JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **24**, No. 2, June 2011

CLASSIFICATION OF EQUIVARIANT VECTOR BUNDLES OVER REAL PROJECTIVE PLANE

Min Kyu Kim*

Abstract. We classify equivariant topological complex vector bundles over real projective plane under a compact Lie group (not necessarily effective) action. It is shown that nonequivariant Chern classes and isotropy representations at (at most) three points are sufficient to classify equivariant vector bundles over real projective plane except one case. To do it, we relate the problem to classification on two-sphere through the covering map because equivariant vector bundles over two-sphere have been already classified.

1. Introduction

In the previous paper [4], we have classified equivariant topological complex vector bundles over S^2. In this paper, we classify equivariant topological complex vector bundles over real projective plane. This explains for one of two exceptional cases of [4].

We will consider only projective linear actions on real projective plane. We explain for this. Real projective plane $\mathbf{RP}^2$ is defined as $\mathbb{R}^3 \setminus (0,0,0) \Big/ \sim$ where the equivalence relation $\sim$ is given by

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$$

for nonzero real number λ, and let

$$o : S^2 \to \mathbf{RP}^2$$

be the usual covering map. The projective linear group $\mathrm{PGL}(3,\mathbb{R})$ is defined as the quotient group $\mathrm{GL}(3,\mathbb{R}) \Big/ Z\big(\mathrm{GL}(3,\mathbb{R})\big)$ where $Z\big(\mathrm{GL}(3,\mathbb{R})\big)$

Received March 19, 2011; Accepted May 16, 2011.
2010 Mathematics Subject Classification: Primary 57S25, 55P91; Secondary 20C99.
Key words and phrases: equivariant vector bundle, equivariant homotopy, isotropy representation.

is the center of $\mathrm{GL}(3,\mathbb{R})$, and let

$$o_{\mathrm{GL}(3,\mathbb{R})} : \mathrm{GL}(3,\mathbb{R}) \to \mathrm{PGL}(3,\mathbb{R})$$

be the qutient map. Let $\mathrm{PGL}(3,\mathbb{R})$ act usually on $\mathbf{RP}^2$. For a compact Lie group G, a topological G-action on $\mathbf{RP}^2$ is called *projective linear* if the action is given by a continuous homomorphism $\rho : G \to \mathrm{PGL}(3,\mathbb{R})$. Since $\mathrm{PO}(3,\mathbb{R}) = o_{\mathrm{GL}(3,\mathbb{R})}\big(\mathrm{O}(3)\big)$ is a maximal compact subgroup of $\mathrm{PGL}(3,\mathbb{R})$, we henceforward may assume

$$\rho(G) \subset \mathrm{PO}(3,\mathbb{R}). \tag{1.1}$$

In Section 2, we show that a topological action on $\mathbf{RP}^2$ by a compact Lie group is conjugate to a projective linear action.

To state main results, we need introduce some terminologies and notations. Let a compact Lie group G act projective linearly (not necessarily effectively) on $\mathbf{RP}^2$ through a homomorphism $\rho : G \to \mathrm{PGL}(3,\mathbb{R})$ to satisfy (1.1). Let $\mathrm{Vect}_G(\mathbf{RP}^2)$ be the set of isomorphism classes of topological complex G-vector bundles over $\mathbf{RP}^2$. For a bundle E in $\mathrm{Vect}_G(\mathbf{RP}^2)$ and a point x in $\mathbf{RP}^2$, denote by E_x the isotropy G_x-representation on the fiber at x. Put $H = \ker \rho$, i.e. the kernel of the G-action on $\mathbf{RP}^2$. Let $\mathrm{Irr}(H)$ be the set of characters of irreducible complex H-representations which has a G-action defined as

$$(g \cdot \chi)(h) = \chi(g^{-1}hg)$$

for $\chi \in \mathrm{Irr}(H)$, $g \in G$, $h \in H$. For $\chi \in \mathrm{Irr}(H)$, an H-representation is called χ-isotypical if its character is a multiple of χ. We slightly generalize this concept. For $\chi \in \mathrm{Irr}(H)$ and a compact Lie group K satisfying $H \lhd K < G$ and $K \cdot \chi = \chi$, a K-representation W is called χ-*isotypical* if $\mathrm{res}_H^K W$ is χ-isotypical, and we denote by $\mathrm{Vect}_K(\mathbf{RP}^2, \chi)$ the set

$$\Big\{ \ [E] \in \mathrm{Vect}_K(\mathbf{RP}^2) \ \Big| \ E_x \text{ is } \chi\text{-isotypical for each } x \in S^2 \ \Big\}$$

where $\mathbf{RP}^2$ delivers the restricted K-action. As in [4], our classification is reduced to $\mathrm{Vect}_{G_\chi}(\mathbf{RP}^2, \chi)$ for each $\chi \in \mathrm{Irr}(H)$. Details are found in [2, Section 2].

The classification of $\mathrm{Vect}_{G_\chi}(\mathbf{RP}^2, \chi)$ is highly dependent on the G_χ-action on the base space $\mathbf{RP}^2$. So, we would list all possible actions on $\mathbf{RP}^2$ up to conjugacy in terms of covering action, and then assign an equivariant simplicial (or CW) complex structure on $\mathbf{RP}^2$ to each action when $\rho(G_\chi)$ is finite. For these, we first introduce some polyhedra in $\mathbb{R}^3$. Let P_m for $m \geq 3$ be the regular m-gon on xy-plane in $\mathbb{R}^3$ whose center is the origin and one of whose vertices is $(1, 0, 0)$. And,

1. $|\mathcal{K}_m|$ is defined as the boundary of the convex hull of P_m, $S = (0,0,-1)$, $N = (0,0,1)$,
2. $|\mathcal{K}_\mathrm{T}|$ is defined as the regular tetrahedron which is the boundary of the convex hull of four points $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$, $(-\frac{1}{3},-\frac{1}{3},\frac{1}{3})$, $(-\frac{1}{3},\frac{1}{3},-\frac{1}{3})$, $(\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$, and which is inscribed to $|\mathcal{K}_4|$,
3. $|\mathcal{K}_\mathrm{I}|$ is defined as a regular icosahedron which has the origin as the center.

With these, denote natural simplicial complex structures on $|\mathcal{K}_m|$, $|\mathcal{K}_\mathrm{T}|$, $|\mathcal{K}_\mathrm{I}|$ by $\mathcal{K}_m$, $\mathcal{K}_\mathrm{T}$, $\mathcal{K}_\mathrm{I}$, respectively. It is well-known that each closed subgroup of SO(3) is conjugate to one of the following subgroups [6, Theorem 11]:

1. $\mathbb{Z}_n$ generated by the rotation a_n through the angle $2\pi/n$ around z-axis,
2. D_n generated by a_n and the rotation b through the angle π around x-axis,
3. the tetrahedral group T which is the rotation group of $|\mathcal{K}_\mathrm{T}|$,
4. the octahedral group O which is the rotation group of $|\mathcal{K}_4|$,
5. the icosahedral group I which is the rotation group of $|\mathcal{K}_\mathrm{I}|$,
6. SO(2) which is the set of rotations around z-axis,
7. O(2) which is defined as $\langle \mathrm{SO}(2), b\rangle$,
8. SO(3) itself.

Let $\imath : \mathrm{SO}(3) \to \mathrm{PO}(3) \subset \mathrm{PGL}(3,\mathbb{R})$ be the monomorphism defined as

$$o_{\mathrm{GL}(3,\mathbb{R})} \circ i_{\mathrm{GL}(3,\mathbb{R})} \big|_{\mathrm{SO}(3)}$$

where $i_{\mathrm{GL}(3,\mathbb{R})} : \mathrm{O}(3) \to \mathrm{GL}(3,\mathbb{R})$ is the usual inclusion. Put

$$\bar{\rho} : G_\chi \longrightarrow \mathrm{SO}(3), \quad g \longmapsto \imath^{-1}\big(\rho(g)\big)$$

for $g \in G_\chi$ which is defined by (1.1). And, define the homomorphism

$$\hat{\rho} : G_\chi \times Z \longrightarrow \mathrm{O}(3), \quad (g, g_0^j) \longmapsto \bar{\rho}(g) \cdot g_0^j$$

for $g \in G_\chi$, $j \in \mathbb{Z}_2$ where Z is the centralizer $\{\mathrm{id}, -\mathrm{id}\}$ of O(3) and we denote $-\mathrm{id} \in Z$ by g_0 to avoid symbolic confusion. In Section 2, it is shown that each compact subgroup of $\mathrm{PGL}(3,\mathbb{R})$ is conjugate to $\imath(\bar{R})$ for an $\bar{R}$-entry of Table 1.1. Denote $\imath(\bar{R})$ by R, and put $\hat{R} = \bar{R} \times Z$ where the notation $\times$ means the internal direct product of two subgroups in O(3). Henceforward, it is assumed that $\rho(G_\chi) = R$ for some $\bar{R}$. Then, images of ρ, $\bar{\rho}$, $\hat{\rho}$ are R, $\bar{R}$, $\hat{R}$, respectively. Note that the case of $\bar{R} = \mathbb{Z}_2$ is conjugate to $\bar{R} = \mathrm{D}_1$. So, we exclude D_1 in Table 1.1. To each finite $\bar{R}$, we assign a simplicial complex $\hat{\mathcal{K}}_{\hat{R}}$ of Table 1.1 where $\mathcal{K}_\mathrm{O}$ is defined in the below. The reason for our choice of $\hat{\mathcal{K}}_{\hat{R}}$ is explained in Section 3.

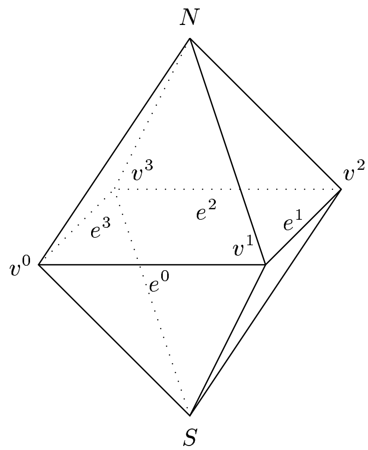
Let O(3) and their subgroups act usually on $\mathbb{R}^3$. Then, the underlying space $|\hat{\mathcal{K}}_{\hat{R}}|$ of $\hat{\mathcal{K}}_{\hat{R}}$ is invariant under the $\hat{R}$-action on $\mathbb{R}^3$ so that $|\hat{\mathcal{K}}_{\hat{R}}|$ inherits the $\hat{R}$-action from which $\hat{\mathcal{K}}_{\hat{R}}$ also carries the induced $\hat{R}$-action. And, $\hat{\mathcal{K}}_{\hat{R}}$ and $|\hat{\mathcal{K}}_{\hat{R}}|$ deliver the $(G_\chi \times Z)$-action through $\hat{\rho}$. These actions induce R- and G_χ-actions on $|\hat{\mathcal{K}}_{\hat{R}}|/Z$. Then, $(G_\chi \times Z)$- and G_χ- actions on $|\hat{\mathcal{K}}_{\hat{R}}|$ and $|\hat{\mathcal{K}}_{\hat{R}}|/Z$ are equal to $(G_\chi \times Z)$- and G_χ- actions on S^2 and $\mathbf{RP}^2$ when we regard $|\hat{\mathcal{K}}_{\hat{R}}|$ and $|\hat{\mathcal{K}}_{\hat{R}}|/Z$ as S^2 and $\mathbf{RP}^2$, respectively. And, we regard o as the orbit map from $|\hat{\mathcal{K}}_{\hat{R}}|$ to $|\hat{\mathcal{K}}_{\hat{R}}|/Z$. We can give a natural equivariant simplicial (or CW) complex structure on $|\hat{\mathcal{K}}_{\hat{R}}|/Z$ so that o is an equivariant cellular map. Sometimes, we consider the restricted G_χ-action on S^2 and $|\hat{\mathcal{K}}_{\hat{R}}|$.

TABLE 1.1. $\hat{\mathcal{K}}_{\hat{R}}$, $\hat{D}_{\hat{R}}$, $\hat{d}^{-1}$ for each $\bar{R}$

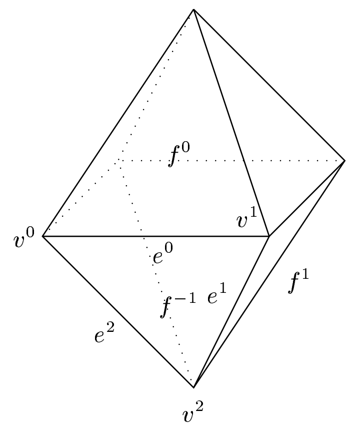
$\bar{R}$	$\hat{\mathcal{K}}_{\hat{R}}$	$\hat{D}_{\hat{R}}$	$\hat{d}^{-1}$
$\mathbb{Z}_n$, odd n	$\mathcal{K}_{2n}$	$\|e^0\|$	S
$\mathbb{Z}_n$, even n	$\mathcal{K}_n$	$\|e^0\|$	S
D_n, odd n, $n>1$	$\mathcal{K}_{2n}$	$[v^0, b(e^0)]$	S
D_n, even n	$\mathcal{K}_n$	$[v^0, b(e^0)]$	S
T	$\mathcal{K}_{\mathrm{O}}$	$\|e^0\|$	$b(f^{-1})$
O	$\mathcal{K}_{\mathrm{O}}$	$[v^0, b(e^0)]$	$b(f^{-1})$
I	$\mathcal{K}_{\mathrm{I}}$	$[v^0, b(e^0)]$	$b(f^{-1})$
SO(2)		$\{v^0\}$	S
O(2)		$\{v^0\}$	S
SO(3)		$\{S\}$	S

In dealing with equivariant vector bundles over $\mathbf{RP}^2$, we need to consider isotropy representations at (at most) three points. We will express those points as the image of some points in S^2 under o. To specify those points, we introduce some more notations. When $m \geq 3$, denote by v^i the vertex $\exp\left(\frac{2\pi i\sqrt{-1}}{m}\right)$ of $\mathcal{K}_m$, and by e^i the edge of $\mathcal{K}_m$ connecting v^i and v^{i+1} for $i \in \mathbb{Z}_m$. These notations are illustrated in Figure 1.1.(a). When we use the notation $\mathbb{Z}_m$ to denote an index set, it is just the group $\mathbb{Z}/m\mathbb{Z}$ of integers modulo m. In [4], $\mathcal{K}_m$ and its v^i's, e^i's for $m=2$ are also defined. We would define similar notations for $\mathcal{K}_{\mathrm{T}}$ and $\mathcal{K}_{\mathrm{I}}$. For $\mathcal{K}_{\mathrm{T}}$ and $\mathcal{K}_{\mathrm{I}}$, pick two adjacent faces in each case, and call them f^{-1} and f^0. And, label vertices of f^{-1} as v^i for $i \in \mathbb{Z}_3$ to satisfy

1. v^0, v^1, v^2 are arranged in the clockwise way around f^{-1},
2. v^0, v^1 are contained in $f^{-1} \cap f^0$.



(a) $\mathcal{K}_4$



(b) $\mathcal{K}_{\mathrm{O}}$

FIGURE 1.1. $\mathcal{K}_4$ and $\mathcal{K}_{\mathrm{O}}$

For $i \in \mathbb{Z}_3$, denote by e^i be the edge connecting v^i and v^{i+1}, and by f^i the face which is adjacent to f^{-1} and contains the edge e^i. We distinguish the superscripts -1 and 2 only for f^i, i.e. $f^{-1} \neq f^2$ in contrast to $v^{-1} = v^2$, $e^{-1} = e^2$. Here, we define one more simplicial complex denoted by $\mathcal{K}_{\mathrm{O}}$ which is the same simplicial complex with $\mathcal{K}_4$ but has the same convention of notations v^i, e^i, f^{-1}, f^i with $\mathcal{K}_{\mathrm{T}}, \mathcal{K}_{\mathrm{I}}$. Also, put $|\mathcal{K}_{\mathrm{O}}| = |\mathcal{K}_4|$. These notations are illustrated in Figure 1.1.(b). With these notations, we explain for $\hat{D}_{\hat{R}}$-entry of Table 1.1. To each finite $\bar{R}$, we assign a path $\hat{D}_{\hat{R}}$ (called the (closed) *one-dimensional fundamental domain*) in $|\hat{\mathcal{K}}_{\hat{R}}|$ which is listed in the third column of Table 1.1 where $b(\sigma)$ is the barycenter of σ for any simplex σ and $[x, y]$ is the shortest path in the underlying space $|\mathcal{K}|$ for any simplicial complex $\mathcal{K}$ and two points x, y in $|\mathcal{K}|$. And, let $\hat{d}^0$ and $\hat{d}^1$ be boundary points of $\hat{D}_{\hat{R}}$ such that $\hat{d}^0$ is nearer to v^0 than $\hat{d}^1$. For each finite $\bar{R}$, we define one more point $\hat{d}^{-1} \in |\hat{\mathcal{K}}_{\hat{R}}|$ which is listed in the fourth column in Table 1.1. If $\bar{R}$ is one-dimensional, then denote by $\hat{D}_{\hat{R}}$ the one point set $\{v^0 = (1, 0, 0)\}$, and let $\hat{d}^{-1}$, $\hat{d}^0$, $\hat{d}^1$ be equal to S, v^0, v^0, respectively. Similarly, if $\bar{R}$ is three-dimensional, then denote by $\hat{D}_{\hat{R}}$ the one point set $\{S\}$, and let $\hat{d}^{-1}$, $\hat{d}^0$, $\hat{d}^1$ be all equal to S. So far, we have defined $\hat{d}^{-1}$, $\hat{d}^0$, $\hat{d}^1$ for each $\bar{R}$. Put $d^i = o(\hat{d}^i)$ for $i \in I^+$ where $I = \{0, 1\}$ and $I^+ = \{-1, 0, 1\}$. Then, d^{-1}, d^0, d^1 are wanted points of $\mathbf{RP}^2$ according to $\bar{R}$, and we will consider the restriction $E|_{\{d^{-1},\ d^0,\ d^1\}}$ for each E in $\mathrm{Vect}_{G_\chi}(\mathbf{RP}^2,\chi)$. We define the semigroup which will be shown to be equal to the set of all the restrictions. If $\bar{R}$ is finite, let $C(\hat{d}^i)$ be the shortest path in $|\hat{\mathcal{K}}_{\hat{R}}|$ connecting $\hat{d}^{-1}$ with $\hat{d}^i$ for $i \in I$. Otherwise, let $C(\hat{d}^i)$ be the shortest path in S^2 connecting $\hat{d}^{-1}$ with $\hat{d}^i$ for $i \in I$. And, denote images

$$o(\hat{D}_{\hat{R}}), \quad o\Big(C(\hat{d}^0)\Big), \quad o\Big(C(\hat{d}^1)\Big)$$

by

$$D_R, \qquad C(d^0), \qquad C(d^1),$$

respectively.

Definition 1.1. For $\chi \in \mathrm{Irr}(H)$, assume that $\rho(G_\chi) = R$ for some $\bar{R}$ of Table 1.1. Let $A_{G_\chi}(\mathbf{RP}^2,\chi)$ be the semigroup of all th triples $(W_{d^{-1}}, W_{d^0}, W_{d^1})$ in $\mathrm{Rep}\Big((G_\chi)_{d^{-1}}\Big) \times \mathrm{Rep}\Big((G_\chi)_{d^0}\Big) \times \mathrm{Rep}\Big((G_\chi)_{d^1}\Big)$ satisfying

i) $W_{d^{-1}}$ is χ-isotypical,

ii) $\mathrm{res}^{(G_\chi)_{d^{-1}}}_{(G_\chi)_{C(d^i)}} W_{d^{-1}} \cong \mathrm{res}^{(G_\chi)_{d^i}}_{(G_\chi)_{C(d^i)}} W_{d^i}$ for $i \in I$,

iii) $\mathrm{res}^{(G_\chi)_{d^0}}_{(G_\chi)_{D_R}} W_{d^0} \cong \mathrm{res}^{(G_\chi)_{d^1}}_{(G_\chi)_{D_R}} W_{d^1}$,

iv) $W_{d^1} \cong {}^{g}W_{d^0}$ if there exists $g \in G_\chi$ such that $gd^0 = d^1$

where $(G_\chi)_T$ is the subgroup of G_χ fixing all points of a subset $T \subset \mathbf{RP}^2$. And, let $p'_{\mathrm{vect}} : \mathrm{Vect}_{G_\chi}(\mathbf{RP}^2,\chi) \to A_{G_\chi}(\mathbf{RP}^2,\chi)$ be the map defined as $E \mapsto (E_{d^{-1}}, E_{d^0}, E_{d^1})$.

Remark 1.2. It might seem that $A_{G_\chi}(\mathbf{RP}^2,\chi)$ is defined in a different way with $A_{G_\chi}(S^2,\chi)$ of [4]. But, these two definitions are defined in the exactly same way by [4, Lemma 3.10.].

Well-definedness of p'_{vect} is proved in Proposition 2.4. For notational simplicity, denote a triple $(W_{d^{-1}}, W_{d^0}, W_{d^1})$ by $(W_{d^i})_{i\in I^+}$. Now, we can state main results. Let $c_1 : \mathrm{Vect}_{G_\chi}(\mathbf{RP}^2,\chi) \to H^2(\mathbf{RP}^2)$ be the map defined as $[E] \mapsto c_1(E)$.

Theorem A. *Assume that $\rho(G_\chi)$ is equal to R for some $\bar{R}$ of Table 1.1 except $\mathbb{Z}_n$ with odd n. Then, p'_{vect} is an isomorphism.*

Theorem B. *Assume that $\rho(G_\chi) = R$ for $\bar{R} = \mathbb{Z}_n$ with odd n. Then, the preimage $p'^{-1}_{\mathrm{vect}}(\mathbf{W})$ has exactly two elements for each $\mathbf{W}$ in $A_{G_\chi}(\mathbf{RP}^2,\chi)$. And, $[E \oplus E_1] \neq [E \oplus E_2]$ for any bundles E, E_1, E_2 in $\mathrm{Vect}_{G_\chi}(\mathbf{RP}^2,\chi)$ such that $p'_{\mathrm{vect}}([E_1]) = p'_{\mathrm{vect}}([E_2])$ and $[E_1] \neq [E_2]$. Also, the following hold:*

1. *if* $\chi(\mathrm{id})$ *is even, then Chern classes of two elements of* $p'^{-1}_{\mathrm{vect}}(\mathbf{W})$ *are the same,*
2. *if* $\chi(\mathrm{id})$ *is odd, then Chern classes of two elements of* $p'^{-1}_{\mathrm{vect}}(\mathbf{W})$ *are different.*

COROLLARY 1.3. *Assume that* $\rho(G_\chi) = R$ *for* $\bar{R} = \mathbb{Z}_n$ *with odd* n. *Then, we have*

$$\mathrm{Vect}_{G_\chi}(\mathbf{RP}^2, \chi) \cong \mathrm{Vect}_R(\mathbf{RP}^2)$$

as semigroups, and $\mathrm{Vect}_R(\mathbf{RP}^2)$ *is generated by line bundles. Also,* $A_R(\mathbf{RP}^2, \mathrm{id})$ *is generated by all the elements with one-dimensional entries where we simply denote by* id *the trivial character of the trivial group. The number of such elements in* $A_R(\mathbf{RP}^2, \mathrm{id})$ *is equal to* n.

This paper is organized as follows. In Section 2, we list all closed subgroups of $\mathrm{PGL}(3,\mathbb{R})$ up to conjugacy, and show that a topological action on $\mathbf{RP}^2$ by a compact Lie group is conjugate to a projective linear action. Also, we show that p'_{vect} is well-defined. In Section 3, we prove Theorem A through covering action. In Section 4, we prove Theorem B and Corollary 1.3 through equivariant clutching construction.

2. Compact subgroups of $\mathrm{PGL}(3,\mathbb{R})$

In this section, we list all compact subgroups of $\mathrm{PGL}(3,\mathbb{R})$ up to conjugacy, and show that a topological action on $\mathbf{RP}^2$ by a compact Lie group is conjugate to a projective linear action. Also, we show that p'_{vect} is well-defined.

First, we define covering action of the G_χ-action on $\mathbf{RP}^2$. We call $G_\chi \times Z$ and its action on S^2 through $\hat{\rho}$ the covering group of G_χ and the covering action of the G_χ-action on $\mathbf{RP}^2$, respectively. Here, we note that two kernels of the $(G_\chi \times Z)$-action on S^2 and the G_χ-action on $\mathbf{RP}^2$ are equal because $\imath$ is injective. By using covering action, we list all closed subgroups of $\mathrm{PGL}(3,\mathbb{R})$ up to conjugacy.

PROPOSITION 2.1. *Each compact subgroup of* $\mathrm{PGL}(3,\mathbb{R})$ *is conjugate to one of* R *for some* $\bar{R}$*-entry in Table 1.1.*

Proof. Let K be an arbitrary compact subgroup of $\mathrm{PGL}(3,\mathbb{R})$, and let $\rho : K \to \mathrm{PGL}(3,\mathbb{R})$ be the inclusion. We may assume that $K \subset \mathrm{PO}(3,\mathbb{R})$ by (1.1). Since the codomain of $\bar{\rho}$ is equal to $\mathrm{SO}(3)$, the image of $\bar{\rho}$ is conjugate to one of

$$\mathbb{Z}_n, \quad \mathrm{D}_n, \quad \mathrm{T}, \quad \mathrm{O}, \quad \mathrm{I}, \quad \mathrm{SO}(2), \quad \mathrm{O}(2), \quad \mathrm{SO}(3).$$

This shows that the image of ρ is conjugate to one of R for some $\bar{R}$-entry in Table 1.1 because $\imath$ is injective. So, we obtain a proof. □

Now, we prove that a compact Lie group action on $\mathbf{RP}^2$ is conjugate to a projective linear action.

PROPOSITION 2.2. *If a compact Lie group G act topologically on $\mathbf{RP}^2$, then it is conjugate to a projective linear action.*

Proof. By [1, Theorem I.9.3], the G-action on $\mathbf{RP}^2$ has a covering G'-action on S^2 where G' satisfies a short exact sequence

$$\langle \mathrm{id} \rangle \to \mathbb{Z}_2 \longrightarrow G' \xrightarrow{\mathrm{pr}} G \to \langle \mathrm{id} \rangle$$

for some homomorphism pr and the G'-action on S^2 satisfies

$$o(g' \cdot \hat{x}) = \mathrm{pr}(g') \cdot o(\hat{x})$$

for $g' \in G'$ and $\hat{x} \in S^2$. It is well-known that a topological action on S^2 by a compact Lie group is conjugate to a linear action [5, Theorem 1.2], [3]. By using this, we may assume that the G'-action is linear, and we obtain a proof. □

To prove well-definedness of p'_{vect}, we state a basic lemma.

LEMMA 2.3. *Let G be a compact Lie group acting topologically on a topological space X. And, let E be an equivariant vector bundle over X. Then, ${}^gE_x \cong E_{gx}$ for each $g \in G$ and $x \in X$. Also,*

$$\mathrm{res}^{G_x}_{G_x \cap G_{x'}} E_x \cong \mathrm{res}^{G_{x'}}_{G_x \cap G_{x'}} E_{x'}$$

for any two points x, x' in the same component of the fixed set $X^{G_x \cap G_{x'}}$.

PROPOSITION 2.4. *p'_{vect} is well-defined.*

Proof. To show well-definedness, we should show that $(E_{d^i})_{i \in I^+}$ is contained in $A_{G_\chi}(\mathbf{RP}^2, \chi)$ for any E in $\mathrm{Vect}_{G_\chi}(\mathbf{RP}^2, \chi)$. By definition of $\mathrm{Vect}_{G_\chi}(\mathbf{RP}^2, \chi)$, it is easy that $E_{d^{-1}}$ is χ-isotypical so that $(E_{d^i})_{i \in I^+}$ satisfies Definition 1.1.i). By the first statement of Lemma 2.3, $(E_{d^i})_{i \in I^+}$ satisfies Definition 1.1.iv). The remaining are Definition 1.1.ii), iii). These are satisfied by the second statement of Lemma 2.3. □

3. Relation between equivariant vector bundles over $\mathbf{RP}^2$ and S^2.

In this section, we investigate the relation between $\mathrm{Vect}_{G_\chi}(\mathbf{RP}^2,\chi)$ and $\mathrm{Vect}_{G_\chi\times Z}(S^2,\chi)$, and the relation between semigroups $A_{G_\chi}(\mathbf{RP}^2,\chi)$ and $A_{G_\chi\times Z}(S^2,\chi)$. By these, we can prove Theorem A.

The first relation is easy. Pullback through the covering map gives the semigroup homomorphism

$$O:\mathrm{Vect}_{G_\chi}(\mathbf{RP}^2)\longrightarrow \mathrm{Vect}_{G_\chi\times Z}(S^2),\quad E\longmapsto o^*E. \tag{3.1}$$

This is an isomorphism because the Z-action on S^2 is free and $\mathbf{RP}^2\cong S^2/Z$, see [7, p. 132]. Since equivariant vector bundles over S^2 have been classified in [4], we can classify equivariant vector bundles over $\mathbf{RP}^2$ by using this isomorphism. This is the reason why $\hat{\mathcal{K}}_{\hat{R}}$ of Table 1.1 is defined as the simplicial complex $\mathcal{K}_{\hat{R}}$ of [4, Table 1.1]. In [4, Section 3], it is shown that the $(G_\chi\times Z)$-action on $\mathbb{R}^3$ through $\hat\rho$ preserves $\hat{\mathcal{K}}_{\hat{R}}$ and its underlying space $|\hat{\mathcal{K}}_{\hat{R}}|$. Also, it is shown that the $(G_\chi\times Z)$-orbit of $\hat{D}_{\hat{R}}$ covers the underlying space $|\hat{\mathcal{K}}^{(1)}_{\hat{R}}|$ of the 1-skeleton $\hat{\mathcal{K}}^{(1)}_{\hat{R}}$, and that $\hat{D}_{\hat{R}}$ is a minimal path satisfying such a property.

Next, we show that $A_{G_\chi}(\mathbf{RP}^2,\chi)$ is isomorphic to $A_{G_\chi\times Z}(S^2,\chi)$. For this, we state an easy lemma on isotropy subgroups of G_χ and $G_\chi\times Z$. For any $\hat{x}\in S^2$ and $x=o(\hat{x})$, let

$$p_1:G_\chi\times Z\to G_\chi \qquad\text{and}\qquad p_{1,\hat{x}}:(G_\chi\times Z)_{\hat{x}}\to (G_\chi)_x$$

be the projection to the first components. And, let $p^*_{1,\hat{x}}:R\Big((G_\chi)_x\Big)\to R\Big((G_\chi\times Z)_{\hat{x}}\Big)$ be the isomorphism sending a $(G_\chi)_x$-representation W to W itself regarded as a $(G_\chi\times Z)_{\hat{x}}$-representation through $p_{1,\hat{x}}$.

LEMMA 3.1. *For any points $\hat{x},\hat{y}$ in S^2 and their images $x=o(\hat{x})$, $y=o(\hat{y})$ in $\mathbf{RP}^2$, the following hold*

1. $p_1\Big((G_\chi\times Z)_{\hat{x}}\Big)=(G_\chi)_x$ *and* $p_1\Big|_{(G_\chi\times Z)_{\hat{x}}}$ *is injective,*
2. $p_1\Big((G_\chi\times Z)_{C(\hat{d}^i)}\Big)=(G_\chi)_{C(d^i)}$ *and* $p_1\Big|_{(G_\chi\times Z)_{C(\hat{d}^i)}}$ *is injective for* $i\in I$,
3. $p_1\Big((G_\chi\times Z)_{\hat{D}_{\hat{R}}}\Big)=(G_\chi)_{D_R}$ *and* $p_1\Big|_{(G_\chi\times Z)_{\hat{D}_{\hat{R}}}}$ *is injective,*
4. $p^*_{1,\hat{x}}(E_x)\cong(o^*E)_{\hat{x}}$ *as* $(G_\chi\times Z)_{\hat{x}}$*-representations for each* E *in the set* $\mathrm{Vect}_{G_\chi}(\mathbf{RP}^2,\chi)$.

Proof. (1) It is easy that $p_1\Big((G_\chi \times Z)_{\hat{x}}\Big) \subset (G_\chi)_x$. For each $g \in (G_\chi)_x$, one of (g, id) and (g, g_0) fixes $\hat{x}$ because $g \cdot \hat{x} = \pm\hat{x}$. So, $p_1\Big((G_\chi \times Z)_{\hat{x}}\Big) = (G_\chi)_x$ is obtained. Now, we prove injectivity of $p_1\Big|_{(G_\chi \times Z)_{\hat{x}}}$. Pick arbitrary two elements (g, g_0^j) and $(g', g_0^{j'})$ of $(G_\chi \times Z)_{\hat{x}}$ for $j, j' \in \mathbb{Z}_2$. If $p_1(g, g_0^j) = p_1(g', g_0^{j'})$, i.e. $g = g'$, then j should be equal to j'. This shows injectivity.

(2) and (3) are obtained by applying (1) to each point $\hat{x}$ in $C(\hat{d}^i)$ and $\hat{D}_{\hat{R}}$.

(4) is easy by definition of pullback. □

Proposition 3.2. *Let $P_1 : A_{G_\chi}(\mathbf{RP}^2, \chi) \to A_{G_\chi \times Z}(S^2, \chi)$ be the map defined as*

$$\big(W_{d^i}\big)_{i \in I^+} \longmapsto \big(p^*_{1,\hat{d}^i} W_{d^i}\big)_{i \in I^+}.$$

Then, P_1 is bijective.

Proof. For each $\mathbf{W} = (W_{d^{-1}}, W_{d^0}, W_{d^1})$ in $A_{G_\chi}(\mathbf{RP}^2, \chi)$, Lemma 3.1 says that the element $P_1(\mathbf{W})$ satisfies Definition 1.1.i)∼iii). Put $W_{\hat{d}^i} = p^*_{1,\hat{d}^i}(W_{d^i})$ for $i \in I$. It is easily shown that there exists g in $G_\chi \times Z$ such that $g\hat{d}^0 = \hat{d}^1$ if and only if there exists g' in G_χ such that $g'd^0 = d^1$ where g, g' satisfy $p_1(g) = g'$. Then, we can also show that if ${}^{g'}W_{d^0} \cong W_{d^1}$ holds, then ${}^{g}W_{\hat{d}^0} \cong W_{\hat{d}^1}$. So, $P_1(\mathbf{W})$ satisfies Definition 1.1.iv), and $P_1(\mathbf{W})$ is contained in $A_{G \times Z}(S^2, \chi)$. Similarly, we can also show that P_1 has an inverse. □

Now, we can prove Theorem A.

Proof of Theorem A. Observe that $p'_{\mathrm{vect}} = P_1^{-1} \circ p_{\mathrm{vect}} \circ O$ where the map p_{vect} on $\mathrm{Vect}_{G_\chi \times Z}(S^2, \chi)$ is defined in [4]. Note that $O, p_{\mathrm{vect}}, P_1$ are all isomorphisms by (3.1), [4, Theorem C], Proposition 3.2, respectively. Therefore, p'_{vect} is also an isomorphism.

$$\begin{array}{ccc} \mathrm{Vect}_{G_\chi}(\mathbf{RP}^2, \chi) & \xrightarrow{O} & \mathrm{Vect}_{G_\chi \times Z}(S^2, \chi) \\ \Big\downarrow{\scriptstyle p'_{\mathrm{vect}}} & & \Big\downarrow{\scriptstyle p_{\mathrm{vect}}} \\ A_{G_\chi}(\mathbf{RP}^2, \chi) & \xrightarrow{P_1} & A_{G_\chi \times Z}(S^2, \chi) \end{array} \tag{3.2}$$

□

4. Proof of Theorem B

In this section, we deal with the case of $\bar{R} = \mathbb{Z}_n$ with odd n. In this case, we can check that

$$R_{d^{-1}} \cong \mathbb{Z}_n \qquad \text{and} \qquad R_{d^0} = R_{d^1} = \langle \mathrm{id} \rangle.$$

Pick an arbitrary $\mathbf{W} = (W_{d^i})_{i \in I^+}$ in $A_{G_\chi}(\mathbf{RP}^2, \chi)$. By [4, Theorem B], the map p_{vect} of diagram (3.2) is surjective and $p_{\mathrm{vect}}^{-1}\Big(P_1(\mathbf{W})\Big)$ consists of two elements. Also since O and P_1 are isomorphic by (3.1) and Proposition 3.2, the diagram (3.2) shows that p'_{vect} is surjective and $p'^{-1}_{\mathrm{vect}}(\mathbf{W})$ consists of two elements. To prove Theorem B, we should calculate Chern classes of these two. Pick an element g_1 of G_χ satisfying $\bar{\rho}(g_1) = a_n$. To calculate Chern classes, we would describe bundles in $p'^{-1}_{\mathrm{vect}}(\mathbf{W})$ as equivariant clutching construction. For this, we first express $\mathbf{RP}^2 \cong |\hat{\mathcal{K}}_{\hat{R}}|/Z$ as the quotient of more simple equivariant simplicial complex. To use results of [4], we do similar things for $p_{\mathrm{vect}}^{-1}\Big(P_1(\mathbf{W})\Big)$ and $S^2 \cong |\hat{\mathcal{K}}_{\hat{R}}|$.

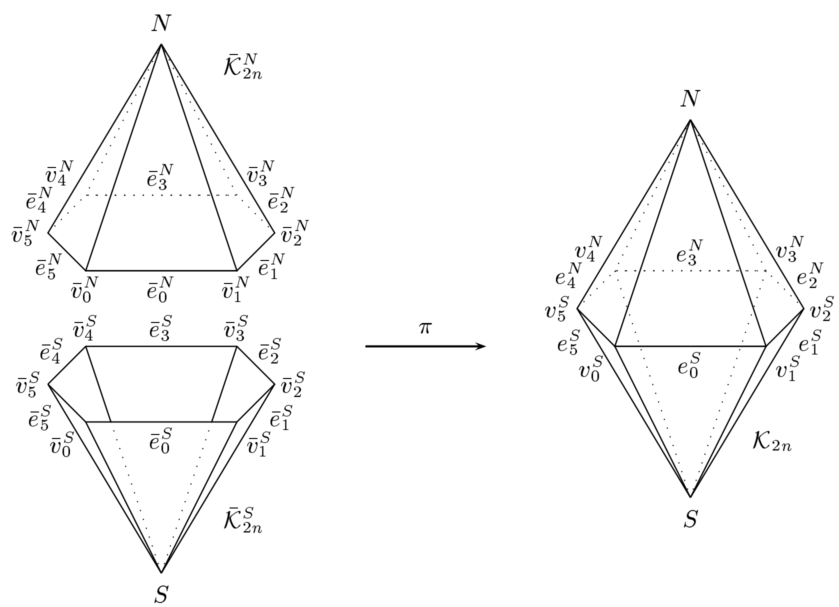


FIGURE 4.1. $\pi : \bar{\mathcal{K}}_{2n} \to \mathcal{K}_{2n}$ when $n = 3$

In Table 1.1, $\hat{\mathcal{K}}_{\hat{R}} = \mathcal{K}_{2n}$. Denote by $\mathcal{K}_{2n}^S$ the lower simplicial subcomplex of $\mathcal{K}_{2n}$ such that

$$|\mathcal{K}_{2n}^S| \;=\; |\mathcal{K}_{2n}| \;\cap\; \Big\{ \;(x,y,z) \in \mathbb{R}^3 \;\Big|\; z \le 0 \;\Big\},$$

and by $\mathcal{K}_{2n}^N$ the upper part. Denote by B the set $\{S, N\}$, and by $\bar{\mathcal{K}}_{2n}$ the disjoint union $\amalg_{q\in B}\, \mathcal{K}_{2n}^q$. Denote $\mathcal{K}_{2n}^q \subset \bar{\mathcal{K}}_{2n}$ by $\bar{\mathcal{K}}_{2n}^q$. The $(G_\chi \times Z)$-action and its restricted G_χ-action on $\mathcal{K}_{2n}$ induce $(G_\chi \times Z)$- and G_χ-actions on $\bar{\mathcal{K}}_{2n}$ and $|\bar{\mathcal{K}}_{2n}|$ where the G_χ-action on $\bar{\mathcal{K}}_{2n}$ preserves $\bar{\mathcal{K}}_{2n}^q$. Then, $\mathcal{K}_{2n}$ can be regarded as a quotient of $\bar{\mathcal{K}}_{2n}$. This is expressed by the simplicial map $\pi : \bar{\mathcal{K}}_{2n} \to \mathcal{K}_{2n}$ such that $\pi|_{\mathcal{K}_{2n}^q}$ is the identity to $\mathcal{K}_{2n}^q$ for $q \in B$. Denote by $|\pi| : |\bar{\mathcal{K}}_{2n}| \to |\mathcal{K}_{2n}|$ the underlying space map of π. Easily, π and $|\pi|$ are $(G_\chi \times Z)$-maps. Here, we introduce the following notations:

$$\begin{aligned}
\bar{P}_{2n}^q &= |\bar{\mathcal{K}}_{2n}^q| \cap |\pi|^{-1}(P_{2n}),\\
\bar{P}_{2n} &= \bar{P}_{2n}^S \cup \bar{P}_{2n}^N,\\
\bar{v}_q^i &= \bar{\mathcal{K}}_{2n}^q \cap \pi^{-1}(v^i),\\
\bar{e}_q^i &= \bar{\mathcal{K}}_{2n}^q \cap \pi^{-1}(e^i)
\end{aligned}$$

for $q \in B$ and $i \in I$. Also, let $c : \bar{P}_{2n} \to \bar{P}_{2n}$ be the map satisfying $c(\bar{x}) \ne \bar{x}$ and $|\pi|(\bar{x}) = |\pi|(c(\bar{x}))$ for each $\bar{x} \in \bar{P}_{2n}$. It is easy that c is $(G_\chi \times Z)$-equivariant. In this time, we do the similar thing for $|\mathcal{K}_{2n}|/Z$.

$$\begin{array}{ccc}
|\bar{\mathcal{K}}_{2n}| & \xrightarrow{\;\bar{o}\;} & |\mathcal{K}_R| \\
{\scriptstyle |\pi|}\big\downarrow & & \big\downarrow{\scriptstyle |\pi'|} \\
|\mathcal{K}_{2n}| & \xrightarrow{\;o\;} & |\mathcal{K}_{2n}|/Z
\end{array} \tag{4.1}$$

Put $\mathcal{K}_R = \bar{\mathcal{K}}_{2n}^S$. Let $|\pi'| : |\mathcal{K}_R| \to |\mathcal{K}_{2n}|/Z$ be the map defined by $\bar{x} \mapsto \big(o \circ |\pi|\big)(\bar{x})$. Then, $|\pi'|$ is a cellular G_χ-map, and we consider $|\mathcal{K}_{2n}|/Z$ as the quotient of $|\mathcal{K}_R|$ through $|\pi'|$. Also, let $\bar{o} : |\bar{\mathcal{K}}_{2n}| \to |\mathcal{K}_R|$ be the map defined by

$$\bar{o}(\bar{x}) = \begin{cases} \bar{x} & \text{for } \bar{x} \in |\bar{\mathcal{K}}_{2n}^S|, \\ g_0\bar{x} & \text{for } \bar{x} \in |\bar{\mathcal{K}}_{2n}^N|. \end{cases}$$

So defined maps $|\pi'|$ and $\bar{o}$ satisfy the commutative diagram (4.1).

Now, we describe equivariant vector bundles over $|\mathcal{K}_{2n}|/Z$ as an equivariant clutching construction of an equivariant vector bundle over $|\mathcal{K}_R|$, and then do the similar thing for equivariant vector bundle over $|\mathcal{K}_{2n}|$. Put $\mathbf{W} = (W_{d^{-1}}, W_{d^0}, W_{d^1})$. Let F be the equivariantly trivial G_χ-bundle $|\mathcal{K}_R| \times W_{d^{-1}}$ over $|\mathcal{K}_R|$. Here, note that $|\pi'|(S) = d^{-1}$, and that both $(G_\chi)_S$ and $(G_\chi)_{d^{-1}}$ are equal to G_χ. Then, we can show that

$$F \cong |\pi'|^* E' \qquad \text{for each } E' \in p'^{-1}_{\mathrm{vect}}(\mathbf{W}) \tag{4.2}$$

because $\left(|\pi'|^* E'\right)_S \cong E'_{d^{-1}} \cong W_{d^{-1}}$ and $|\mathcal{K}_R|$ is equivariant homotopically equivalent to the one point set $\{S\} \subset |\mathcal{K}_R|$. Let $\Phi : \bar{P}^S_{2n} \to \mathrm{Iso}(W_{d^{-1}})$ be an arbitrary continuous map which is called a *preclutching map* with respect to F where Iso means the set of nonequivariant isomorphisms. We try to glue F along $\bar{P}^S_{2n}$

$$\bar{P}^S_{2n} \times W_{d^{-1}} \longrightarrow \bar{P}^S_{2n} \times W_{d^{-1}}, \qquad (\bar{x}, u) \mapsto \left(g_0 c(\bar{x}), \Phi(\bar{x})u\right)$$

for $\bar{x} \in \bar{P}^S_{2n}$ and $u \in W_{d^{-1}}$ via a preclutching map Φ. This gluing gives an equivariant vector bundle in $p'^{-1}_{\mathrm{vect}}(\mathbf{W})$ if and only if the following two conditions hold:

E1$'$. $\Phi\left(g_0 c(\bar{x})\right) = \Phi(\bar{x})^{-1}$ for each $\bar{x} \in \bar{P}^S_{2n}$,
E2$'$. $\Phi(g\bar{x}) = g\Phi(\bar{x})g^{-1}$ for each $g \in G_\chi$ and $\bar{x} \in \bar{P}^S_{2n}$.

A preclutching map satisfying these two conditions is called an *equivariant clutching map* with respect to F. By (4.2), two elements of $p'^{-1}_{\mathrm{vect}}(\mathbf{W})$ can be constructed in this way. We do the similar thing for equivariant vector bundle over $|\mathcal{K}_{2n}|$. Let $\bar{F}$ be the $(G_\chi \times Z)$-bundle $\bar{o}^* F$ over $|\bar{\mathcal{K}}_{2n}|$. Easily, $\bar{F}$ is equal to the equivariantly trivial bundle $|\bar{\mathcal{K}}_{2n}| \times p_1^* W_{d^{-1}}$. The $(G_\chi \times Z)$-action on it is expressed by

$$g \cdot (\bar{x}, u) = (g\bar{x}, gu),$$
$$g_0 \cdot (\bar{x}, u) = (g_0\bar{x}, u)$$

for $g \in G_\chi$, $\bar{x} \in |\mathcal{K}_{2n}|$, $u \in p_1^* W_{d^{-1}}$. And, we can show that

$$\bar{F} \cong |\pi|^* E \qquad \text{for each } E \in p^{-1}_{\mathrm{vect}}\left(P_1(\mathbf{W})\right) \tag{4.3}$$

because $|\bar{\mathcal{K}}_{2n}|$ is equivariant homotopically equivalent to the set $B \subset |\bar{\mathcal{K}}_{2n}|$. Let $\bar{\Phi} : \bar{P}_{2n} \to \mathrm{Iso}(p_1^* W_{d^{-1}})$ be an arbitrary continuous map. We call it a *preclutching map* with respect to $\bar{F}$. We try to glue $\bar{F}$ along $\bar{P}_{2n}$

$$\bar{P}_{2n} \times p_1^* W_{d^{-1}} \longrightarrow \bar{P}_{2n} \times p_1^* W_{d^{-1}}, \qquad (\bar{x}, u) \mapsto (c(\bar{x}), \bar{\Phi}(\bar{x})u)$$

for $\bar{x} \in \bar{P}_{2n}$ and $u \in p_1^* W_{d^{-1}}$ via a preclutching map $\bar{\Phi}$. This gluing gives an equivariant vector bundle in $p^{-1}_{\mathrm{vect}}\left(P_1(\mathbf{W})\right)$ if and only if the following two conditions hold:

E1. $\bar{\Phi}(c(\bar{x})) = \bar{\Phi}(\bar{x})^{-1}$ for each $\bar{x} \in \bar{P}_{2n}$,
E2. $\bar{\Phi}(\bar{g}\bar{x}) = \bar{g}\bar{\Phi}(\bar{x})\bar{g}^{-1}$ for each $\bar{g} \in G_\chi \times Z$ and $\bar{x} \in \bar{P}_{2n}$.

A preclutching map satisfying these two conditions is called an *equivariant clutching map* with respect to $\bar{F}$. Note that equivariant clutching maps $\bar{\Phi}$, Φ with respect to $\bar{F}$, F are actually $\mathrm{Iso}_H(p_1^* W_{d^{-1}})$, $\mathrm{Iso}_H(W_{d^{-1}})$-valued by equivariance, respectively. Here, $\mathrm{Iso}_H(\cdot)$ is the subgroup of H-equivariant elements in $\mathrm{Iso}(\cdot)$. Also since

$$\mathrm{res}_H^{G_\chi \times Z} p_1^* W_{d^{-1}} = \mathrm{res}_H^{G_\chi} W_{d^{-1}},$$

we have $\mathrm{Iso}_H(p_1^* W_{d^{-1}}) = \mathrm{Iso}_H(W_{d^{-1}})$. Details on equivariant clutching construction can be found in [4, Section 4].

Let $\Omega_{\bar{F}}$ and Ω_F be sets of equivariant clutching maps with respect to $\bar{F}$ and F, respectively. To use results of [4], we need relate Ω_F to $\Omega_{\bar{F}}$. Consider the map

$$q_\Omega : \Omega_{\bar{F}} \to \Omega_F, \quad \bar{\Phi} \longmapsto \Phi(\bar{x}) = g_0 \bar{\Phi}(\bar{x})$$

for $\bar{x} \in \bar{P}_{2n}^S$. Then, we have the following:

LEMMA 4.1. *q_Ω is a well-defined one-to-one correspondence.*

Proof. First, we show that $q_\Omega(\bar{\Phi})$ for each $\bar{\Phi}$ is actually contained in Ω_F. Put $\Phi = q_\Omega(\bar{\Phi})$. For this, we show that Φ satisfies Condition E1$'$., E2$'$. Condition E1$'$. is proved as follows:

$$\begin{aligned} \Phi\big(g_0 c(\bar{x})\big) &= g_0 \bar{\Phi}\big(g_0 c(\bar{x})\big) \\ &= \bar{\Phi}\big(c(\bar{x})\big) g_0^{-1} \\ &= \bar{\Phi}(\bar{x})^{-1} g_0^{-1} \\ &= \Big(g_0^{-1}\Phi(\bar{x})\Big)^{-1} g_0^{-1} \\ &= \Phi(\bar{x})^{-1} \end{aligned}$$

for $\bar{x} \in \bar{P}_{2n}^S$. And, Condition E2$'$. is proved as follows:

$$\begin{aligned} \Phi\big(g\bar{x}\big) &= g_0 \bar{\Phi}\big(g\bar{x}\big) \\ &= g_0 g \bar{\Phi}\big(\bar{x}\big) g^{-1} \\ &= g g_0 \bar{\Phi}\big(\bar{x}\big) g^{-1} \\ &= g \Phi\big(\bar{x}\big) g^{-1} \end{aligned}$$

for each $g \in G_\chi$ and $\bar{x} \in \bar{P}_{2n}^S$. So, $q_\Omega(\bar{\Phi})$ is contained in Ω_F.

Second, we construct the inverse of q_Ω to show bijectivity of it. Consider the map

$$q_\Omega^{-1} : \Omega_F \longrightarrow \Omega_{\bar{F}}, \quad \Phi \mapsto \bar{\Phi}(\bar{x}) = \begin{cases} g_0^{-1}\Phi(\bar{x}) & \text{for } \bar{x} \in \bar{P}_{2n}^S, \\ \Big(g_0^{-1}\Phi\big(c(\bar{x})\big)\Big)^{-1} & \text{for } \bar{x} \in \bar{P}_{2n}^N. \end{cases}$$

We show that $\bar{\Phi} = q_\Omega^{-1}(\Phi)$ for each Φ is actually contained in $\Omega_{\bar{F}}$. For this, we show that $\bar{\Phi}$ satisfies Condition E1., E2. Condition E1. is proved by definition of $\bar{\Phi}$. We prove Condition E2. only when $\bar{x} \in \bar{P}_{2n}^S$ and $\bar{g} = gg_0$ for some $g \in G_\chi$ as follows:

$$\begin{aligned}
\bar{\Phi}(\bar{g}\bar{x}) = \bar{\Phi}(gg_0\bar{x}) &= \Big(g_0^{-1}\Phi\big(c(gg_0\bar{x})\big)\Big)^{-1} \\
&= \Phi\big(c(gg_0\bar{x})\big)^{-1} g_0 \\
&= \Phi\big(g_0 c(g\bar{x})\big)^{-1} g_0 \\
&= \Phi\big(g\bar{x}\big) g_0 \\
&= g\Phi\big(\bar{x}\big) g^{-1} g_0 \\
&= gg_0 g_0^{-1}\Phi\big(\bar{x}\big) g^{-1} g_0 \\
&= gg_0 \bar{\Phi}\big(\bar{x}\big) g_0^{-1} g^{-1} \\
&= \bar{g}\bar{\Phi}\big(\bar{x}\big)\bar{g}^{-1}
\end{aligned}$$

where we use equivariance of c. Proof of Condition E2. for other $\bar{x}$'s and $\bar{g}$'s is proved similarly. Since q_Ω and q_Ω^{-1} are inverses of each other, we obtain a proof. ☐

For convenience in calculation, we parameterize each $|e^i|$, $|\bar{e}_q^i|$ in $|\mathcal{K}_{2n}|$, $|\bar{\mathcal{K}}_{2n}|$ for $0 \le i \le 2n-1$ by $t \in [i, i+1]$ linearly to satisfy $v_S^i \mapsto i$ and $\bar{v}_S^i \mapsto i$. Vertices v_S^0, $\bar{v}_S^0$ are parameterized by 0 or $2n$ according to context. Pick a path $\sigma : [0,1] \to \mathrm{Iso}_H(W_{d^{-1}})$ such that $\sigma(0) = \sigma(1) = \mathrm{id}$ and $[\sigma]$ is a generator of $\pi_1\Big(\mathrm{Iso}_H(W_{d^{-1}}), \mathrm{id}\Big)$. For a path γ defined on $[0,1]$, define a path γ^{tr} on $[0,1]$ as $\gamma^{tr}(t) = \gamma(1-t)$ for $t \in [0,1]$, and a path γ_{-i} on $[i, i+1]$ as $\gamma_{-i}(t) = \gamma(t-i)$ for $i \in \mathbb{Z}$. Now, we prove Theorem B and Corollary 1.3.

Proof of Theorem B. We easily obtain the first and second statements through the isomorphism (3.1) because the preimage $p_{\mathrm{vect}}^{-1}\big(P_1(\mathbf{W})\big)$ contained in $\mathrm{Vect}_{G_\chi \times Z}(S^2, \chi)$ has two elements by [4, Theorem B]. In [4, Proof of Theorem B], these two are described as equivariant clutching constructions of $\bar{F}$ via equivariant clutching maps $\bar{\Phi}$, $\bar{\Phi}'$ with respect to $\bar{F}$ satisfying the following:

1. $\bar{\Phi}|_{\bar{P}_{2n}^S} \equiv \mathrm{id}$,
2. $\bar{\Phi}'|_{\bar{P}_{2n}^S}(t) = \begin{cases} \sigma(t) & \text{for } t \in [0,1], \\ (\sigma^{tr})_{-1}(t) & \text{for } t \in [1,2]. \end{cases}$

Since $\bar{\Phi}'|_{\bar{P}_{2n}^S}(t) = \sigma \vee (\sigma^{tr})_{-1}\ (t)$ for $t \in [0,2]$ and $\bar{\Phi}'$ is equivariant, we also have

$$\bar{\Phi}'|_{\bar{P}_{2n}^S}(t) = g_1^i\Big(\ \sigma \vee (\sigma^{tr})_{-1}\ (t-2i)\Big)g_1^{-i}$$

for an integer $0 \le i \le n-1$ and $t \in [2i, 2i+2]$ where $\bar{\rho}(g_1) = a_n$. Put $\Phi = q_\Omega(\bar{\Phi})$ and $\Phi' = q_\Omega(\bar{\Phi}')$. By Lemma 4.1, they are equivariant clutching maps with respect to F. Denote by E and E' the equivariant vector bundles determined by Φ and Φ', respectively. It is easy that the E is equivariantly trivial. Since Φ' is equivariant, it is determined by its values on the half $[0,n]$ of $\bar{P}_{2n}^S$. Observe that Φ' on $[1,n]\cup[n+1,2n]$ does not contribute to $c_1(E')$ because n is odd and the pair $g_1^i\sigma g_1^{-i}$ and $g_1^{i'}(\sigma^{tr})_{-1}g_1^{-i'}$ for $0 \le i, i' \le n-1$ cancel each other in the fundamental group $\pi_1\Big(\ \mathrm{Iso}_H(W_{d^{-1}}), \mathrm{id}\ \Big)$. In other words, E' is nonequivariantly isomorphic to the nonequivariant vector bundle E'' determined by the preclutching map Φ'' with respect to F defined by

$$\begin{aligned}
\Phi''(t) &= \mathrm{id} && \text{for } t \in [1,n]\cup[n+1,2n],\\
\Phi''(t) &= \sigma(t) && \text{for } t \in [0,1],\\
\Phi''(t) &= \sigma(t-n)^{-1} && \text{for } t \in [n, n+1].
\end{aligned}$$

Then, we can show that

$$c_1(E'') \equiv \chi(\mathrm{id}) \mod 2$$

in $H^2\big(\mathbf{RP}^2, \mathbb{Z}\big)$ by [4, Lemma 7.1] because $[\sigma]$ is a generator of the fundamental group $\pi_1\Big(\ \mathrm{Iso}_H(W_{d^{-1}}), \mathrm{id}\ \Big)$. Therefore, we obtain a proof. □

Proof of Corollary 1.3. By [4, Theorem D], $\mathrm{Vect}_{G_\chi \times Z}(S^2, \chi)$ has a rank $\chi(\mathrm{id})$-bundle so that $\mathrm{Vect}_{G_\chi}(\mathbf{RP}^2, \chi)$ has a rank $\chi(\mathrm{id})$-bundle by the isomorphism (3.1). From this, we have

$$\mathrm{Vect}_{G_\chi}(\mathbf{RP}^2, \chi) \cong \mathrm{Vect}_R(\mathbf{RP}^2)$$

as semigroups by [2, Lemma 2.2]. Also, we obtain

$$\mathrm{Vect}_{\bar{R}\times Z}(S^2) \cong \mathrm{Vect}_R(\mathbf{RP}^2)$$

by the isomorphism (3.1). It is easy that $\mathrm{Vect}_R(\mathbf{RP}^2)$ is generated by line bundles and that $A_R(\mathbf{RP}^2, \mathrm{id})$ is generated by all the elements with one-dimensional entries by [4, Theorem D]. Since $R_x = \langle \mathrm{id} \rangle$ for each $x \in o(P_{2n})$, any triple $(W_{d^i})_{i \in I^+}$ in $\mathrm{Rep}\Big(R_{d^{-1}}\Big) \times \mathrm{Rep}\Big(R_{d^0}\Big) \times \mathrm{Rep}\Big(R_{d^1}\Big)$ is contained in $A_R(\mathbf{RP}^2, \mathrm{id})$ if and only if W_{d^i}'s are same dimensional. So, the number of all the elements in $A_R(\mathbf{RP}^2, \mathrm{id})$ with one-dimensional entries is equal to n because $R_{d^{-1}} \cong \mathbb{Z}_n$ and $R_{d^0} = R_{d^0} = \langle \mathrm{id} \rangle$. Therefore, we obtain a proof. $\square$

References

[1] G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York and London, 1972.

[2] J.-H. Cho, S. S. Kim, M. Masuda, D. Y. Suh, *Classification of equivariant complex vector bundles over a circle*, J. Math. Kyoto Univ. **41** (2001), 517-534.

[3] A. Constantin, B. Kolev, *The theorem of Kerekjarto on periodic homeomorphisms of the disk and the sphere*, Enseign. Math. (2) **40** (1994), 193-204.

[4] M. K. Kim, *Classification of equivariant vector bundles over two-sphere*, arXiv:1005.0681v2.

[5] B. Kolev, *Sous-groupes compacts d'homeomorphismes de la sphere*, Enseign. Math. (2) **52** (2006), no. 3-4, 193-214.

[6] E. G. Rees, *Notes on geometry*, Universitext, Springer-Verlag, Berlin-New York, 1983.

[7] G. Segal, *Equivariant K-theory*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 129-151.

*

Department of Mathematics Education
Gyeongin National University of Education
Incheon 407-753, Republic of Korea
E-mail: mkkim@kias.re.kr