_0 = 0, a_1 = 0, b_1 = \pm\sqrt{\frac{m^2-1}{m^2+1}},\\ \omega = \pm\sqrt{\frac{m^2+3}{m^2+1}}, \mu = \pm 1\Big\}.\end{aligned} \tag{3.28}$$

From Eqs.(3.22) and (3.23),we obtain a Jacobi doubly periodic wave solution

$$u_1(x,t) = \pm m\sqrt{\frac{2}{1+m^2}} \cdot \mathrm{sn}\left(x \pm m\sqrt{\frac{1}{1+m^2}}t\right). \tag{3.29}$$

As $m \to 1$, the solution (3.29) degenerates to a following soliton solution

$$u_{11}(x,t) = \pm\tanh\left(x - \sqrt{\frac{1}{2}}t\right).$$

From Eqs.(3.22) and (3.24),we obtain a Jacobi doubly periodic wave solution

$$u_2(x,t) = \pm(m-1)\sqrt{\frac{1}{C_1}} + \frac{\pm 2(m-1)\sqrt{\frac{m}{C_1}} \cdot \mathrm{sn}\left(x \pm \sqrt{\frac{C_1+2}{C_1}}t\right)}{\pm C_1\sqrt{\frac{1}{C_1}}\sqrt{\frac{m}{C_1}} \cdot \mathrm{sn}\left(x \pm \sqrt{\frac{C_1+2}{C_1}}t\right) + 1}.$$

From Eqs.(3.22) and (3.25),we obtain a Jacobi doubly periodic wave solution

$$\begin{aligned}u_3(x,t) = &\pm (m+1)\sqrt{\frac{1}{C_2}}\\ &+ \frac{\pm 2(m+1)\sqrt{-\frac{m}{C_2}} \cdot \mathrm{sn}\left(x \pm \sqrt{\frac{C_2+2}{C_2}}t\right)}{\pm C_2\sqrt{\frac{1}{C_2}}\sqrt{-\frac{m}{C_2}} \cdot \mathrm{sn}\left(x \pm \sqrt{\frac{C_2+2}{C_2}}t\right) + 1}.\end{aligned} \tag{3.30}$$

As $m \to 1$, the solution (3.30) degenerates to a following soliton solution

$$u_{31}(x,t) = \pm i + \frac{\pm 2\ \tanh\big(x - \sqrt{\frac{1}{2}}t\big)}{\pm i \cdot \tanh\big(x - \sqrt{\frac{1}{2}}t\big) + 1}.$$

From Eqs.(3.22) and (3.26),we obtain a Jacobi doubly periodic wave solution

$$u_4(x,t) = \pm m\sqrt{\frac{2}{2m^2-1}} \cdot \mathrm{cn}\left(x \pm m\sqrt{\frac{2}{2m^2-1}}t\right). \tag{3.31}$$

As $m \to 1$, the solution (3.31) degenerates to a following soliton solution

$$u_{41}(x,t) = \pm\sqrt{2}\mathrm{sech}(x - \sqrt{2}t).$$

From Eqs.(3.22) and (3.27),we obtain a Jacobi doubly periodic wave solution

$$\begin{aligned} u_5(x,t) = &\pm m\sqrt{-\frac{1}{m^2-2}} \cdot \mathrm{sn}\left(x \pm m\sqrt{\frac{1}{m^2-2}}t\right) \\ &\pm m\sqrt{\frac{1}{m^2-2}} \cdot \mathrm{cn}\left(x \pm m\sqrt{\frac{1}{m^2-2}}t\right). \end{aligned} \tag{3.32}$$

As $m \to 1$, this solution (3.32) degenerates to a following soliton solution

$$u_{51}(x,t) = \pm\ \tanh(x - it) \pm i \cdot \mathrm{sech}(x - it).$$

From Eqs.(3.22) and (3.28), we obtain the following $\mathrm{sn}(\xi)$ and $\mathrm{cn}(\xi)$ rational formal doubly periodic wave solutions

$$u_6(x,t) = \frac{\pm\sqrt{\frac{m^2-1}{m^2+1}} \cdot \mathrm{cn}\left(x \pm \sqrt{\frac{m^2+3}{m^2+1}}t\right)}{\pm\mathrm{sn}\left(x \pm \sqrt{\frac{m^2+3}{m^2+1}}t\right) + 1}. \tag{3.33}$$

As $m \to 1$, the solution (3.33) do not degenerate to the soliton solution. But when $m \to 0$, it degenerates to a triangular periodic solution of the form

$$u_{60}(x,t) = \frac{\pm i\cos(x \pm \sqrt{3}t)}{\pm i\sin(x \pm \sqrt{3}t) + 1}.$$

Remark 3.1. Especially, since when $m \to 1$, $\mathrm{sn}(\xi) \to \tanh(\xi)$ and $\mathrm{cn}(\xi) \to \mathrm{sech}(\xi)$; while $m \to 0$, $\mathrm{sn}(\xi) \to \sin(\xi)$ and $\mathrm{cn}(\xi) \to \cos(\xi)$; thus it is easy to see that the present method is used to obtain Jacobi doubly periodic wave solutions, triangular periodic solutions and soliton solutions. Therefore, it is easy to see that the solutions derived from the present method include the results of the sine-cosine method.

4. Conclusion

Based on the Jacobi elliptic function rational expansion method and computerized symbolic computation, we have obtained new doubly periodic wave solutions of combined KdV-mKdV and Klein-Gordon equations. When the modulus $m \to 1$ and $m \to 0$, some of the obtained solutions degenerate solitary wave and trigonometric function solutions respectively. The result reveals that the Jacobi elliptic function rational expansion method is a promising tool since it can provide a variety of new solutions of distinct physical structures when compared with existing methods.

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