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MINIMAL GRAPHS WITH PLANAR ENDS

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Abstract. In this article, we consider an unbounded minimal graph $M \subset \mathbf{R}^3$ which is contained in a slab. Assume that ∂M consists of two Jordan curves lying in parallel planes, which is symmetric with the reflection under a plane. If the asymptotic behavior of M is also symmetric in some sense, then we prove that the minimal graph is itself symmetric along the same plane.

1. Introduction

In 1956, M. Shiffman [8] proved three elegant theorems about a minimal annulus A lying on a slab $S(-1, 1)$, where

$$S(a, b) := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, a \leq x_3 \leq b,\ a, b \in \mathbf{R}\},$$

such that the boundary curves are continuous convex Jordan curves contained in parallel planes at height ± 1, respectively, as follows:

1. For all $-1 < t < 1$, the intermediate curve $A \cap P_t$ contained on the horizontal plane P_t is a strictly convex Jordan curve where
$$P_t := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, x_3 = t\}.$$
In particular, the minimal annulus A is an embedding.
2. If ∂A is a union of circles, then $A \cap P_t$ is a circle for all $-1 \leq t \leq 1$.
3. If ∂A is symmetric with respect to a plane perpendicular to x_1x_2-plane, then A is symmetric with respect to the same plane.

We will consider the third result of Shiffman in the case of unbounded minimal graphs. Recall that the asymptotic behavior of a complete, embedded minimal surface in $\mathbf{R}^3$ with finite total curvature is well-understood. Particularly, R. Schoen [7] demonstrated that, after a rotation, each embedded end of a complete minimal surface with finite total

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curvature can be parametrized as a graph u over the exterior of a disk in the (x_1, x_2)-plane by;

$$u(x_1, x_2) = \beta_1 + \beta_2 \log r + \frac{\alpha_1 x_1 + \alpha_2 x_2}{r^2} + O(r^{-2}) \tag{1.1}$$

for $r = (x_1^2 + x_2^2)^{1/2}$ sufficiently large, where α_1, α_2, β_1 and β_2 are real constants and $O(r^{-2})$ denotes a function such that $r^2 O(r^{-2})$ is bounded as $r \to \infty$. If $\beta_2 = 0$ then the end is asymptotic to a plane, we say it a *planar end*, otherwise it is asymptotic to a catenoid. From the physical point of view, minimal surfaces in $\mathbf{R}^3$ are objects submitted to a balanced force system. Precisely, the action of the unit conormal vector around a closed curve in the surface relates a tendency of rotation around an axis, i.e., angular momentum, which is expressed by the torque vector of the surface around the closed curve.

In this paper, we will prove a symmetry result about an unbounded minimal graph which has an asymptotic symmetric property around its end as following:

THEOREM 1.1. *Let $M \subset S(a, b)$ be a minimal graph with an end E such that ∂M consists of two Jordan curves Γ_1 and Γ_2 lying in parallel planes of $\partial S(a, b)$. And let $\Gamma_1 \cup \Gamma_2$ be symmetric with respect to the reflection under a plane P. If the torque of E has the normal direction of P, then M is itself symmetric under the reflection through the same plane.*

2. Main results

Since M is contained in a slab, the end E must close to a horizontal plane by [7]. We may assume that $a < 0 < b$ of $S(a, b)$ and $\beta_1 = 0$ in (1.1), i.e., E closes to P_0, asymptotically. Notice that, the minimal graph M is conformally equivalent to the punctured annulus

$$A_R^* := \{z \in \mathbf{C} \mid \frac{1}{R} < |z| < R\} \setminus \{p\}, \quad \frac{1}{R} < |p| < R$$

where $R > 1$. We have a conformal harmonic map

$$X = (X^1, X^2, X^3) : A_R^* \to \mathbf{R}^3$$

such that $X(A_R^*) = M$. Then, after a suitable conformal change of the domain, we can say that

$$X^3(z) = \frac{1}{\log r} \log |z - p|$$

for some $r > 1$. Moreover, except at P_0, M meets every horizontal plane in a compact Jordan curve, and hence X^3 can be continuously extended to the whole domain A_R. It is clear that

$$2\frac{\partial X^3}{\partial z} = \frac{d}{dz}\left(\frac{1}{\log r}\log(z-p)\right) = \frac{1}{\log r}\frac{1}{(z-p)}.$$

By the Weierstrass representation theorem, see [2], it is well known that

$$g(z)f(z) = 2\frac{\partial X^3}{\partial z}$$

where g is the stereographic projection of the Gauss map of X with respect to the north pole, just say the Gauss map of X, and f is a holomorphic function on A_R. Since both z and dz have no a zero or a pole at $z = 0$, the (extended) Gauss map has the minimum branching order two at the puncture. In general, for a planar end E which closes to a plane Π, $(E \cap \Pi) \setminus B$, B being a large ball, consists of $2k-2$ curves which are asymptotic to $2k-2$ rays on Π making an equal angle of $\pi/(k-1)$. In particular, if the (extended) Gauss map g has a zero (or a pole) of the minimum branching order 2, i.e., $g(z) = z^2$, at the puncture, then $M \cap \Pi$ is an immersion of $\mathbf{R}^1$ which is asymptotically parallel to the line in Π. In our case, $M \cap P_0$ approximate to the line ℓ, where

$$\ell: \quad \alpha_1 x_1 + \alpha_2 x_2 = 0, \quad x_3 = 0 \tag{2.1}$$

as the notations in (1.1), see [2]. Observe that we can define the flux and the torque of the planar end as following;

Proposition 2.1 ([3]). *We define the flux and the torque associated to the planar end E as those of one representative curve γ. With the simple calculation, we can compute that;*

$$\begin{aligned} Flux(E) &= \int_\gamma \nu\, ds = (0,0,0) \\ Torque(E) &= \int_\gamma X \wedge \nu\, ds = -\pi(-\alpha_2, \alpha_1, 0) \end{aligned} \tag{2.2}$$

where α_i, $i = 1, 2$, is defined as in (1.1). Observe that the torque of E does not depend upon the base point of X since the flux vanishes.

Note that we can take a representative curve of the planar end E, for example, $E \cap \partial B$. The above proposition shows us that the torque of E has the direction of $M \cap P_0$ asymptotically. Thus $Torque(E)$ describes the asymptotic behavior of the end. Now, without loss of generality, we may assume that $\partial M = \Gamma_1 \cup \Gamma_2$ is symmetric under the x_1x_3-plane, called P. Then $\alpha_2 = 0$, since the torque of E must be normal to P. Let us denote the minimal graph of E by;

$$u : \Omega \to \mathbf{R}$$

where $\Omega \subset P_0 = \mathbf{R}^2$ is an unbounded domain whose boundary consists of two disjoint Jordan curves. Until now, we have shown that

$$u(x_1, x_2) = \frac{\alpha_1 x_1}{r^2} + O(r^{-2}) \tag{2.3}$$

for large $r > 0$. Consider the reflection of M along the x_1x_3-plane, denoted by $Ref(M)$. Take another minimal graph $\tilde{u} : \Omega \to \mathbf{R}$ such that

$$\tilde{u}(x_1, x_2) = u(x_1, -x_2)$$

then the image of $\tilde{u}$ is $Ref(M)$. By the definition of $\tilde{u}$ and (2.3), it is clear that

$$\tilde{u}(x_1, x_2) = \frac{\alpha_1 x_1}{r^2} + O(r^{-2}).$$

Now we can conclude that

$$u - \tilde{u} = O(r^{-2}) \tag{2.4}$$

near the infinity. Since every minimal graph is harmonic on a conformal domain,

$$\Delta(u - \tilde{u}) = 0 \quad \text{on } \Omega$$

where Δ is the Laplace operator. Additionally,

$$u \equiv \tilde{u} \quad \text{on } \partial\Omega \tag{2.5}$$

since ∂M is invariant under the reflection along the x_1x_3-plane. Recall Ω is conformally equivalent to the punctured annulus A_R^*. Let

$$\phi : A_R^* \to \Omega \cup \{\infty\}$$

be a conformal mapping such that $\phi(p) = \infty$. By (2.4) the harmonic function

$$(u - \tilde{u}) \circ \phi : A_R \setminus \{p\} \to \mathbf{R}$$

has the removable singularity at the puncture p, so we can define the extending harmonic function $\tilde{\phi}$ on the whole domain A_R such that

$$\tilde{\phi}(z) = \begin{cases} (u - \tilde{u})(\phi(z)) & \text{if } z \in A_R^* \\ 0 & \text{if } z = p \end{cases}$$

Observe that $\tilde{\phi}|_{\partial A_R} \equiv 0$ by (2.5). Therefore the harmonic function $\tilde{\phi}$ is equal to zero on the whole domain A_R, and we can say that

$$u \equiv \tilde{u} \quad \text{on } A_R.$$

It shows that the minimal surface M is itself symmetric under the reflection along the x_1x_3-plane.

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