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ANOTHER PROOF THAT $A_\gamma(G)$ AND $A_\Delta(G)$ ARE BANACH ALGEBRAS

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Abstract. We provide another unified proof that $A_\gamma(G)$ and $A_\Delta(G)$ are Banach algebras for a compact group G, where $A_\gamma(G)$ and $A_\Delta(G)$ are images of the convolution and the twisted convolution, respectively, on $A(G \times G)$. Our new approach heavily depends on analysis of co-multiplication on $VN(G)$, the group von-Neumann algebra of G.

1. Introduction

Let G be a compact group. A well-known fact about $L^1(G)$, the space of integrable functions on G with respect to the normalized Haar measure, is a Banach algebra under the convolution product. Recall that the convolution $*$ is defined by

$$f * g(x) = \int_G f(y)g(y^{-1}x)dy, \quad f, g \in L^1(G),$$

where dy denotes the normalized Haar measure on G. There is another natural Banach algebra associated to G, namely the Fourier algebra $A(G)$. Among many equivalent definitions of $A(G)$ we present the following definition using Fourier transform. Let $\widehat{G}$ be the collection of equivalence classes of irreducible unitary representations of G. Then for any $f \in L^1(G)$ we define Fourier transform of f by the collection of matrices $(\widehat{f}(\pi))_{\pi\in\widehat{G}}$ given by

$$\widehat{f}(\pi) = \int_G f(x)\overline{\pi}(x)dx \in M_{d_\pi},$$

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where d_π is the dimension of $\pi \in \widehat{G}$ and $\overline{\pi}$ is the conjugate representation of π given by $\overline{\pi}(x) = (\pi(x^{-1}))^t$, $x \in G$.

Now we define the Fourier algebra $A(G)$ by

$$A(G) := \{f \in C(G) : \|f\|_{A(G)} = \sum_{\pi \in \widehat{G}} d_\pi \left\| \widehat{f}(\pi) \right\|_1 < \infty\},$$

where $\|\cdot\|_1$ is the Schatten 1-norm. It is well-known that $A(G)$ is a Banach algebra under the pointwise multiplication. $A(G)$ can be defined for general locally compact group, but compactness of G gives us another option, namely the fact that $A(G)$ is also a Banach algebra under the convolution product. Thus, we may consider convolution product $f*g$ of two functions $f, g \in A(G)$ and twisted convolution product $f * \check{g}$, where $\check{g}(x) = g(x^{-1})$, $x \in G$, which we have $f * g, f * \check{g} \in A(G)$. Note that the map $f \mapsto \check{f}$ is an isometry on $A(G)$. Then, it is natural to be interested in the ranges of the above two operations, namely the ranges of Φ and Ψ given by

$$\Phi : A(G \times G) \to A(G), f \otimes g \mapsto f * g$$

and

$$\Psi : A(G \times G) \to A(G), f \otimes g \mapsto f * \check{g}.$$

Recall that simple tensors of the form $f \otimes g$ spans a dense subspace of $A(G \times G)$. Although, the operations Φ and Ψ look quite similar, it turned out that their images are quite different, and actually they are the spaces $A_\gamma(G)$ and $A_\Delta(G)$, which will be defined below. Note that $A_\gamma(G)$ is the object originally considered by B. E. Johnson to get an example of a compact group whose Fourier algebra is not an amenable Banach algebra ([3]). The spaces $A_\gamma(G)$ and $A_\Delta(G)$ are defined as follows.

$$A_\gamma(G) := \{f \in C(G) : \|f\|_{A_\gamma(G)} = \sum_{\pi \in \widehat{G}} d_\pi^2 \left\| \widehat{f}(\pi) \right\|_1 < \infty\}$$

and

$$A_\Delta(G) := \{f \in C(G) : \|f\|_{A_\Delta(G)} = \sum_{\pi \in \widehat{G}} d_\pi^{\frac{3}{2}} \left\| \widehat{f}(\pi) \right\|_2 < \infty\},$$

where $\|\cdot\|_2$ is the Schatten 2-norm (or Hilbert-Schimidt norm).

The spaces $A_\gamma(G)$ and $A_\Delta(G)$ are actually Banach algebras under pointwise multiplication as is proved in [1] and [3] in quite different styles. The main theme of this paper is to give a unified approach which can be applied for both of the cases using a quantum group (or Kac algebra) style of formulation.

This paper is organized as follows. In section 2 we will collect preliminaries which we will need in the proof of our main results, and our main results will be presented in the latter section. We will assume that the reader is familiar with standard functional analysis concepts including Banach space tensor products.

2. Preliminaries

In this paper G is always a fixed compact group. A standard reference for harmonic analysis on compact groups is [2]. The group von Neumann algebra $VN(G)$ is defined by the von Neumann algebra generated by $\{\lambda(x) : x \in G\} \subset B(L^2(G))$, where λ is the left regular representation of G, or equivalently, $\lambda(x)$ is the left translation operator with respect to $x \in G$. The group structure of G is encoded in the following co-multiplication.

$$\Gamma : VN(G) \to VN(G \times G) \cong VN(G)\overline{\otimes}VN(G), \;\; \lambda(x) \mapsto \lambda(x) \otimes \lambda(x),$$

where $\overline{\otimes}$ is the von Neumann algebra tensor product. Recall that Γ is a normal (or weak*-continuous) injective $*$-homomorphism, in particular it is a contraction.

Using a well-developed representation theory of compact groups we could provide another realization of $VN(G)$. Indeed, we have a unitary equivalence

$$VN(G) \cong \bigoplus_{\pi\in\widehat{G}} M_{d_\pi} \subseteq B(H), \tag{2.1}$$

where $H = \bigoplus_{\pi\in\widehat{G}} \ell^2_{d_\pi}$. Note that the above direct sums over $\widehat{G}$ assume the repetition of the same component d_π-times for $\pi \in \widehat{G}$. Note also that the left regular representation λ has the decomposition

$$\lambda \cong \bigoplus_{\pi\in\widehat{G}} \overline{\pi}, \tag{2.2}$$

where the unitary equivalence coincide with the one in (2.1).

For notational convenience we will frequently use vector-valued weighted ℓ^p spaces indexed by $\widehat{G}$. The space $\ell^1(\alpha_\pi; X_\pi)$ for $\alpha_\pi > 0$, $\pi \in \widehat{G}$ and Banach spaces X_π's, refers to the space of sequences $(x_\pi)_{\pi\in\widehat{G}}$ with the norm

$$\left\|(x_\pi)_{\pi\in\widehat{G}}\right\|_{\ell^1(\alpha_\pi;X_\pi)} := \sum_{\pi\in\widehat{G}} \alpha_\pi \left\|x_\pi\right\|_{X_\pi}.$$

Similarly, $\ell^\infty(\alpha_\pi; X_\pi)$ for $\alpha_\pi > 0$, $\pi \in \widehat{G}$ and Banach spaces X_π's, is the space of sequences $(x_\pi)_{\pi\in\widehat{G}}$ with the norm

$$\left\|(x_\pi)_{\pi\in\widehat{G}}\right\|_{\ell^\infty(\alpha_\pi;X_\pi)} := \sup_{\pi\in\widehat{G}} \alpha_\pi \left\|x_\pi\right\|_{X_\pi}.$$

With the above notation we can write

$$\bigoplus_{\pi\in\widehat{G}} M_{d_\pi} = \ell^\infty(1; M_{d_\pi}). \tag{2.3}$$

Since $A(G)$ can be identified as the predual of $VN(G)$, we have the following isometric isomorphism.

$$A(G) \cong \ell^1(d_\pi; S^1_{d_\pi}), \tag{2.4}$$

where S^1_n refers to the Schatten 1-class over ℓ^2_n. The additional factor d_π comes from the duality bracket

$$\langle (A(\pi))_\pi, (B(\pi))_\pi \rangle = \sum_{\pi\in\widehat{G}} d_\pi \mathrm{tr}(A(\pi)B(\pi))$$

for $A = (A(\pi))_\pi \in \ell^\infty(1; M_{d_\pi})$ and $B = (B(\pi))_\pi \in \ell^1(d_\pi; S^1_{d_\pi})$, which is usual in analysis of compact groups.

We also need vector-valued weighted ℓ^p spaces indexed by $\widehat{G} \times \widehat{G}$. The definition is similar to the above case, so we omit it. One example is $VN(G\times G) \cong VN(G)\overline{\otimes}VN(G)$. With this notation we can write

$$VN(G\times G) \cong \bigoplus_{\pi,\pi'\in\widehat{G}} M_{d_\pi}(M_{d_{\pi'}}) = \ell^\infty(1\times 1; M_{d_\pi}(M_{d_{\pi'}})) \tag{2.5}$$

and

$$A(G\times G) \cong \ell^1(d_\pi \times d_{\pi'}; S^1_{d_\pi}(S^1_{d_{\pi'}})). \tag{2.6}$$

Recall that $M_{d_\pi}(M_{d_{\pi'}}) \cong M_{d_\pi d_{\pi'}}$ and $S^1_{d_\pi}(S^1_{d_{\pi'}}) \cong S^1_{d_\pi d_{\pi'}}$ isometrically. We will denote an element $B \in \ell^\infty(1\times 1; M_{d_\pi}(M_{d_{\pi'}}))$ by

$$B = (B(\pi,\pi'))_{\pi,\pi'\in\widehat{G}}.$$

The Fourier transform as a mapping is defined as follows.

$$\mathcal{F} : L^1(G) \to \ell^\infty(1; M_{d_\pi}), \quad f \mapsto \mathcal{F}(f),$$

where $\mathcal{F}(f)(\pi) = \widehat{f}(\pi)$, $\pi \in \widehat{G}$. The corresponding Fourier inverse transform $\mathcal{F}^{-1}$ is defined as follows.

$$\mathcal{F}^{-1} : \ell^1(d_\pi; S^1_{d\pi}) \to L^\infty(G), \quad A = (A(\pi))_{\pi\in\widehat{G}} \mapsto \mathcal{F}^{-1}(A)$$

and

$$\mathcal{F}^{-1}(A)(x) = \sum_{\pi\in\widehat{G}} d_\pi \mathrm{tr}(A(\pi)\pi(x)),\ x \in G.$$

As in the case of the Fourier transform on the real line we have the following Fourier inversion formula. Note that the situation is simpler this case since $L^\infty(G) \subset L^1(G)$ due to the compactness of G.

PROPOSITION 2.1. *If $A \in \ell^1(d_\pi; S^1_{d\pi})$, then we have $\mathcal{F}(\mathcal{F}^{-1}(A)) = A$.*

We close this section with a notation concerning representation theory of compact groups. For any $\pi, \pi' \in \widehat{G}$ we can consider their tensor product $\pi \otimes \pi'$. It is well-known that

$$\pi \otimes \pi' \cong \bigoplus_{k=1}^{n} \tau_k \tag{2.7}$$

for some $\tau_k \in \widehat{G}$.

DEFINITION 2.2. Let $\tau, \pi, \pi' \in \widehat{G}$. We denote $\tau \subset \pi \otimes \pi'$ if $\tau \cong \tau_k$ for some k, where τ_k's are from (2.7).

3. Main results

We begin with an observation how the co-multiplication $\Gamma : VN(G) \to VN(G \times G)$ is translated into

$$\Gamma : \ell^\infty(1; M_{d_\pi}) \to \ell^\infty(1 \times 1; M_{d_\pi}(M_{d_{\pi'}})).$$

We will still denote the translation again by Γ.

Let $A = (A(\pi))_{\pi\in\widehat{G}} \in \ell^\infty(1; M_{d_\pi})$ with $F = \{\pi \in \widehat{G} : A(\pi) \neq 0\}$ is a finite set. Then we have $A = \mathcal{F}(f)$, where $f(x) = \sum_{\rho\in\widehat{G}} d_\rho \mathrm{tr}(A(\rho)\rho(x))$, $x \in G$ by the Fourier inversion formula (Proposition 2.1). Let $\Gamma(A) = (\Gamma(A)(\pi, \pi'))_{\pi,\pi'\in\widehat{G}}$, then the decomposition (2.2) tells us that

$$\begin{aligned}
\Gamma(A)(\pi, \pi') &= \int_G f(x)\overline{\pi(x)} \otimes \overline{\pi'(x)}dx \\
&= \sum_{\rho\in\widehat{G}} d_\rho \int_G \mathrm{tr}(A(\rho)\rho(x))\overline{\pi(x)} \otimes \overline{\pi'(x)}dx \\
&= \sum_{\rho\in\widehat{G}} d_\rho \sum_{i,j=1}^{d_\rho} \int_G A_{ij}(\rho)\rho_{ji}(x)\overline{\pi(x)} \otimes \overline{\pi'(x)}dx
\end{aligned}$$

$$\cong \bigoplus_{\tau\subset\pi\otimes\pi'} \sum_{\rho\in\widehat{G}} d_\rho \sum_{i,j=1}^{d_\rho} \int_G A_{ij}(\rho)\rho_{ji}(x)\overline{\tau(x)}dx$$
$$= \bigoplus_{\tau\subset\pi\otimes\pi'} A(\tau).$$

The last equality comes from the Schur orthogonality relation and $(\overline{\tau(x)})_{ij} = \overline{\tau(x)_{ji}}$. Note that we are taking multiplicity into account in the notation $\tau \subset \pi \otimes \pi'$. From the weak*-continuity of Γ we have shown the following.

PROPOSITION 3.1. *For any $A \in \ell^\infty(1; M_{d_\pi})$ we have*

$$\Gamma(A)(\pi,\pi') \cong \bigoplus_{\tau\subset\pi\otimes\pi'} A(\tau), \quad \forall \pi,\pi' \in \widehat{G}.$$

Recall that Γ is the adjoint map of the pointwise multiplication map $A(G\times G) \to A(G)$. Since we will consider pointwise multiplication as algebra multiplications of $A_\gamma(G)$ and $A_\Delta(G)$, we need to check that Γ can be extended to contractions on $A^*_\gamma(G) \to (A_\gamma(G)\otimes_\gamma A_\gamma(G))^*$ and $A^*_\Delta(G) \to (A_\Delta(G)\otimes_\gamma A_\Delta(G))^*$, where $\otimes_\gamma$ is the projective tensor product of Banach spaces. Note that

$$A_\gamma(G) \cong \ell^1(d_\pi^2, S^1_{d_\pi}) \text{ and } A_\Delta(G) \cong \ell^1(d_\pi^{\frac{3}{2}}, S^2_{d_\pi})$$

isometrically, so that we have

$$\begin{cases} A^*_\gamma(G) \cong \ell^\infty(d_\pi^{-1}, M_{d_\pi}), \\ A^*_\Delta(G) \cong \ell^\infty(d_\pi^{-\frac{1}{2}}, S^2_{d_\pi}), \\ (A_\gamma(G)\otimes_\gamma A_\gamma(G))^* \cong \ell^\infty(d_\pi^{-1}\times d_{\pi'}^{-1}, M_{d_\pi}\otimes_\epsilon M_{d_\pi}), \\ (A_\Delta(G)\otimes_\gamma A_\Delta(G))^* \cong \ell^\infty(d_\pi^{-\frac{1}{2}}\times d_{\pi'}^{-\frac{1}{2}}, S^2_{d_\pi}\otimes_\epsilon S^2_{d_{\pi'}}) \end{cases}$$

isometrically, where $\otimes_\epsilon$ implies the injective tensor product of Banach spaces.

THEOREM 3.2. *$A_\gamma(G)$ and $A_\Delta(G)$ are Banach algebras under pointwise multiplication.*

Proof. First, we consider the case of $A_\gamma(G)$. From the above observations it is enough to check that

$$d_\pi^{-1}d_{\pi'}^{-1}\left\|\Gamma(A)(\pi,\pi')\right\|_{M_{d_\pi}\otimes_\epsilon M_{d_{\pi'}}} \le \sup_{\rho\in\widehat{G}} d_\rho^{-1}\left\|A(\rho)\right\|_{M_{d_\rho}}$$

for any $A \in \ell^\infty(d_\pi^{-1}, M_{d_\pi})$ and $\pi, \pi' \in \widehat{G}$. Indeed, we have

$$\begin{aligned}\left\|\Gamma(A)(\pi,\pi')\right\|_{M_{d_\pi}\otimes_\epsilon M_{d_{\pi'}}} &\leq \left\|\bigoplus_{\tau\subset\pi\otimes\pi'} A(\tau)\right\|_{M_{d_\pi d_{\pi'}}} = \sup_{\tau\subset\pi\otimes\pi'} \|A(\tau)\|_{M_{d_\tau}}\\ &\leq (\sup_{\tau\subset\pi\otimes\pi'} d_\tau) \sup_{\rho\in\widehat{G}} d_\rho^{-1} \|A(\rho)\|_{M_{d_\rho}}\\ &\leq d_\pi d_{\pi'} \sup_{\rho\in\widehat{G}} d_\rho^{-1} \|A(\rho)\|_{M_{d_\rho}}.\end{aligned}$$

The first ineqality comes from the contractive embedding $M_{nm} \subset M_n \otimes_\epsilon M_m$, and the last inequality comes from $\sum_{\tau\subset\pi\otimes\pi'} d_\tau = d_\pi d_{\pi'}$. Note again that we are taking multiplicity into account in the notation $\tau \subset \pi\otimes\pi'$.

Similarly, for the case of $A_\Delta(G)$ we need to check that

$$d_\pi^{-\frac{1}{2}} d_{\pi'}^{-\frac{1}{2}} \left\|\Gamma(A)(\pi,\pi')\right\|_{S^2_{d_\pi}\otimes_\epsilon S^2_{d_{\pi'}}} \leq \sup_{\rho\in\widehat{G}} d_\rho^{-\frac{1}{2}} \|A(\rho)\|_{S^2_{d_\rho}}$$

for any $A \in \ell^\infty(d_\pi^{-\frac{1}{2}}, S^2_{d_\pi})$ and $\pi, \pi' \in \widehat{G}$. Indeed, we have

$$\begin{aligned}\left\|\Gamma(A)(\pi,\pi')\right\|^2_{S^2_{d_\pi}\otimes_\epsilon S^2_{d_{\pi'}}} &\leq \left\|\bigoplus_{\tau\subset\pi\otimes\pi'} A(\tau)\right\|^2_{S^2_{d_\pi d_{\pi'}}} = \sum_{\tau\subset\pi\otimes\pi'} \|A(\tau)\|^2_{S^2_{d_\tau}}\\ &= \sum_{\tau\subset\pi\otimes\pi'} d_\tau d_\tau^{-1} \|A(\tau)\|^2_{S^2_{d_\tau}}\\ &\leq (\sum_{\tau\subset\pi\otimes\pi'} d_\tau)\Big[\sup_{\rho\in\widehat{G}} d_\rho^{-\frac{1}{2}} \|A(\rho)\|_{S^2_{d_\rho}}\Big]^2\\ &= d_\pi d_{\pi'}\Big[\sup_{\rho\in\widehat{G}} d_\rho^{-\frac{1}{2}} \|A(\rho)\|_{S^2_{d_\rho}}\Big]^2.\end{aligned}$$

The first ineqality comes from the contractive embedding $S^2_{nm} \subset S^2_n \otimes_\epsilon S^2_m$. □

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