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THE UNIFORM DISTANCE SYSTEM WITH THE SYMMETRIC DISTANCE FUNCTION

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Abstract. Every topology is generated by a quasi-uniform distance system. The quasi-uniform distance system is a uniform distance system. The topology is uniformizable if and only if it is generated by the uniform distance system with the symmetric distance function.

1. Introduction

It is already shown that every topology will be generated by the quasi-uniform structure ([1],[3]). In [2] a mathematical system $\alpha = (W, \leq, P, d)$ called a *topological distance system for a set* X is introduced and shown that any topology will be also generated by this system. Hereby we observe that P is a quasi-uniform structure which is a uniform structure without a symmetric condition, and the distance function d from $X \times X$ into W should not be symmetric, either.

In this paper we will study the characteristics of topologies generated by the uniform distance system $\alpha = (W, \leq, P, d)$ which satisfies the so called 'half condition'(3.3 (b)). This half condition can be considered as a replacement of a triangle property of a metric. The name 'uniform' is not directly derived from the concept of the uniform structure. However, as a main result (3.10) of this paper, if the symmetric condition is added to the distance function d of uniform distance system α, we will see that the generating concept of topologies by this uniform distance system is equivalent to that of uniform structure.

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It is well known that a topology is generated by a uniform structure iff it is T_{3a} space([4]). Due to this theorem an interesting counterexample of a topology that will be generated by the topological distance system with symmetric distance function d but not by a uniform structure is given. This clarify the fact that the symmetric property alone of a distance function in a topological distance system is not enough to establish an equivalent relationships of both systems.

Here we recall some necessary notions appeared in [2].

A mathematical system $\alpha = (W, \leq, P, d)$ is called a *distance system for a set* X if it satisfies the following properties.

(a) $(W, \leq)$ is a partially ordered set.

(b) P is a nonempty subset of W and for each $\epsilon, \delta \in P$ there is $\sigma \in P$ such that $\sigma \leq \epsilon, \delta$. P is called a *positive area of* W and $(W, \leq, P)$ *a range area.*

(c) d is a function from $X \times X$ into W which is called a *distance function from* $X \times X$ *into* W.

We define the $\epsilon-$*ball* and the $\alpha-$*open* subset in the usual way of the general topology.

For every $\epsilon \in P$ and $x \in X$ $B^{\alpha}_{\epsilon}(x) := \{y | y \in X, d(x, y) \leq \epsilon\}$ (briefly $B_{\epsilon}(x)$ if clear which α is meant) and a subset T of X is $\alpha-$*open* if to each $x \in T$ there is $\epsilon \in P$ such that $B^{\alpha}_{\epsilon}(x) \subset T$.

Finally, if $\alpha = (W, \leq, P, d)$ is a distance system for a set X, then the set $\mathcal{O}_{\alpha} = \{T | T \subset X, T \text{ is } \alpha\text{−open}\}$ is a topology which is called *the topology generated by* α. Throughout this paper X is always a set, $\iota := \{(x, x) | x \in X\}$ and $\mathcal{P}(X)$ is a power set of X.

2. The quasi-uniform distance system

The triangle property of a metric plays an essential role for an $\epsilon-$ball to be a neighborhood of a point. However, in a weakened form of a metric it should not be satisfied automatically. Hence we require this property to the distance systems as a hypothesis as follows.

Definition 2.1. *$\alpha = (W, \leq, P, d)$ is called topological distance system for X if for all $x \in X$ and $\epsilon \in P$, $B^{\alpha}_{\epsilon}(x)$ is a neighborhood of x relative to the generated topology $\mathcal{O}_{\alpha}$.*

The next lemma is a useful tool to verify the topology generated by a topological distance system. We give this theorem without proof, because it folllows directly from the definition above.

Notation 2.2. *For a set X of sets and arbitrary x we set $X(x) := \{S|S \in X, x \in S\}$*

Lemma 2.3. *Let $(X, \mathcal{O})$ be a topological space. Let $(W, \leq, P)$ be a range area and d a distance function from $X \times X$ into W. Then $\alpha = (W, \leq, P, d)$ is a topological distance system for X with $\mathcal{O} = \mathcal{O}_\alpha$ iff for all $x \in X$ the following hold.*

(a) To each $\epsilon \in P$ there is $U \in \mathcal{O}(x)$ with $U \subset B_\epsilon(x)$.

(b) To each $U \in \mathcal{O}(x)$ there is $\epsilon \in P$ with $B_\epsilon(x) \subset U$

Definition 2.4. *Let $(X, \mathcal{O})$ be a topological space and $\Delta = \{\mathcal{U}|\mathcal{U} \subset \mathcal{O}, \mathcal{U}$ is finite and $X \in \mathcal{U}\}$. For all $x \in X$ and $\mathcal{U} \in \Delta$ put*

$$D^{\mathcal{U}}(x) := \bigcap_{U \in \mathcal{U}(x)} U$$

and define a relation $R_{\mathcal{U}}$ on X as follows:

$$xR_{\mathcal{U}}y \longleftrightarrow y \in D^{\mathcal{U}}(x) \text{ for all } x, y \in X.$$

With the definition above we are able to show that every topology will be generated by a topological distance system $\alpha = (W, \leq, P, d)$ (2.8, [2]) and this system is called a quasi-uniform distance system, because P in α is a quasi-uniform structure as the next theorem shows.

Notation 2.5. *If A, B are subsets of $X \times X$, then $A \cdot B := \{(x, y)|$ there is $z \in X$ such that $(x, z) \in A, (z, y) \in B\}$.*

Definition 2.6. *A nonempty subset $\mathcal{R}$ of $\mathcal{P}(X \times X)$ is called a quasi-uniform structure on X if the following conditions are satisfied.*

(QU1) If $A \subset X \times X$ and there exists $R \in \mathcal{R}$ such that $R \subset A$, then $A \in \mathcal{R}$.

(QU2) If $R, S \in \mathcal{R}$, then $R \cap S \in \mathcal{R}$.

(QU3) For all $S \in \mathcal{R}$, $\iota \subset S$

(QU4) For all $S \in \mathcal{R}$, there is $Q \in \mathcal{R}$ such that $Q \cdot Q \subset S$

Theorem 2.7. *Let $W = \{S|\iota \subset S \subset X \times X\}$ and $P = \{\epsilon|\epsilon \subset X \times X$, there is $\mathcal{U} \in \Delta$ such that $R_{\mathcal{U}} \subset \epsilon\}$. Then $(W, \subset, P)$ is a range area and P is a quasi-uniform structure on X.*

Proof. We prove only the latter case. Note first $P \neq \emptyset$.

(QU1) Obvious.

(QU2)The finite intersection property follows from the fact that $\mathcal{U}, \mathcal{V} \in \Delta$ implies $R_{\mathcal{U} \cup \mathcal{V}} \subset R_{\mathcal{U}} \cap R_{\mathcal{V}}$.

(QU3) Straightforawd.

(QU4) Let $\epsilon \in P$. There is one $\mathcal{U} \in \Delta$ such that $R_{\mathcal{U}} \subset \epsilon$. Putting $\delta := R_{\mathcal{U}}$, it sufficies to show $\delta = \delta \cdot \delta$.

Let $(x, y) \in \delta \cdot \delta$. There is $z \in X$ such that $(x, z), (z, y) \in \delta$, i.e. $z \in D^{\mathcal{U}}(x), y \in D^{\mathcal{U}}(z)$. For all $V \in \mathcal{U}(x), z \in D^{\mathcal{U}}(x)$ implies $z \in V$, thus $\mathcal{U}(x) \subset \mathcal{U}(z)$. From $y \in D^{\mathcal{U}}(z)$ follows $y \in D^{\mathcal{U}}(x)$. Therefore $xR_{\mathcal{U}}y$ which means $(x, y) \in R_{\mathcal{U}} = \delta$.

The other direction is obvious by $\iota \subset R_{\mathcal{U}}$ for all $\mathcal{U} \in \Delta$. □

Next we rewrite the Theorem and Definition 2.8 in [2] which says that every topology will be generated by a topological distance system. Since the positive area of this system is a quasi-uniform structure, we call the system *a quasi-uniform distance system.*

THEOREM AND DEFINITION 2.8. *Let $(X, \mathcal{O})$ be a topological space, $W = \{S|\iota \subset S \subset X \times X\}$ and $P = \{\epsilon|\epsilon \subset X \times X$, there is $\mathcal{U} \in \Delta$ such that $R_{\mathcal{U}} \subset \epsilon\}$. Let $d : X \times X \to W, (x, y) \mapsto \iota \cup \{(x, y)\}$. Then $\alpha = (W, \subset, P, d)$ is a topological distance system for X which generates $\mathcal{O}$, i.e. $\mathcal{O}_\alpha = \mathcal{O}$.*

$\alpha = (W, \subset, P, d)$ is called a quasi-uniform distance system for X.

THEOREM AND DEFINITION 2.9. *Let $\mathcal{R}$ be a quasi-uniform structure on $X \times X$. Let $R(x) := \{y|y \in X, (x, y) \in R\}$ for $R \in \mathcal{R}, x \in X$.*

We say that s subset T of X is $\mathcal{R}$-open if to each $x \in T$ there exists $R \in \mathcal{R}$ such that $R(x) \subset T$. Then $\mathcal{O}(\mathcal{R}) = \{T|T \subset X, T$ is $\mathcal{R}-$open$\}$ is a topology on X and we call it a topology generated by $\mathcal{R}$.

A topology $\mathcal{O}$ on X is said to be quasi-uniformizable if there exists a quasi-uniform structure $\mathcal{R}$ on X such that $\mathcal{O} = \mathcal{O}(\mathcal{R})$.

In 3.5, [2] it is shown that every topology is quasi-uniformizable. Hence $\mathcal{O}(\mathcal{R}) = \mathcal{O}_\alpha = \mathcal{O}$.

3. Uniform distance system

In this section we introduce a uniform distance system and survey the characteristics of topologies generated by this systems.

NOTATION 3.1. *For a subset A of $X \times X$, $A^{-1} := \{(y, x)|(x, y) \in A\}$.*

DEFINITION 3.2. *A nonempty subset $\mathcal{R}$ of $\mathcal{P}(X \times X)$ is called a uniform structure on X if the following conditions are satisfied.*

(U1) If $A \subset X \times X$ and there exists $R \in \mathcal{R}$ such that $R \subset A$, then $A \in \mathcal{R}$.

(U2) If $R, S \in \mathcal{R}$, then $R \cap S \in \mathcal{R}$.

(U3) For all $S \in \mathcal{R}$, $\iota \subset S$

(U4) For all $S \in \mathcal{R}$, there is $Q \in \mathcal{R}$ such that $Q \cdot Q \subset S$
(U5) $R \in \mathcal{R}$ implies $R^{-1} \in \mathcal{R}$.

Definition 3.3. *Let $\alpha := (W, \leq, P, d)$ be a distance system for a set X. We say α is uniform if it satisfies the following.*
(a) The reflexive condition: For all $\epsilon \in P, x \in X, d(x,x) \leq \epsilon$
(b) The half condition: To each $\epsilon \in P$ there is $\delta \in P$ *such that if $d(x,y)$, $d(y,z) \leq \delta$, then $d(x,z) \leq \epsilon$ for all $x, y, z \in X$.*

Example 3.4. *Every quasi-uniform distance system is a uniform distance system.*

Proof. The reflexive condition (a) of the distance function d of 2.8 is obviously satisfied. The half condition of d also can be easily proved, since to each $\epsilon \in P$ there exists $\delta \in P$ such that $\delta^2 \subset \epsilon$. □

Since every topology is generated by a quasi-uniform distance system, we obtain:

Remark 3.5. *Every topology will be generated by a uniform distance system*

Theorem 3.6. *Every uniform distance system is topological.*

Proof. Let $\alpha := (W, \leq, P, d)$ be a uniform distance system for a set X. We have to show that $B_\epsilon(x)$ is a neighborhood of x for all $x \in X$ and $\epsilon \in P$. Let $x \in X, \epsilon \in P$. Define

$$U := \{y | y \in X, \text{ there exists } \delta \in P \text{ such that } B_\delta(y) \subset B_\epsilon(x)\}$$

and we show (1) $U \subset B_\epsilon(x)$ and (2) $U \in \mathcal{O}(x)$ sequentially.

(1) Let $y \in U$. There exists one $\delta \in P$ such that $B_\delta(y) \subset B_\epsilon(x)$. By the reflection of d, $d(y,y) \leq \delta$ so that $y \in B_\delta(y)$. Hence $y \in B_\epsilon(x)$.

(2) It is obvious $x \in U$. Let $y \in U$. It is to show that there exists $\sigma \in P$ such that $B_\sigma(y) \subset U$. There exists $\delta \in P$ such that $B_\delta(y) \subset B_\epsilon(x)$. Since α satisfies the half condition, there exists $\sigma \in P$ such that $d(y,t), d(t,z) \leq \sigma$ implies $d(y,z) \leq \delta$ for all $t, z \in X$. Let $z \in B_\sigma(y)$. Then $d(y,z) \leq \sigma$. From $d(z,z) \leq \sigma$ and half condition of α follows $d(y,z) \leq \delta$, hence $z \in B_\delta(y) \subset B_\epsilon(x)$. Therefore $B_\sigma(y) \subset B_\epsilon(x)$. □

Definition 3.7. *A distance function d of a distance system $\alpha = (W, \leq, P, d)$ for X is said to be symmetric if $d(x,y) = d(y,x)$ for all $x, y \in X$.*

Definition 3.8. *A topology $\mathcal{O}$ on X is said to be uniformizable if there exists a uniform structure $\mathcal{R}$ on X such that $\mathcal{O} = \mathcal{O}(\mathcal{R})$.*

THEOREM 3.9. *A topology is uniformizable if and only if it is generated by a uniform distance system with a symmetric distance function.*

Proof. Let $(X, \mathcal{O})$ be a topological space.

'$\longrightarrow$': Let $\mathcal{O}$ be uniformizable. Then there exists a uniform structure $\mathcal{R}$ on X such that $\mathcal{O} = \mathcal{O}(\mathcal{R})$. Let $W := \{S | \iota \subset S \subset X \times X\}, P := \mathcal{R}$. Since P satisfies property (U2), $(W, \subset, P)$ is a range area. Let $d : X \times X \to \iota \cup \{(x, y), (y, x)\}$.

Then we will show the following sequentially.

(a) d is symmetric

(b) $\alpha := (W, \subset, P, d)$ is a uniform distance system on X

(c) α generates $\mathcal{O}$.

(a) d is obviously symmetric.

(b) We claim that α satisfies the reflexive and half condition. Since for all $x \in X$ and $\epsilon \in P, d(x, x) = \iota \subset \epsilon$, the reflexive condition of α is satisfied.

Let $\epsilon \in P$. We have to show that there exists $\delta \in P$ such that $d(x, y), d(y, z) \subset \delta$ implies $d(x, z) \subset \epsilon$ for all $x, y, z \in X$. Since P is a uniform structure on X, there exists $\delta \in P$ such that $\delta \cdot \delta \subset \epsilon$. Let $x, y, z \in X$ and $d(x, y), d(y, z) \subset \delta$. Then $(x, y), (y, z) \in \delta$, thus $(x, z) \in \delta \cdot \delta \subset \epsilon$. By the symmetric property of d, $(y, x), (z, y) \in \delta$, hence $(z, x) \in \delta \cdot \delta \subset \epsilon$. Therefore $d(x, z) \subset \epsilon$.

(c) By $\mathcal{O} = \mathcal{O}(R)$, by the lemma 2.3 it is enough to show for all $x \in X$:

(1) For all $\epsilon \in P$ there exists $\delta \in P$ such that $B_\delta(x) \subset \epsilon(x)$

(2) For all $\epsilon \in P$ there exists $\delta \in P$ such that $\delta(x) \subset B_\epsilon(x)$.

However, it is easy to see, because for all $x \in X$ and $\epsilon \in P$,

$$B_\epsilon(x) \subset \epsilon(x) \text{ and } (\epsilon \cap \epsilon^{-1})(x) \subset B_\epsilon(x).$$

'$\longleftarrow$': Let $\mathcal{O}$ be generated by a uniform distance system $\alpha := (W, \leq, P, d)$ where d is symmetric. We should show that there exists a uniform structure $\mathcal{R}$ on X such that $\mathcal{O} = \mathcal{O}(\mathcal{R})$. For this for all $\epsilon \in P$ we define a relation R_ϵ on X such as

$$x R_\epsilon y :\longleftrightarrow d(x, y) \leq \epsilon.$$

Let

$$\mathcal{R} := \{S | S \subset X \times X, \text{ there exists } \epsilon \in P \text{ with } R_\epsilon \subset S\}.$$

Then the following hold.

(a) $\mathcal{R}$ is a uniform struture on X.

(b) $\mathcal{O}$ will be generated by $\mathcal{R}$.

(a) First $\mathcal{R} \neq \emptyset$. $\mathcal{R}$ should satisfy the properies (U1)-(U5).

(U1): Obvious.

(U2) Let $S, T \in \mathcal{R}$. We should show that there exists $\sigma \in P$ such that $R_\sigma \subset S \cap T$. Then there are $\epsilon, \delta \in P$ such that $R_\epsilon \subset S$ and $R_\delta \subset T$. Choose $\sigma \in P$ such that $\sigma \leq \epsilon, \delta$ and we show $R_\sigma \subset R_\epsilon \cap R_\delta$. Let $(x, y) \in R_\sigma$. Then $d(x, y) \leq \sigma \leq \epsilon, \delta$ so that $(x, y) \in R_\epsilon \cap R_\delta \subset S \cap T$. Hence $R_\sigma \subset S \cap T$.

(U3) It follows directly from the reflexive condition of α.

(U4) Let $S \in \mathcal{R}$. It is to show that there exists $Q \in \mathcal{R}$ such that $Q \cdot Q \subset S$. There exists $\delta \in P$ such that $R_\delta \subset S$. By the half condition of α there exists $\sigma \in P$ such that $d(x, y), d(y, z) \leq \sigma$ implies $d(x, z) \leq \delta$ for all $x, y, z \in X$. Choose $Q := R_\sigma$. Let $(x, y) \in Q \cdot Q$. Then there exists $z \in X$ such that $(x, z), (z, y) \in Q = R_\sigma$ which means $d(x, z), d(z, y) \leq \sigma$, thus $d(x, y) \leq \delta$. Hence $(x, y) \in R_\delta \subset S$ that means $Q \cdot Q \subset S$.

(U5) Let $S \in \mathcal{R}$. We have to show that $S^{-1} \in \mathcal{R}$, i.e., there exists $\delta \in P$ such that $R_\delta \subset S^{-1}$. There exists $\epsilon \in P$ such that $R_\epsilon \subset S$. Since d is symmetric, the following are equivalent.

$$xR_\epsilon y; d(x, y) \leq \epsilon; d(y, x) \leq \epsilon; yR_\epsilon x.$$

Hence $R_\epsilon = R_\epsilon^{-1} \subset S^{-1}$.

(b) It suffices to show that it holds $B_\epsilon(x) = R_\epsilon(x)$ for all $x \in X$ and $\epsilon \in R$. Let $x \in X$ and $\epsilon \in P$. Then the following are equivalent.

$$y \in B_\epsilon(x); d(x, y) \leq \epsilon; (x, y) \in R_\epsilon; y \in R_\epsilon(x)$$

Hence $\mathcal{O}_\alpha = \mathcal{O}(\mathcal{R})$. Therefore $\mathcal{O} = \mathcal{O}(\mathcal{R})$. □

Theorem 3.10. *The following statements are equivalent.*

(a) A topological space $(X, \mathcal{O})$ is a T_{3a}-space.

(b) $\mathcal{O}$is uniformizable.

(c) $\mathcal{O}$ will be generated by a uniform distance system with a symmetric distance function.

In following a counterexample of a topology which is generated by a topological distance system with the symmetric distance function but not uniformizable is given.

Example 3.11. *Let $\mathcal{O}_{cof}$ be the cofinite topology on $\mathbb{N}$. Then*

(a) $\mathcal{O}_{cof}$ is generated by a topological distance system $\alpha := (\mathbb{N}, \mathbb{R}_{\geq 0}, \mathbb{R}_{>0}, d)$ where d is a real symmetric distance function defined as follows.

$$d : X \times X \to \mathbb{R}\geq 0, (x, y) \mapsto \begin{cases} 0 & \text{if } x = y \\ \frac{1}{|x-y|} & \text{if } x \neq y \end{cases}$$

(b) $\mathcal{O}_{cof}$ is not a T_{3a}-space, hence it is not uniformizable.

Proof. Througout this proof the notations $[a,b], [a,b[(a,b \in \mathbb{R}, a < b)$ are usual intervals on $\mathbb{R}$.

(a) Since d is obviously symmetric, it is enough to show the conditions (1),(2) of lemma 2.3.

(1) Let $x \in \mathbb{N}$. First we determine ϵ-ball at x. The following are equivalent.

$$y \in B_\epsilon(x); d(x,y) \leq \epsilon; (x = y) \text{ or } (\frac{1}{|x-y|} \leq \epsilon); (x = y) \text{ or } (|x-y| \geq \frac{1}{\epsilon});$$

$$(x = y) \text{ or } (y \geq x + \frac{1}{\epsilon} \text{ or } y \leq x - \frac{1}{\epsilon})$$

Hence

$$B_\epsilon(x) = \{x\} \cup (\mathbb{N} \cap [x + \frac{1}{\epsilon}, \infty[) \cup (\mathbb{N} \cap [x - \frac{1}{\epsilon}, \infty[)$$

itself is a cofinite subset of $\mathbb{N}$ so that (1) is proved.

(2) Let $U \in \mathcal{O}_{cof}(x)$. It is enough to show the existence of $\epsilon > 0$ satisfying $B_\epsilon(x) \subset U$. Since $\mathbb{N} \setminus U$ is finite, there is $m \in \mathbb{N}$ such that $\mathbb{N} \cap [m, \infty[\subset U$. Choose $\epsilon = \min\{\frac{1}{x}, \frac{1}{|m-x|+1}\}$. Then $[1, x - \frac{1}{\epsilon}] = \emptyset$ and

$$x + \frac{1}{\epsilon} \geq x + |m - x| + 1 \geq m + 1 > m.$$

Hence $\mathbb{N} \cap [x + \frac{1}{\epsilon}, \infty[\subset \mathbb{N} \cap [m, \infty[$. Therefore

$$B_\epsilon(x) = \{x\} \cup (\mathbb{N} \cap [x + \frac{1}{\epsilon}, \infty[) \subset \{x\} \cup (\mathbb{N} \cap [m, \infty[\subset U.$$

(b) Since for all $U, V \in \mathcal{O}_{\text{cof}} \setminus \{\emptyset\}$ $U \cap V \neq \emptyset$, $(\mathbb{N}, \mathcal{O}_{\text{cof}})$ is not T_3 space, hence not T_{3a}. Therefore it can not be uniformizable. □

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