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FRACTIONAL POLYA-SZEGÖ INEQUALITY

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ABSTRACT. Let $0 < s < 1$. For f^* representing the symmetric radial decreasing rearrangement of f, we build up a fractional version of Polya-Szegö inequality:

$$\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f^*(x)|^2 dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 dx.$$

1. Introduction and the main theorem

Rearrangement technique has long been a basic tool in the calculus of variations and in the theory of those partial differential equations that arise as Euler-Lagrange equations of variational problems. It permits one to concentrate on radial monotone decreasing functions and thereby reduces many problems to simple one dimensional ones. One of the most important applications of the concept of the symmetric radial decreasing rearrangement would be the kinetic energy reduction. In fact, Polya-Szegö inequality claims that the symmetric decreasing rearrangement diminishes the L^2-norm of the gradient of a function f:

$$\int_{\mathbb{R}^n} |\nabla f^*(x)|^2 dx \leq \int_{\mathbb{R}^n} |\nabla f(x)|^2 dx, \tag{1.1}$$

where f^* represents the symmetric decreasing rearrangement of f. Inequality (1.1) itself has numerous applications in physics. For instance, applying the symmetric decreasing rearrangement, G. Polya and G. Szegö proved that the capacity of a condenser decreases or remains unchanged [10]. It was also employed to show that among all bounded bodies with fixed measure, balls have the minimal capacity [8], and it also has played a crucial role in the solution of the famous Choquards

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conjecture [7, 5]. Moreover, it turned out that inequality (1.1) is extremely useful to establish the existence of ground states solutions of a nonlinear Schrödinger equation [5].

The *relativistic kinetic energy* statement of the inequality (1.1) is

$$\int_{\mathbb{R}^n} |\sqrt{-\Delta} f^*(x)|^2 dx \leq \int_{\mathbb{R}^n} |\sqrt{-\Delta} f(x)|^2 dx$$

(page 174 in [8]). With this motivation we ask a question on the non-expansivity of symmetric decreasing rearrangement of functions with respect to the fractional actions $(-\Delta)^{s/2}$ for $0 < s < 1$. Our main theorem is stated as follows:

THEOREM 1.1. *Let $0 < s < 1$. Let f^* denote the symmetric radial decreasing rearrangement of f. Then we have*

$$\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f^*(x)|^2 dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 dx, \tag{1.2}$$

in the sense that the finiteness of of the right side implies the finiteness of the left side. Furthermore the equality occurs for radial decreasing functions and the best constant is 1.

For the basic terminology and some properties of symmetric decreasing rearrangement, we refer [6] and Chapter 3 in [8].

2. The proof

It suffices to prove the following:

$$\int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}[f^*](\xi)|^2 d\xi \leq \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi, \tag{2.1}$$

where $\hat{f} = \mathcal{F}(f)$ represents the Fourier transform of f on $\mathbb{R}^n$ defined by

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx,$$

if $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. (We present this only for the explicit expression of the Bessel kernel for later use in (2.3).) We first prove the following lemma.

LEMMA 2.1. *For $0 < s < 1$, we have*

$$\int_{\mathbb{R}^n} \left(\frac{|\xi|^2}{1+|\xi|^2}\right)^s |\mathcal{F}[f^*](\xi)|^2 d\xi \leq \int_{\mathbb{R}^n} \left(\frac{|\xi|^2}{1+|\xi|^2}\right)^s |\hat{f}(\xi)|^2 d\xi. \tag{2.2}$$

Suppose that Lemma 2.1 has been proved. Take $\varepsilon > 0$, and replace $f(x)$ by $f(\varepsilon x)$. We have $[f(\varepsilon x)]^* = f^*(\varepsilon x)$, since rearrangement commutes with uniform dilation. Then, (2.2) becomes

$$\frac{1}{\varepsilon^{2n}}\int_{\mathbb{R}^n}\left(\frac{|\xi|^2}{1+|\xi|^2}\right)^s |\mathcal{F}[f^*](\xi/\varepsilon)|^2 d\xi \le \frac{1}{\varepsilon^{2n}}\int_{\mathbb{R}^n}\left(\frac{|\xi|^2}{1+|\xi|^2}\right)^s |\hat{f}(\xi/\varepsilon)|^2 d\xi.$$

Changing variables $\xi/\varepsilon = \eta$ gives

$$\int_{\mathbb{R}^n}\left(\frac{|\eta|^2}{1+\varepsilon^2|\eta|^2}\right)^s |\mathcal{F}[f^*](\eta)|^2 d\eta \le \int_{\mathbb{R}^n}\left(\frac{|\eta|^2}{1+\varepsilon^2|\eta|^2}\right)^s |\hat{f}(\eta)|^2 d\eta.$$

Take the limit of both sides as $\varepsilon \to 0$ to get (2.1).

Proof of Lemma 2.1. From the following expression

$$\left(\frac{|\xi|^2}{1+|\xi|^2}\right)^s = \left(1-\frac{1}{1+|\xi|^2}\right)^s = 1-\sum_{k=1}^{\infty}(-1)^{k+1}\begin{pmatrix} s \\ k \end{pmatrix}\left(\frac{1}{1+|\xi|^2}\right)^k$$

with

$$\begin{pmatrix} s \\ k \end{pmatrix} = \frac{s(s-1)\cdots(s-(k-1))}{k!},$$

inequality (2.2) now becomes

$$\int_{\mathbb{R}^n}|\mathcal{F}[f^*](\xi)|^2 d\xi - \sum_{k=1}^{\infty}(-1)^{k+1}\begin{pmatrix} s \\ k \end{pmatrix}\int_{\mathbb{R}^n}\frac{1}{(1+|\xi|^2)^k}|\mathcal{F}[f^*](\xi)|^2 d\xi$$
$$\le \int_{\mathbb{R}^n}|\mathcal{F}[f](\xi)|^2 d\xi - \sum_{k=1}^{\infty}(-1)^{k+1}\begin{pmatrix} s \\ k \end{pmatrix}\int_{\mathbb{R}^n}\frac{1}{(1+|\xi|^2)^k}|\mathcal{F}[f](\xi)|^2 d\xi.$$

It is enough to show that for each positive integer k

$$\int_{\mathbb{R}^n}\frac{1}{(1+|\xi|^2)^k}|\mathcal{F}[f^*](\xi)|^2 d\xi \ge \int_{\mathbb{R}^n}\frac{1}{(1+|\xi|^2)^k}|\mathcal{F}[f](\xi)|^2 d\xi$$

since $(-1)^{k+1}\begin{pmatrix} s \\ k \end{pmatrix} > 0$ with $0 < s < 1$. We consider a function g_k such that $\mathcal{F}[\tilde{g}_k](\xi) = (1+|\xi|^2)^{-k/2}$, where $\tilde{g}(x) = \bar{g}(-x)$. In fact, if we let

$$G_k(x) = \frac{1}{\Gamma(k/2)}\frac{1}{(4\pi)^{k/2}}\int_0^{\infty} e^{-\frac{\pi|x|^2}{\delta}} e^{-\frac{\delta}{4\pi}}\delta^{-\frac{(n-k)}{2}}\frac{d\delta}{\delta},$$

then we have

$$\mathcal{F}[G_k](\xi) = (1+4\pi^2|\xi|^2)^{-k/2}. \tag{2.3}$$

It can be easily shown that

$$(1+|\xi|^2)^{-k} = \mathcal{F}[G_{2k}](\xi/2\pi) = (2\pi)^k\mathcal{F}[G_{2k}](2\pi\xi).$$

So, we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^k} |\mathcal{F}[f](\xi)|^2 d\xi &= (2\pi)^k \int_{\mathbb{R}^n} \hat{G}_{2k}(2\pi\xi)\hat{f}(\xi)\overline{\hat{f}}(\xi)d\xi \\
&= (2\pi)^k \int_{\mathbb{R}^n} G_{2k}(-2\pi x)(f * \tilde{f})(x)dx \\
&= (2\pi)^k [G_{2k}(2\pi x) * (f * \tilde{f})(x)](0) \\
&= (2\pi)^k \int_{\mathbb{R}^n \times \mathbb{R}^n} G_{2k}[2\pi(y-z)]f(z)\bar{f}(y)dydz \\
&\leq (2\pi)^k \int_{\mathbb{R}^n \times \mathbb{R}^n} G_{2k}[2\pi(y-z)]f^*(z)\bar{f}^*(y)dydz \qquad (2.4) \\
&= \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^k} |\mathcal{F}[f^*](\xi)|^2 d\xi,
\end{aligned}
$$

where

$$(g * h)(x) = \int_{R^n} g(x-y)h(y)dy.$$

The *Symmetrization lemma* by W. Beckner in [4] takes care of the inequality (2.4), which is the only place that inequality occurs. So it is clear that the extremals are radial decreasing functions and the best constant is 1. The proof is now completed. □

REMARK 2.2. *Theorem 1.1 implies the nonexpansivity of symmetric decreasing rearrangement on fractional Sobolev spaces* $H^s(\mathbb{R}^n)$, $0 < s < 1$; *for* $f \in H^s(\mathbb{R}^n)$,

$$\|f^*\|_{H^s(\mathbb{R}^n)} \leq \|f\|_{H^s(\mathbb{R}^n)}, \qquad (2.5)$$

where the fractional Sobolev norm $\|\cdot\|_{H^s(\mathbb{R}^n)}$ *is employed by*

$$\|f\|_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^2 \, dx\right)^{1/2} + \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi\right)^{1/2}.$$

REMARK 2.3. *The argument of the proof has been taken from the author's own paper* [9].

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