JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 24, No. 2, June 2011

THE STABILITY OF LINEAR MAPPINGS IN BANACH MODULES ASSOCIATED WITH A GENERALIZED JENSEN MAPPING

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Abstract. Let X and Y be vector spaces. It is shown that a mapping $f : X \to Y$ satisfies the functional equation

$$(\ddagger) \qquad dk\; f\Big(\frac{\sum_{j=1}^{dk} x_j}{dk}\Big) = \sum_{j=1}^{dk} f(x_j)$$

if and only if the mapping $f : X \to Y$ is Cauchy additive, and prove the Cauchy-Rassias stability of the functional equation ($\ddagger$) in Banach modules over a unital C^*-algebra. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. As an application, we show that every almost homomorphism $h : \mathcal{A} \to \mathcal{B}$ of $\mathcal{A}$ into $\mathcal{B}$ is a homomorphism when $h((k-1)^n uy) = h((k-1)^n u)h(y)$ for all unitaries $u \in \mathcal{A}$, all $y \in \mathcal{A}$, and $n = 0, 1, 2, \cdots$.

Moreover, we prove the Cauchy-Rassias stability of homomorphisms in C^*-algebras.

1. Introduction

In 1940, S. M. Ulam [20] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Hyers [3] showed that if $\epsilon > 0$ and $f : X \to Y$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \epsilon$$

Received March 8, 2011; Accepted May 16, 2011.
2010 *Mathematics Subject Classifications*: Primary 39B52, 46L05, 47B48, 17A36.
Key words and phrases: Cauchy-Rassias stability, C^*-algebra homomorphism, Banach module over C^*-algebra.

for all $x \in X$.

Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$(*) \qquad \|f(x + y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th.M. Rassias [15] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in X$. The inequality $(*)$ that was introduced for the first time by Th.M. Rassias [12] is called *Cauchy-Rassias inequality* and the stability of the functional equation *Cauchy-Rassias stability.* This inequality has provided a lot of influence in the development of what is known as *Hyers-Ulam-Rassias stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was taken up by a number of mathematicians (cf. [4], [6], [9], [14]–[19]). Th.M. Rassias [13] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Z. Gajda [1] following the same approach as in Th.M. Rassias [15], gave an affirmative solution to this question for $p > 1$.

Găvruta [2] generalized the Rassias' result: Let G be an abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in G$. C. Park [7] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra.

Jun and Lee [5] proved the following: Denote by $\varphi : X \setminus \{0\} \times X \setminus \{0\} \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 3^{-j} \varphi(3^j x, 3^j y) < \infty$$

for all $x, y \in X \setminus \{0\}$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in X \setminus \{0\}$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - f(0) - T(x)| \leq \frac{1}{3}(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x))$$

for all $x \in X \setminus \{0\}$. C. Park and W. Park [11] applied the Jun and Lee's result to the Jensen's equation in Banach modules over a C^*-algebra.

Throughout this paper, assume that d, k are positive integers with $k \geq 2$.

In this paper, we solve the following functional equation

$$dk\ f(\frac{\sum_{j=1}^{dk} x_j}{dk}) = \sum_{j=1}^{dk} f(x_j), \tag{1.i}$$

which is called a *generalized Jensen functional equation*. Each solution of the functional equation (1.i) is called a *generalized Jensen mapping*. We moreover prove the Cauchy-Rassias stability of the functional equation (1.i) in Banach modules over a unital C^*-algebra. The main purpose of this paper is to investigate homomorphisms between C^*-algebras and between Poisson C^*-algebras, and to prove their Cauchy-Rassias stability.

2. A generalized Jensen's mapping

Throughout this section, assume that X and Y are linear spaces.

LEMMA 2.1. *An odd mapping $f : X \to Y$ satisfies* (1.i) *for all* $x_1, x_2, \cdots, x_{dk} \in X$ *if and only if f is Cauchy additive.*

Proof. Assume that $f : X \to Y$ satisfies (1.i) for all $x_1, x_2, \cdots, x_{dk} \in X$. Putting $x_2 = \cdots = x_{dk} = 0$ in (1.i), we get

$$dk\ f(\frac{x_1}{dk}) = f(x_1) \tag{2.1}$$

for all $x_1 \in X$. Putting $x_3 = \cdots = x_{dk} = 0$ in (1.i), it follows from (2.1) that

$$f(x_1 + x_2) = dk\ f(\frac{x_1 + x_2}{dk}) = f(x_1) + f(x_2)$$

for all $x_1, x_2 \in X$. Thus f is Cauchy additive.

The converse is obviously true. □

3. Cauchy-Rassias stability of the generalized Jensen's mapping in Banach modules over a C^*-algebra

Throughout this section, assume that $\mathcal{A}$ is a unital C^*-algebra with norm $|\cdot|$ and unitary group $\mathcal{U}(\mathcal{A})$, and that X and Y are left Banach modules over $\mathcal{A}$ with norms $||\cdot||$ and $\|\cdot\|$, respectively.

Given a mapping $f : X \to Y$, we set

$$D_u f(x_1, \cdots, x_{dk}) := dk\ f(\frac{\sum_{j=1}^{dk} u x_j}{dk}) - \sum_{j=1}^{dk} u f(x_j)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \cdots, x_{dk} \in X$.

THEOREM 3.1. *Let $f : X \to Y$ be an odd mapping for which there is a function $\varphi : X^{dk} \to [0, \infty)$ such that*

(3.i)
$$\widetilde{\varphi}(x_1, \cdots, x_{dk}) := \sum_{j=0}^{\infty} \frac{1}{(k-1)^j} \varphi((k-1)^j x_1, \cdots, (k-1)^j x_{dk}) < \infty,$$

(3.ii)
$$\|D_u f(x_1, \cdots, x_{dk})\| \le \varphi(x_1, \cdots, x_{dk})$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \cdots, x_{dk} \in X$. Then there exists a unique $\mathcal{A}$-linear generalized Jensen's mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{1}{d(k-1)} \widetilde{\varphi}(\underbrace{(k-1)x, \cdots, (k-1)x}_{d \text{ times}}, \underbrace{-x, \cdots, -x}_{d(k-1) \text{ times}}) \tag{3.iii}$$

for all $x \in X$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd mapping. Let $u = 1 \in \mathcal{U}(\mathcal{A})$. Putting $x_1 = \cdots = x_d = (k-1)x$ and $x_{d+1} = \cdots = x_{dk} = -x$ in (3.ii), we have

$$\begin{aligned}(3.1)\qquad &\| -d\, f((k-1)x) - d(k-1)f(-x)\| \\ &\leq \varphi(\underbrace{(k-1)x, \cdots, (k-1)x}_{d \text{ times}}, \underbrace{-x, \cdots, -x}_{d(k-1) \text{ times}})\end{aligned}$$

for all $x \in X$. So

$$\begin{aligned}&\|f(x) - \frac{1}{k-1} f((k-1)x)\| \\ &\leq \frac{1}{d(k-1)} \varphi(\underbrace{(k-1)x, \cdots, (k-1)x}_{d \text{ times}}, \underbrace{-x, \cdots, -x}_{d(k-1) \text{ times}})\end{aligned}$$

for all $x \in X$. Hence

(3.2)

$$\begin{aligned}&\|\frac{1}{(k-1)^n} f((k-1)^n x) - \frac{1}{(k-1)^{n+1}} f((k-1)^{n+1} x)\| \\ &= \frac{1}{(k-1)^n} \|f((k-1)^n x) - \frac{1}{k-1} f((k-1)(k-1)^n x)\| \\ &\leq \frac{1}{d(k-1)^{n+1}} \varphi(\underbrace{(k-1)^{n+1}x, \cdots, (k-1)^{n+1}x}_{d \text{ times}}, \\ &\qquad \underbrace{-(k-1)^n x, \cdots, -(k-1)^n x}_{d(k-1) \text{ times}})\end{aligned}$$

for all $x \in X$ and all positive integers n. By (3.2), we have

$$\begin{aligned}&\|\frac{1}{(k-1)^m} f((k-1)^m x) - \frac{1}{(k-1)^n} f((k-1)^n x)\| \leq \sum_{l=m}^{n-1} \frac{1}{d(k-1)^{l+1}} \\ (3.3)\qquad &\times \varphi(\underbrace{(k-1)^{l+1}x, \cdots, (k-1)^{l+1}x}_{d \text{ times}}, \underbrace{-(k-1)^l x, \cdots, -(k-1)^l x}_{d(k-1) \text{ times}})\end{aligned}$$

for all $x \in X$ and all positive integers m and n with $m < n$. This shows that the sequence $\{\frac{1}{(k-1)^n} f((k-1)^n x)\}$ is a Cauchy sequence for all $x \in X$.

Since Y is complete, the sequence $\{\frac{1}{(k-1)^n} f((k-1)^n x)\}$ converges for all $x \in X$. So we can define a mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty} \frac{1}{(k-1)^n} f((k-1)^n x)$$

for all $x \in X$. Since $f(-x) = -f(x)$ for all $x \in X$, we have $L(-x) = L(x)$ for all $x \in X$. Also, we get

$$\begin{aligned}
&\|D_1 L(x_1, \cdots, x_{dk})\| \\
&\qquad = \lim_{n\to\infty} \frac{1}{(k-1)^n} \|D_1 f((k-1)^n x_1, \cdots, (k-1)^n x_{dk})\| \\
&\qquad \le \lim_{n\to\infty} \frac{1}{(k-1)^n} \varphi((k-1)^n x_1, \cdots, (k-1)^n x_{dk}) = 0
\end{aligned}$$

for all $x_1, \cdots, x_{dk} \in X$. By Lemma 2.1, L is Cauchy additive. Putting $m = 0$ and letting $n \to \infty$ in (3.2), we get (3.iii).

Now, let $L' : X \to Y$ be another generalized Jensen's mapping satisfying (3.iii). Then we have

$$\begin{aligned}
\|L(x) - L'(x)\| &= \frac{1}{(k-1)^n} \|L((k-1)^n x) - L'((k-1)^n x)\| \\
&\le \frac{1}{(k-1)^n} (\|L((k-1)^n x) - f((k-1)^n x)\| \\
&\quad + \|L'((k-1)^n x) - f((k-1)^n x)\|) \\
&\le \frac{2}{d(k-1)^{n+1}} \widetilde{\varphi}(\underbrace{(k-1)^{n+1} x, \cdots, (k-1)^{n+1} x}_{d \text{ times}}, \\
&\qquad \underbrace{-(k-1)^n x, \cdots, -(k-1)^n x}_{d(k-1) \text{ times}}),
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $L(x) = L'(x)$ for all $x \in X$. This proves the uniqueness of L.

By the assumption, for each $u \in \mathcal{U}(\mathcal{A})$, we get

$$\begin{aligned}
\|D_u L(x, \underbrace{0, \cdots, 0}_{dk-1 \text{ times}})\| &= \lim_{n\to\infty} \frac{1}{(k-1)^n} \|D_u f((k-1)^n x, \underbrace{0, \cdots, 0}_{dk-1 \text{ times}})\| \\
&\le \lim_{n\to\infty} \frac{1}{(k-1)^n} \varphi((k-1)^n x, \underbrace{0, \cdots, 0}_{dk-1 \text{ times}}) = 0
\end{aligned}$$

for all $x \in X$. So

$$dkL(\frac{ux}{dk}) = uL(x)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x \in X$. Since L is Cauchy additive,

$$L(ux) = dkL(\frac{ux}{dk}) = uL(x)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x \in X$.

By the same reasoning as in the proofs of [8] and [10], one can show that the unique generalized Jensen's mapping $L : \mathcal{A} \to \mathcal{B}$ is an $\mathcal{A}$-linear mapping.

□

COROLLARY 3.2. *Let θ and $p < 1$ be positive real numbers. Let $f : X \to Y$ be an odd mapping such that*

$$\|D_u f(x_1, \cdots, x_{dk})\| \le \theta \sum_{j=1}^{dk} ||x_j||^p$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \cdots, x_{dk} \in X$. Then there exists a unique $\mathcal{A}$-linear generalized Jensen's mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{(k-1) + (k-1)^p}{(k-1) - (k-1)^p} \theta ||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \cdots, x_{dk}) = \theta \sum_{j=1}^{dk} ||x_j||^p$, and apply Theorem 3.1.

□

THEOREM 3.3. *Let $f : X \to Y$ be an odd mapping for which there is a function $\varphi : X^{dk} \to [0, \infty)$ such that*

(3.iv)
$$\widetilde{\varphi}(x_1, \cdots, x_{dk}) := \sum_{j=1}^{\infty} (k-1)^j \varphi(\frac{x_1}{(k-1)^j}, \cdots, \frac{x_{dk}}{(k-1)^j}) < \infty,$$

(3.v)
$$\|D_u f(x_1, \cdots, x_{dk})\| \le \varphi(x_1, \cdots, x_{dk})$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \cdots, x_{dk} \in X$. Then there exists a unique $\mathcal{A}$-linear generalized Jensen's mapping $L : X \to Y$ such that

(3.vi)
$$\|f(x) - L(x)\| \le \frac{1}{d(k-1)} \widetilde{\varphi}(\underbrace{(k-1)x, \cdots, (k-1)x}_{d \text{ times}}, \underbrace{-x, \cdots, -x}_{d(k-1) \text{ times}})$$

for all $x \in X$.

Proof. Replacing x by $\frac{x}{k-1}$ in (3.1), we have

$$\|f(x)-(k-1)f(\frac{x}{k-1})\| \le \frac{1}{d}\varphi(\underbrace{x,\cdots,x}_{d\text{ times}},\underbrace{-\frac{x}{k-1},\cdots,-\frac{x}{k-1}}_{d(k-1)\text{ times}})$$

for all $x \in X$. So

$$\begin{aligned}(3.4)\qquad &\|(k-1)^n f(\frac{x}{(k-1)^n})-(k-1)^{n+1}f(\frac{x}{(k-1)^{n+1}})\| \\ &= (k-1)^n\|f(\frac{x}{(k-1)^n})-(k-1)f(\frac{1}{k-1}\cdot\frac{x}{(k-1)^n})\| \\ &\le \frac{(k-1)^n}{d}\varphi(\underbrace{\frac{x}{(k-1)^n},\cdots,\frac{x}{(k-1)^n}}_{d\text{ times}}, \\ &\qquad \underbrace{-\frac{x}{(k-1)^{n+1}},\cdots,-\frac{x}{(k-1)^{n+1}}}_{d(k-1)\text{ times}})\end{aligned}$$

for all $x \in X$ and all positive integers n. By (3.4), we have

$$\begin{aligned}&\|(k-1)^m f(\frac{x}{(k-1)^m})-(k-1)^n f(\frac{x}{(k-1)^n})\| \\ &\le \sum_{l=m}^{n-1}\frac{(k-1)^l}{d}\varphi(\underbrace{\frac{x}{(k-1)^l},\cdots,\frac{x}{(k-1)^l}}_{d\text{ times}}, \\ (3.5)\qquad &\qquad \underbrace{-\frac{x}{(k-1)^{l+1}},\cdots,-\frac{x}{(k-1)^{l+1}}}_{d(k-1)\text{ times}})\end{aligned}$$

for all $x \in X$ and all positive integers m and n with $m < n$. This shows that the sequence $\{(k-1)^n f(\frac{x}{(k-1)^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(k-1)^n f(\frac{x}{(k-1)^n})\}$ converges for all $x \in X$. So we can define a mapping $L : X \to Y$ by

$$L(x) := \lim_{n\to\infty}(k-1)^n f(\frac{x}{(k-1)^n})$$

for all $x \in X$. Also, we get

$$\begin{aligned}\|D_1L(x_1,\cdots,x_{dk})\| &= \lim_{n\to\infty}(k-1)^n\|D_1f(\frac{x_1}{(k-1)^n},\cdots,\frac{x_{dk}}{(k-1)^n})\| \\ &\le \lim_{n\to\infty}(k-1)^n\varphi(\frac{x_1}{(k-1)^n},\cdots,\frac{x_{dk}}{(k-1)^n})=0\end{aligned}$$

for all $x_1,\cdots,x_{dk} \in X$. By Lemma 2.1, L is Cauchy additive. Putting $m=0$ and letting $n\to\infty$ in (3.5), we get (3.vi).

The rest of the proof is similar to the proof of Theorem 3.1. □

COROLLARY 3.4. *Let θ and $p>1$ be positive real numbers. Let $f : X \to Y$ be an odd mapping such that*

$$\|D_uf(x_1,\cdots,x_{dk})\| \le \theta\sum_{j=1}^{dk}||x_j||^p$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1,\cdots,x_{dk} \in X$. Then there exists a unique $\mathcal{A}$-linear generalized Jensen's mapping $L : X \to Y$ such that

$$\|f(x)-L(x)\| \le \frac{(k-1)^p+(k-1)}{(k-1)^p-(k-1)}\theta||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1,\cdots,x_{dk}) = \theta\sum_{j=1}^{dk}||x_j||^p$, and apply Theorem 3.3. □

4. Isomorphisms in unital C^*-algebras

Throughout this section, assume that $\mathcal{A}$ is a unital C^*-algebra with norm $||\cdot||$, unit e and unitary group $\mathcal{U}(\mathcal{A})$, and that $\mathcal{B}$ is a unital C^*-algebra with norm $\|\cdot\|$.

We investigate C^*-algebra isomorphisms in unital C^*-algebras.

THEOREM 4.1. *Let $h : \mathcal{A} \to \mathcal{B}$ be an odd bijective mapping satisfying $h((k-1)^nuy) = h((k-1)^nu)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and $n = 0,1,2,\cdots$, for which there is a function $\varphi : \mathcal{A}^{dk} \to [0,\infty)$ satisfying* (3.i)

such that
(4.i)
$$\|dk\ h(\frac{\sum_{j=1}^{dk}\mu x_j}{dk}) - \sum_{j=1}^{dk}\mu h(x_j)\| \le \varphi(x_1, \cdots, x_{dk}),$$
(4.ii)
$$\|h((k-1)^n u^*) - h((k-1)^n u)^*\| \le \varphi(\underbrace{(k-1)^n u, \cdots, (k-1)^n u}_{dk \text{ times}})$$
for all $u \in \mathcal{U}(\mathcal{A})$*, all* $x_1, \cdots, x_{dk} \in \mathcal{A}$*, all* $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid \ |\lambda| = 1\}$ *and* $n = 0, 1, 2, \cdots$. *Assume that*
$$\text{(4.iii)} \qquad \lim_{n\to\infty} \frac{h((k-1)^n e)}{(k-1)^n} \text{ is invertible.}$$
Then the odd bijective mapping $h : \mathcal{A} \to \mathcal{B}$ *is a* C^**-algebra isomorphism.*

Proof. We can consider a C^*-algebra as a Banach module over a unital C^*-algebra $\mathbb{C}$. So by Theorem 3.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ such that
$$\text{(4.iv)} \quad \|h(x) - H(x)\| \le \frac{1}{d(k-1)}\widetilde{\varphi}(\underbrace{(k-1)x, \cdots, (k-1)x}_{d \text{ times}}, \underbrace{-x, \cdots, -x}_{d(k-1) \text{ times}})$$
for all $x \in \mathcal{A}$. The mapping $H : \mathcal{A} \to \mathcal{B}$ is given by
$$\text{(4.1)} \qquad H(x) = \lim_{n\to\infty} \frac{1}{(k-1)^n} h((k-1)^n x)$$
for all $x \in \mathcal{A}$.

By (3.i) and (4.ii), we get
$$H(u^*) = \lim_{n\to\infty} \frac{h((k-1)^n u^*)}{(k-1)^n} = \lim_{n\to\infty} \frac{h((k-1)^n u)^*}{(k-1)^n}$$
$$= (\lim_{n\to\infty} \frac{h((k-1)^n u)}{(k-1)^n})^* = H(u)^*$$
for all $u \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (cf. [7, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$),
$$H(x^*) = H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j)^*$$
$$= (\sum_{j=1}^m \lambda_j H(u_j))^* = H(\sum_{j=1}^m \lambda_j u_j)^* = H(x)^*$$

for all $x \in \mathcal{A}$.

Since $h((k-1)^n uy) = h((k-1)^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$,

$$
\begin{aligned}
H(uy) &= \lim_{n\to\infty} \frac{1}{(k-1)^n} h((k-1)^n uy) \\
&= \lim_{n\to\infty} \frac{1}{(k-1)^n} h((k-1)^n u)h(y) = H(u)h(y) \qquad (4.2)
\end{aligned}
$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. By the additivity of H and (4.2),

$$(k-1)^n H(uy) = H((k-1)^n uy) = H(u((k-1)^n y)) = H(u)h((k-1)^n y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Hence

$$H(uy) = \frac{1}{(k-1)^n} H(u)h((k-1)^n y) = H(u)\frac{1}{(k-1)^n} h((k-1)^n y) \qquad (4.3)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Taking the limit in (4.3) as $n \to \infty$, we obtain

$$H(uy) = H(u)H(y) \qquad (4.4)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$), it follows from (4.4) that

$$
\begin{aligned}
H(xy) &= H(\sum_{j=1}^m \lambda_j u_j y) = \sum_{j=1}^m \lambda_j H(u_j y) = \sum_{j=1}^m \lambda_j H(u_j)H(y) \\
&= H(\sum_{j=1}^m \lambda_j u_j)H(y) = H(x)H(y)
\end{aligned}
$$

for all $x, y \in \mathcal{A}$.

By (4.2) and (4.4),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in \mathcal{A}$. Since $\lim_{n\to\infty} \frac{h((k-1)^n e)}{(k-1)^n} = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in \mathcal{A}$.

Therefore, the odd bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism. □

Corollary 4.2. *Let $h : \mathcal{A} \to \mathcal{B}$ be an odd bijective mapping satisfying $h((k-1)^n uy) = h((k-1)^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|dkh(\frac{\sum_{j=1}^{dk} \mu x_j}{dk}) - \sum_{j=1}^{dk} \mu h(x_j)\| \leq \theta \sum_{j=1}^{dk} ||x_j||^p,$$
$$\|h((k-1)^n u^*) - h((k-1)^n u)^*\| \leq dk(k-1)^{np}\theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, $n = 0, 1, 2, \cdots$, and all $x_1, \cdots, x_{dk} \in \mathcal{A}$. Assume that

$$\lim_{n\to\infty} \frac{h((k-1)^n e)}{(k-1)^n} \text{ is invertible.}$$

Then the odd bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof. Define $\varphi(x_1, \cdots, x_{dk}) = \theta \sum_{j=1}^{dk} ||x_j||^p$, and apply Theorem 4.1. □

Theorem 4.3. *Let $h : \mathcal{A} \to \mathcal{B}$ be an odd bijective mapping satisfying $h((k-1)^n uy) = h((k-1)^n u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and $n = 0, 1, 2, \cdots$, for which there is a function $\varphi : \mathcal{A}^{dk} \to [0, \infty)$ satisfying (3.i), (4.ii), and (4.iii) such that*

$$\|dkh(\frac{\sum_{j=1}^{dk} \mu x_j}{dk}) - \sum_{j=1}^{dk} \mu h(x_j)\| \leq \varphi(x_1, \cdots, x_{dk}), \tag{4.v}$$

for all $x_1, \cdots, x_{dk} \in \mathcal{A}$ and $\mu = 1, i$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then the odd bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof. Put $\mu = 1$ in (4.v). By the same reasoning as in the proof of Theorem 4.1, there exists a unique generalized Jensen's mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (4.iv). By the same reasoning as in the proof of [15, Theorem], the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (4.v). By the same method as in the proof of Theorem 4.1, one can obtain that

$$H(ix) = \lim_{n\to\infty} \frac{h((k-1)^n ix)}{(k-1)^n} = \lim_{n\to\infty} \frac{ih((k-1)^n x)}{(k-1)^n} = iH(x)$$

for all $x \in \mathcal{A}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) \\ &= sH(x) + itH(x) = (s + it)H(x) = \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{A}$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{A}$. Hence the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 4.1. □

Now we prove the Cauchy-Rassias stability of C^*-algebra homomorphisms in unital C^*-algebras.

THEOREM 4.4. *Let $h : \mathcal{A} \to \mathcal{B}$ be an odd mapping for which there exists a function $\varphi : \mathcal{A}^{dk} \to [0, \infty)$ satisfying* (3.i), (4.i) *and* (4.ii) *such that*

$$\begin{aligned} &\|h((k-1)^n u (k-1)^n v) - h((k-1)^n u) h((k-1)^n v)\| \\ &\qquad \le \varphi((k-1)^n u, (k-1)^n v, \underbrace{0, \cdots, 0}_{dk-2 \text{ times}}) \end{aligned} \tag{4.vi}$$

for all $u, v \in \mathcal{U}(\mathcal{A})$ and $n = 0, 1, 2, \cdots$. Then there exists a unique C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ satisfying (4.iv).

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear involutive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (4.iv).

By (4.vi),

$$\begin{aligned} &\frac{1}{(k-1)^{2n}} \|h((k-1)^n u (k-1)^n v) - h((k-1)^n u) h((k-1)^n v)\| \\ &\quad \le \frac{1}{(k-1)^{2n}} \varphi((k-1)^n u, (k-1)^n v, \underbrace{0, \cdots, 0}_{dk-2 \text{ times}}) \\ &\quad \le \frac{1}{(k-1)^n} \varphi((k-1)^n u, (k-1)^n v, \underbrace{0, \cdots, 0}_{dk-2 \text{ times}}), \end{aligned}$$

which tends to zero by (3.i) as $n \to \infty$. By (4.1),

$$\begin{aligned} H(uv) &= \lim_{n\to\infty} \frac{h((k-1)^n u(k-1)^n v)}{(k-1)^{2n}} \\ &= \lim_{n\to\infty} \frac{h((k-1)^n u)h((k-1)^n v)}{(k-1)^{2n}} \\ &= \lim_{n\to\infty} \frac{h((k-1)^n u)}{(k-1)^n}\frac{h((k-1)^n v)}{(k-1)^n} = H(u)H(v) \end{aligned}$$

for all $u, v \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A}))$,

$$\begin{aligned} H(xv) &= H(\sum_{j=1}^m \lambda_j u_j v) = \sum_{j=1}^m \lambda_j H(u_j v) = \sum_{j=1}^m \lambda_j H(u_j)H(v) \\ &= H(\sum_{j=1}^m \lambda_j u_j)H(v) = H(x)H(v) \end{aligned}$$

for all $x \in \mathcal{A}$ and all $v \in \mathcal{U}(\mathcal{A})$. By the same method as given above, one can obtain that

$$H(xy) = H(x)H(y)$$

for all $x, y \in \mathcal{A}$. So the mapping $H : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra homomorphism. □

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