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# ESTIMATION OF GENUS FOR CERTAIN ARITHMETIC GROUP

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Abstract. In this work, we find the genera of arithmetic subgroups $\langle \Gamma_\Delta(N), \Phi\rangle$ of $\mathrm{GL}_2^+(\mathbb{R})$ generated by congruence subgroup $\Gamma_\Delta(N)$ and the Fricke involution $\Phi$.

## 1. Introduction

Let $\mathbb{H}$ be the complex upper half plane. Then $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Let $\mathbb{H}^*$ be the union of $\mathbb{H}$ and $\mathbb{P}^1(\mathbb{Q})$, and $\Gamma$ be a discrete subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ commensurable with $\mathrm{SL}_2(\mathbb{Z})$. Then the quotient curve $\Gamma\backslash\mathbb{H}^*$ is a projective closure of the affine curve $\Gamma\backslash\mathbb{H}$, which is denoted by $X_\Gamma$ with genus $g(\Gamma)$.

For any positive integer $N$, let $\Gamma_1(N)$, $\Gamma_0(N)$ be the congruence subgroups of $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ consisting of the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ congruent modulo $N$ to $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ respectively. Let $\Delta$ be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^*$. Let $\Gamma_\Delta(N)$ be the modular group defined by

$$\Gamma_\Delta(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \bmod N, (a \bmod N) \in \Delta \right\}.$$

We always assume that $-1 \in \Delta$.

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Kim and Koo determine the genus of the curve $X_\Gamma$ when $\Gamma = \langle \Gamma_1(N), \Phi\rangle$ is the arithmetic subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ generated by $\Gamma_1(N)$ and the Fricke involution $\Phi = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ as follows:

THEOREM 1.1. *For any positive integer $N \geq 1$,*

$$g(\langle \Gamma_1(N), \Phi\rangle) = \frac{g(\Gamma_1(N)) - g(\Gamma_0(N))}{2} + g(\langle \Gamma_0(N), \Phi\rangle).$$

In this paper, we show that the same genus formula holds true for the curve $X_\Gamma$ when $\Gamma = \langle \Gamma_\Delta(N), \Phi\rangle$. Our main result is as follows:

THEOREM 1.2. *For any positive integer $N \geq 1$,*

$$g(\langle \Gamma_\Delta(N), \Phi\rangle) = \frac{g(\Gamma_\Delta(N)) - g(\Gamma_0(N))}{2} + g(\langle \Gamma_0(N), \Phi\rangle).$$

One can find a genus formula of $g(\Gamma_\Delta(N))$ in [1].

## 2. Proof of Theorem 1.2

Kim and Koo [2] obtained their genus formula by showing that the number of $\langle \Gamma_1(N), \Phi\rangle$-inequivalent elliptic points fixed by $\Phi$ is the same as the number of $\langle \Gamma_0(N), \Phi\rangle$-inequivalent elliptic points fixed by $\Phi$. If we can show that it holds for $\Gamma_\Delta(N)$, then we are done. For that we use the exact same notation as in [2].

For any positive integer $N$, let $\Gamma^0(N)$ be the congruence subgroup of $\Gamma(1)$ consisting of the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ congruent $\left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)$ mod $N$, and let $\Gamma^\Delta(N)$ be the subgroup of $\Gamma^0(N)$ consisting of the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a \in \Delta$. Since $\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \langle \Gamma_*(N), \Phi\rangle \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \langle \Gamma^*(N), \Phi'\rangle$ with $\Phi' = \begin{pmatrix} 0 & -N \\ 1 & 0 \end{pmatrix}$, the number of $\langle \Gamma_*(N), \Phi\rangle$-inequivalent elliptic points fixed by $\Phi$ is the same as the number of $\langle \Gamma^*(N), \Phi'\rangle$-inequivalent elliptic points fixed by $\Phi'$.

Observe that for $1 \leq N \leq 4$, $\langle \Gamma^\Delta(N), \Phi'\rangle = \langle \Gamma^0(N), \Phi'\rangle$ and $\Gamma^\Delta(N) = \Gamma^0(N)$. Thus it suffices to prove for the case $N \geq 5$. Let $\mathcal{O}_D$ be an order with discriminant $D$ in a quadratic number field $\mathbb{Q}(\sqrt{-N})$ and $C(\mathcal{O}_D)$ be the group of equivalence classes of proper $\mathcal{O}_D$-lattices. For an elliptic point $w \in \mathbb{H}^*$ of $\langle \Gamma^0(N), \Phi'\rangle$, we put $[w]$ to be an orbit of $w$, i.e., $[w] = \{\gamma w \mid \gamma \in \langle \Gamma^0(N), \Phi'\rangle\}$.

Let $E_2'$ be the set of equivalence classes of elliptic points of $\langle \Gamma^0(N), \Phi'\rangle$ fixed by some elliptic elements in the coset $\Gamma^0(N)\Phi'$. We define $E$(resp.

$E'$) a subset $E'_2$ consisting of $[w]$ where $w$ is an elliptic point fixed by some elliptic element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0(N)\Phi'$ where one of $a$ and $d$ is odd (resp. both $a$ and $d$ are even).

Kim and Koo [2] showed that

$$E'_2 = \begin{cases} E & \text{if } -N \not\equiv 1 \bmod 4 \\ E \dot{\cup} E' & \text{if } -N \equiv 1 \bmod 4 \end{cases}.$$

Let $G = (\mathbb{Z}/N\mathbb{Z})^*/\{\pm 1\}$. We define two maps $\mathfrak{N} : E \to G/G^2$ and $\mathfrak{N}' : E' \to G/G^2$ as follows: for $[w] \in E$(resp. $E'$),

$$\mathfrak{N}([w])(\text{resp. } \mathfrak{N}'([w])) = \overline{N(\{1, w\})^{-1}} \bmod G^2$$

where $N$ denote the norm map from the set of proper $\mathcal{O}_D$-lattice to $\mathbb{Z}$. Note that if $w$ satisfy the quadratic equation $aw^2 + bw + c$ then $N(\{1, w\}) = a^{-1}$. Then Kim and Koo [2] showed that $\mathfrak{N}$ and $\mathfrak{N}'$ are surjective homomorphisms.

Now we let $H = (\mathbb{Z}/N\mathbb{Z})^*/\Delta$. We define two maps $\mathfrak{N}_H : E \to H/H^2$ and $\mathfrak{N}'_H : E' \to H/H^2$ by the exact same manner as $\mathfrak{N}$ and $\mathfrak{N}'$. Since there is a natural surjective homomorphism $G/G^2 \to H/H^2$, the maps $\mathfrak{N}_H$ and $\mathfrak{N}'_H$ are surjective homomorphisms too.

Theorem 2.1. *Let $N \geq 5$. Then the number of elements of $E'_2$ is the same as the number of elements of the set $E_2$ consisting of $\langle \Gamma^{\Delta}(N), \Phi' \rangle$-inequivalent elliptic points fixed by some elliptic elements in the coset $\Gamma^{\Delta}(N)\Phi'$*

*Proof.* Consider a diagram

$$\begin{array}{ccc} \mathbb{H}^* & \xrightarrow{\ id\ } & \mathbb{H}^* \\ \pi \downarrow & & \downarrow \pi' \\ \langle \Gamma^{\Delta}(N), \Phi' \rangle \backslash \mathbb{H}^* & \xrightarrow[\varphi]{} & \langle \Gamma^{0}(N), \Phi' \rangle \backslash \mathbb{H}^* \end{array}$$

Let $[w] \in E'_2$ and $M \in \Gamma^0(N)\Phi'$ fix $w$. Let $\overline{\Gamma}^0(N)/\overline{\Gamma}^{\Delta}(N) = \{\overline{\gamma}_1, \ldots, \overline{\gamma}_{\delta_N}\}$ where $\delta_N$ is the order of $H$. Then $\varphi^{-1}(\pi'(w)) = \{\pi(\gamma_i w) \mid i = 1, \ldots, \delta_N\}$. For each $i$, one can show that $\gamma_i w$ is an elliptic point of $\langle \Gamma^{\Delta}(N), \Phi' \rangle$ if and only $t_w \equiv ad^{-2} \bmod N$ for some $a \in \Delta$ when we write $M = \begin{pmatrix} * & * \\ * & t_w \end{pmatrix} \Phi'$ and $\gamma_i = \begin{pmatrix} * & * \\ * & d \end{pmatrix}$. Thus we know that $\pi(E_2) \cap \varphi^{-1}(\pi'(w)) \neq \emptyset$ if and only $\overline{t_w} \in H^2$.

Furthermore, by the exactly method in the proof of Lemma 11 in [2] we get that

$$\overline{t_w} \in H^2 \Leftrightarrow \begin{cases} [w] \in \ker \mathfrak{N}_H & \text{if } [w] \in E \\ [w] \in ([w'] \cdot \ker \mathfrak{N}'_H) & \text{if } [w] \in E' \text{ and } -N \equiv 1 \bmod 4 \end{cases}$$

where we have chosen $[w'] \in E'$ so that $f'([w']) = \overline{2}^{-1} \bmod H^2$.

Let $\nu_2$ and $\nu'_2$ be the number of $E_2$ and $E'_2$ respectively. Now

$$\begin{aligned} \nu_2 &= \#\pi(E_2) = \sum_{[w]\in E'_2} \#[\pi(E_2) \cap \varphi^{-1}([w])] \\ &= \sum_{[w]\in \ker\mathfrak{N}_H \cup ([w']\cdot \ker\mathfrak{N}'_H)} \#\{d \in H \mid d^2 = \overline{t_w}^{-1}\} \\ &= \sum_{[w]\in \ker\mathfrak{N}_H \cup ([w']\cdot \ker\mathfrak{N}'_H)} \#\{d \in H \mid d^2 = 1\} \\ &= \sum_{[w]\in \ker\mathfrak{N}_H \cup ([w']\cdot \ker\mathfrak{N}'_H)} |H/H^2| \\ &= \begin{cases} |H/H^2| \cdot |\ker\mathfrak{N}_H| & \text{if } -N \not\equiv 1 \bmod 4 \\ |H/H^2| \cdot (|\ker\mathfrak{N}_H| + |\ker\mathfrak{N}'_H|) & \text{if } -N \equiv 1 \bmod 4 \end{cases} \\ &= \begin{cases} \#E & \text{if } -N \not\equiv 1 \bmod 4 \\ \#E + \#E' & \text{if } -N \equiv 1 \bmod 4 \end{cases} \\ &= \#E'_2 = \nu'_2 \end{aligned}$$

□

By Hurwitz formula one can show that $g(\langle \Gamma_\Delta(N), \Phi\rangle) = \dfrac{g(\Gamma_\Delta(N))}{2} + \dfrac{1}{2} - \dfrac{\nu_2}{4}$ and $g(\langle \Gamma_0(N), \Phi\rangle) = \dfrac{g(\Gamma_0(N))}{2} + \dfrac{1}{2} - \dfrac{\nu'_2}{4}$. Theorem 1.2 follows from these equations and Theorem 2.1.

## References

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