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THE GENERALIZED TRAPEZOIDAL FUZZY SETS

BongJu Lee* and Yong Sik Yun**

Abstract. We would like to generalize about trapezoidal fuzzy set and to calculate four operations based on the Zadeh's extension principle for two generalized trapezoidal fuzzy sets. And we roll up triangular fuzzy numbers and generalized triangular fuzzy sets into it. Since triangular fuzzy numbers and generalized triangular fuzzy sets are generalized trapezoidal fuzzy sets, we need no more the separate painstaking calculations of addition, subtraction, multiplication and division for two such kinds once the operations are done for generalized trapezoidal fuzzy sets.

1. Introduction

The purpose of this paper is to generalize the results of four operations for two generalized trapezoidal fuzzy sets. We use four operations, addition $A(+)B$, subtraction $A(-)B$, multiplication $A(\cdot)B$ and division $A(/)B$ for generalized trapezoidal fuzzy sets A and B. These operations for two fuzzy numbers $\big(A, \mu_A\big)$ and $\big(B, \mu_B\big)$ are defined in Definition 2.3 and based on the Zadeh's extension principle ([2], [3], [4]). Addition $A(+)B$ and subtraction $A(-)B$ become generalized trapezoidal fuzzy sets. However, multiplication $A(\cdot)B$ and division $A(/)B$ need not to be generalized trapezoidal fuzzy sets.

There are so many results of above four operations for two triangular fuzzy numbers and two generalized triangular fuzzy sets([1]). Since these fuzzy numbers and fuzzy sets are generalized trapezoidal fuzzy sets, we roll it up into the single concept of generalized trapezoidal fuzzy set. Thus we would like to show that we need no more the separate painstaking calculations of addition, subtraction, multiplication and division of

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Correspondence should be addressed to Yong Sik Yun, yunys@jejunu.ac.kr.

two such kinds once the operations are done for generalized trapezoidal fuzzy sets. And we provide some examples.

2. Preliminaries

Definition 2.1. A *triangular fuzzy number* is a fuzzy set $A = (a_1, a_2, a_3)$ having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_3 \le x \\ \frac{x-a_1}{a_2-a_1}, & a_1 \le x < a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \le x < a_3 \end{cases}$$

Definition 2.2. A *generalized triangular fuzzy set* is a symmetric fuzzy set $A = (a_1, c, a_2)$ having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_2 \le x \\ \frac{2c(x-a_1)}{a_2-a_1}, & a_1 \le x < \frac{a_1+a_2}{2} \\ \frac{-2c(x-a_2)}{a_2-a_1}, & \frac{a_1+a_2}{2} \le x < a_2 \end{cases}$$

Definition 2.3. The addition, subtraction, multiplication and division of two fuzzy sets A and B are defined as

1. Addition $A(+)B$:

$$\mu_{A(+)B}(z) = \sup_{x+y=z} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B$$

2. Subtraction $A(-)B$:

$$\mu_{A(+)B}(z) = \sup_{x-y=z} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B$$

3. Multiplication $A(\cdot)B$:

$$\mu_{A(\cdot)B}(z) = \sup_{xy=z} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B$$

4. Division $A(/)B$:

$$\mu_{A(\cdot)B}(z) = \sup_{\frac{x}{y}=z} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B$$

3. Generalized trapezoidal fuzzy set

Definition 3.1. A fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_4 \le x \\ \frac{c(x-a_1)}{a_2-a_1}, & a_1 \le x < a_2 \\ c, & a_2 \le x < a_3 \\ \frac{c(a_4-x)}{a_4-a_3}, & a_3 \le x < a_4 \end{cases}$$

where $a_i \in \mathbb{R}, i = 1, 2, 3, 4$ and $0 < c < 1$, is called *a generalized trapezoidal fuzzy set* and will be denoted by $A = (a_1, a_2, c, a_3, a_4)$.

Remark 3.2. A triangular fuzzy number $A = (a_1, a_2, a_3)$ is just a special case of a generalized trapezoidal fuzzy set. In fact, $(a_1, a_2, a_3) = (a_1, a_2, 1, a_2, a_3)$.

Remark 3.3. A generalized triangular fuzzy set is also a special case of a generalized trapezoidal fuzzy set. In fact,

$$A = \big((a_1, c_1, a_2)\big) = \big(a_1, \frac{a_1 + a_2}{2}, c_1, \frac{a_1 + a_2}{2}, a_2\big)$$

We generalize about four operations for two generalized trapezoidal fuzzy sets, A and B, in the following 3.1 $\sim$ 3.4. For that, let $A = (a_1, a_2, m_1, a_3, a_4)$ and $B = (b_1, b_2, m_2, b_3, b_4)$, where $a_i, b_i \in \mathbb{R}, i = 1, 2, 3, 4, 0 < m_1 \le m_2 < 1$ and $\mu_B(x) \ge m_1$ in $\big[p, r\big]$.

3.1. Addition

It is convenient to consider $\min\{\mu_A(x), \mu_B(y)\}$ as a function of two variable.

Now

$$\mu_{A(+)B}(z) = \sup_{x+y=z} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B$$

Therefore to find the value of this function we have to look at the values of $(x, y) \mapsto \min\{\mu_A(x), \mu_B(y)\}$ on the line $x + y = z$. In fugure 1, we see that the maximum value of $\min\{\mu_A(x), \mu_B(x)\}$ on the curve $x + y = z$ occurs at the intersection point of the $x + y = z$ and the path joining the four points E, P, R and G.

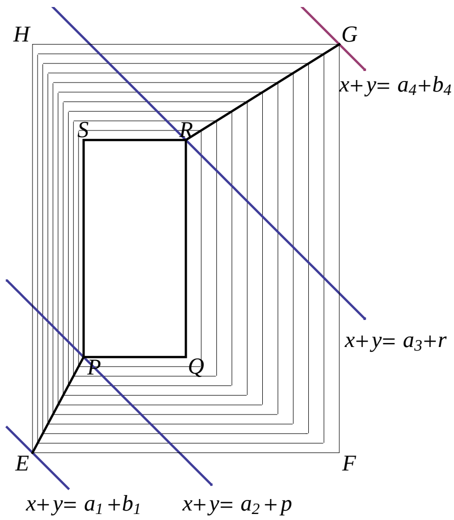

FIGURE 1. Contour plot of $(x, y) \mapsto \min\{\mu_A(x), \mu_B(y)\}$ and $x + y = z$

Then

$$\mu_{A(+)B}(z) = \begin{cases} 0, & z < a_1 + b_1, a_4 + b_4 \leq z \\ \psi_1(z), & a_1 + b_1 \leq z < a_2 + p \\ \eta_1(z), & a_2 + p \leq z < a_3 + r \\ \zeta_1(z), & a_3 + r \leq z < a_4 + b_4 \end{cases}$$

To find the expressions of the functions ψ_1, η_1, ζ_1, we have to find y coordinates of the points P and R. The y-coordinate p of $P(a_1, p)$ is determined by

$$m_1 = m_2 \cdot \frac{p - b_1}{b_2 - b_1}$$

Hence

$$p = b_1 + (b_2 - b_1) \cdot \frac{m_1}{m_2}$$

The y-coordinate r of $R(a_3, r)$ is determined by

$$m_1 = m_2 \cdot \frac{b_4 - r}{b_4 - b_3}$$

Hence

$$r = b_4 - (b_4 - b_3) \cdot \frac{m_1}{m_2}$$

Therefore

- $\psi_1(z)$ is determined by the intersection point of the line $x + y = z$ with the line joining the two points E and P:

$$\begin{cases} x + y &= z \\ y - b_1 &= \frac{p-b_1}{a_2-a_1}(x - a_1) \end{cases}$$

The x-coordinate of the intersection point is

$$x_{EP} = \frac{zm_2(a_2 - a_1) - m_2b_1(a_2 - a_1) + m_1a_1(b_2 - b_1)}{m_2(a_2 - a_1) + m_1(b_2 - b_1)}$$

Hence

$$\begin{aligned} \psi_1(z) &= \mu_A(x_{EP}) \\ &= \frac{m_1m_2(z - a_1 - b_1)}{m_2(a_2 - a_1) + m_1(b_2 - b_1)} \end{aligned}$$

- $\eta_1(z)$ is a constant function m_1 on $\left[a_2 + p, a_3 + r\right]$ because $(x, y) \mapsto \min\{\mu_A(x), \mu_B(y)\}$ takes constant value m_1 on the rectangle $PQRS$.
- $\eta_1(z)$ is determined by the intersection point of the line $x + y = z$ with the line joining the two points R and G:

$$\begin{cases} x + y &= z \\ y - r &= \frac{b_4-r}{a_4-a_3}(x - a_3) \end{cases}$$

The x-coordinate of the intersection point is

$$x_{RG} = \frac{zm_2(a_4 - a_3) - m_2b_4(a_4 - a_3) + m_1a_4(b_4 - b_3)}{m_2(a_4 - a_3) + m_1(b_4 - b_3)}$$

Hence

$$\begin{aligned} \zeta_1(z) &= \mu_A(x_{RG}) \\ &= \frac{m_1m_2(a_4 + b_4 - z)}{m_2(a_4 - a_3) + m_1(b_4 - b_3)} \end{aligned}$$

In summary, the membership function $\mu_{A(+)B}(z)$ is

$$\begin{cases} 0, & z < a_1 + b_1, a_4 + b_4 \le z \\ \frac{m_1 m_2 (z - a_1 - b_1)}{m_2(a_2 - a_1) + m_1(b_2 - b_1)}, & a_1 + b_1 \le z < a_2 + b_1 + (b_2 - b_1) \cdot \frac{m_1}{m_2} \\ m_1, & a_2 + b_1 + (b_2 - b_1) \cdot \frac{m_1}{m_2} \le z \\ & \qquad < a_3 + b_4 - (b_4 - b_3) \cdot \frac{m_1}{m_2} \\ \frac{m_1 m_2 (a_4 + b_4 - z)}{m_2(a_4 - a_3) + m_1(b_4 - b_3)}, & a_3 + b_4 - (b_4 - b_3) \cdot \frac{m_1}{m_2} \le z < a_4 + b_4 \end{cases}$$

i.e. $A(+)B$ is a generalized trapezoidal fuzzy set.

3.2. Subtraction

In fugure 2, we see that the maximum value of $\min\{\mu_A(x), \mu_B(x)\}$ on the curve $x - y = z$ occurs at the intersection point of the $x - y = z$ and the path joining the four points H, S, Q and F.

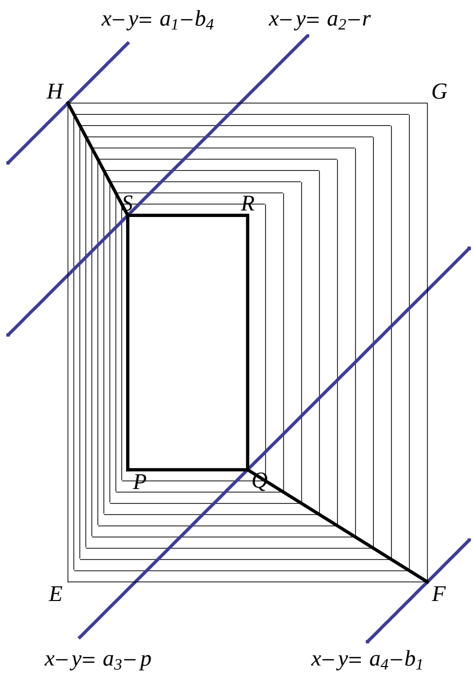

FIGURE 2. Contour plot of $(x, y) \mapsto \min\{\mu_A(x), \mu_B(y)\}$ and $x - y = z$

Then

$$\mu_{A(-)B}(z) = \begin{cases} 0, & z < a_1 - b_4, a_4 - b_1 \leq z \\ \psi_2(z), & a_1 - b_4 \leq z < a_2 - r \\ \eta_2(z), & a_2 - r \leq z < a_3 - p \\ \zeta_2(z), & a_3 - p \leq z < a_4 - b_1 \end{cases}$$

Therefore

- $\psi_2(z)$ is determined by the intersection point of the line $x - y = z$ with the line joining the two points H and S:

$$\begin{cases} x - y & = z \\ y - b_4 & = \frac{r-b_4}{a_2-a_1}(x - a_1) \end{cases}$$

The y-coordinate of the intersection point is

$$y_{HS} = -\frac{zm_1(b_4 - b_3) - m_1a_1(b_4 - b_3) - m_2b_4(a_2 - a_1)}{m_2(a_2 - a_1) + m_1(b_4 - b_3)}$$

Hence

$$\begin{aligned} \psi_2(z) &= \mu_B(y_{HS}) \\ &= \frac{m_1m_2(z + b_4 - a_1)}{m_2(a_2 - a_1) + m_1(b_4 - b_3)} \end{aligned}$$

- $\eta_2(z)$ is a constant function m_1 on $\left[a_2 - r, a_3 - p\right]$ because $(x, y) \mapsto \min\{\mu_A(x), \mu_B(y)\}$ takes constant value m_1 on the rectangle $PQRS$.
- $\eta_2(z)$ is determined by the intersection point of the line $x - y = z$ with the line joining the two points Q and F:

$$\begin{cases} x - y & = z \\ y - p & = \frac{b_1-p}{a_4-a_3}(x - a_3) \end{cases}$$

The y-coordinate of the intersection point is

$$y_{QF} = \frac{zm_2(a_4 - a_3) - m_2b_4(a_4 - a_3) + m_1a_4(b_4 - b_3)}{m_2(a_4 - a_3) + m_1(b_4 - b_3)}$$

Hence

$$\begin{aligned} \zeta_2(z) &= \mu_B(y_{QF}) \\ &= \frac{m_1m_2(a_4 - b_1 - z)}{m_2(a_4 - a_3) + m_1(b_2 - b_1)} \end{aligned}$$

In summary, the membership function $\mu_{A(-)B}(z)$ is

$$\begin{cases} 0, & z < a_1 - b_4, a_4 - b_1 \leq z \\ \frac{m_1 m_2 (z + b_4 - a_1)}{m_2(a_2 - a_1) + m_1(b_4 - b_3)}, & a_1 - b_4 \leq z < a_2 - \left(b_4 - (b_4 - b_3) \cdot \frac{m_1}{m_2}\right) \\ m_1, & a_2 - \left(b_4 - (b_4 - b_3) \cdot \frac{m_1}{m_2}\right) \leq z \\ & \qquad < a_3 - \left(b_1 + (b_2 - b_1) \cdot \frac{m_1}{m_2}\right) \\ \frac{m_1 m_2 (a_4 - b_1 - z)}{m_2(a_4 - a_3) + m_1(b_2 - b_1)}, & a_3 - \left(b_1 + (b_2 - b_1) \cdot \frac{m_1}{m_2}\right) \leq z < a_4 - b_1 \end{cases}$$

i.e. $A(-)B$ is a generalized trapezoidal fuzzy set.

3.3. Multiplication

In fugure 3, we see that the maximum value of $\min\{\mu_A(x), \mu_B(x)\}$ on the curve $xy = z$ occurs at the intersection point of the $xy = z$ and the path joining the four points E, P, R and G.

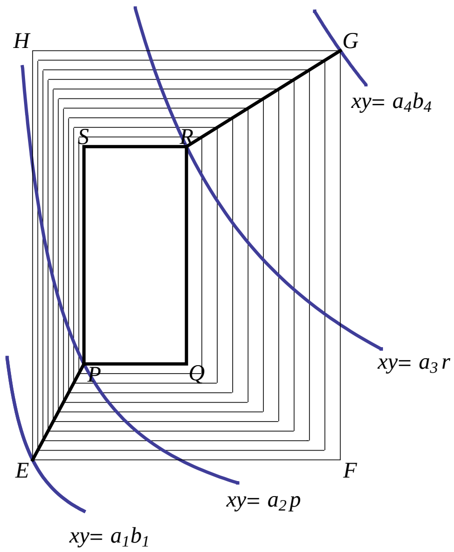

FIGURE 3. Contour plot of $(x, y) \mapsto \min\{\mu_A(x), \mu_B(y)\}$ and $xy = z$

Then

$$\mu_{A(\cdot)B}(z) = \begin{cases} 0, & z < a_1 b_1, a_4 b_4 \le z \\ \psi_3(z), & a_1 b_1 \le z < a_2 p \\ \eta_3(z), & a_2 p \le z < a_3 r \\ \zeta_3(z), & a_3 r \le z < a_4 b_4 \end{cases}$$

Therefore

- $\psi_3(z)$ is determined by the intersection point of the line $xy = z$ with the line joining the two points E and P:

$$\begin{cases} xy & = z \\ y - b_1 & = \frac{p-b_1}{a_2-a_1}(x - a_1) \end{cases}$$

  The x-coordinate of the intersection point is

$$x_{EP} = \frac{D + \sqrt{D^2 + 4m_1 m_2 (b_2 - b_1)(a_2 - a_1)z}}{2m_1(b_2 - b_1)}$$

  Hence

$$\begin{aligned} \psi_3(z) &= \mu_A(x_{EP}) \\ &= \frac{-D_1 + \sqrt{D^2 + 4m_1 m_2 (b_2 - b_1)(a_2 - a_1)z}}{2(b_2 - b_1)(a_2 - a_1)} \end{aligned}$$

  where

$$\begin{aligned} D &= b_1 m_2 (a_2 - a_1) - a_1 m_1 (b_2 - b_1) \\ D_1 &= b_1 m_2 (a_2 - a_1) + a_1 m_1 (b_2 - b_1) \end{aligned}$$

- $\eta_3(z)$ is a constant function m_1 on $\big[a_2 p, a_3 r\big]$ because $(x, y) \mapsto \min\{\mu_A(x), \mu_B(y)\}$ takes constant value m_1 on the rectangle $PQRS$.
- $\eta_3(z)$ is determined by the intersection point of the line $xy = z$ with the line joining the two points R and G:

$$\begin{cases} xy & = z \\ y - r & = \frac{b_4-r}{a_4-a_3}(x - a_3) \end{cases}$$

  The x-coordinate of the intersection point is

$$x_{RG} = \frac{\widetilde{D} + \sqrt{\widetilde{D}^2 + 4m_1 m_2 (b_4 - b_3)(a_4 - a_3)}}{2m_1(b_4 - b_3)}$$

Hence

$$\begin{aligned}\zeta_3(z) &= \mu_A(y_{RG})\\ &= \frac{\widetilde{D}_1 - \sqrt{\widetilde{D}^2 + 4m_1m_2(b_4-b_3)(a_4-a_3)z}}{2(b_4-b_3)(a_4-a_3)}\end{aligned}$$

where

$$\begin{aligned}\widetilde{D} &= a_4m_1(b_4-b_3) - b_4m_2(a_4-a_3)\\ \widetilde{D}_1 &= a_4m_1(b_4-b_3) + b_4m_2(a_4-a_3)\end{aligned}$$

In summary, the membership function $\mu_{A(\cdot)B}(z)$ is

$$\begin{cases} 0, & z < a_1b_1, a_4b_4 \le z\\ \frac{-D_1+\sqrt{D^2+4m_1m_2(b_2-b_1)(a_2-a_1)z}}{2(b_2-b_1)(a_2-a_1)}, & a_1b_1 \le z < a_2\big(b_1 + (b_2-b_1)\cdot\frac{m_1}{m_2}\big)\\ m_1, & a_2\big(b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}\big) \le z\\ & \quad < a_3\big(b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}\big)\\ \frac{\widetilde{D}_1-\sqrt{\widetilde{D}^2+4m_1m_2(b_4-b_3)(a_4-a_3)z}}{2m_1(b_4-b_3)}, & a_3\big(b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}\big) \le z < a_4b_4\end{cases}$$

where

$$\begin{aligned}D &= b_1m_2(a_2-a_1) - a_1m_1(b_2-b_1)\\ D_1 &= b_1m_2(a_2-a_1) + a_1m_1(b_2-b_1)\\ \widetilde{D} &= a_4m_1(b_4-b_3) - b_4m_2(a_4-a_3)\\ \widetilde{D}_1 &= a_4m_1(b_4-b_3) + b_4m_2(a_4-a_3)\end{aligned}$$

i.e. $A(\cdot)B$ is a fuzzy set on $\big(a_1b_1, a_4b_4\big)$, but need not to be a generalized trapezoidal fuzzy set.

3.4. Division

In fugure 4, we see that the maximum value of $\min\{\mu_A(x), \mu_B(x)\}$ on the curve $\frac{x}{y} = z$ occurs at the intersection point of the $\frac{x}{y} = z$ and the path joining the four points H, S, Q and F.

Then

$$\mu_{A(/)B}(z) = \begin{cases} 0, & z < \frac{a_1}{b_4}, \frac{a_4}{b_1} \le z\\ \psi_4(z), & \frac{a_1}{b_4} \le z < \frac{a_2}{r}\\ \eta_4(z), & \frac{a_2}{r} \le z < \frac{a_3}{p}\\ \zeta_4(z), & \frac{a_3}{p} \le z < \frac{a_4}{b_1}\end{cases}$$

Therefore

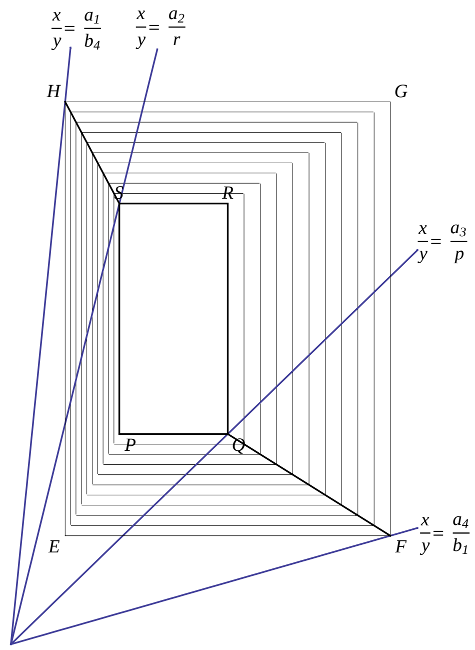

FIGURE 4. Contour plot of $(x, y) \mapsto \min\{\mu_A(x), \mu_B(y)\}$ and $\frac{x}{y} = z$

- $\psi_4(z)$ is determined by the intersection point of the line $\frac{x}{y} = z$ with the line joining the two points H and S:

$$\begin{cases} \frac{x}{y} & = z \\ y - b_4 & = \frac{r - b_4}{a_2 - a_1}(x - a_1) \end{cases}$$

The y-coordinate of the intersection point is

$$y_{HS} = \frac{m_2 b_4 (a_2 - a_1) + m_1 a_1 (b_4 - b_3)}{m_1 (b_4 - b_3) z + m_2 (a_2 - a_1)}$$

Hence

$$\begin{aligned} \psi_4(z) &= \mu_B(y_{HS}) \\ &= \frac{m + 1 m_2 (b_4 z - a_1)}{m_1 (b_4 - b_3) z + m_2 (a_2 - a_1)} \end{aligned}$$

- $\eta_4(z)$ is a constant function m_1 on $\left[\frac{a_2}{r}, \frac{a_3}{p}\right]$ because $(x, y) \mapsto \min\{\mu_A(x), \mu_B(y)\}$ takes constant value m_1 on the rectangle $PQRS$.

- $\eta_4(z)$ is determined by the intersection point of the line $\frac{x}{y} = z$ with the line joining the two points Q and F:

$$\begin{cases} \frac{x}{y} & = z \\ y - p & = \frac{b1-p}{a_4-a_3}(x - a_3) \end{cases}$$

The y-coordinate of the intersection point is

$$y_{QF} = \frac{b_1 m_2(a_4 - a_3) + a_4 m_1(b_2 - b_1)}{(b_2 - b_1)m_1 z + m_2(a_4 - a_3)}$$

Hence

$$\begin{aligned} \zeta_4(z) &= \mu_B(y_{QF}) \\ &= \frac{m_1 m_2(a_4 - b_1 z)}{m_1(b_2 - b_1)z + m_2(a_4 - a_3)} \end{aligned}$$

In summary,

$$\mu_{A(/)B}(z) = \begin{cases} 0, & z < \frac{a_1}{b_4}, \frac{a_4}{b_1} \le z \\ \frac{m_1 m_2(b_4 z - a_1)}{m_1(b_4-b_3)z + m_2(a_2-a_1)}, & \frac{a_1}{b_4} \le z < \frac{a_2}{b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}} \\ m_1, & \frac{a_2}{b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}} \le z < \frac{a_3}{b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}} \\ \frac{m_1 m_2(a_4-b_1 z)}{m_1(b_2-b_1)z + m_2(a_4-a_3)}, & \frac{a_3}{b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}} \le z < \frac{a_4}{b_1} \end{cases}$$

i.e. $A(/)B$ is a fuzzy set on $\left(\frac{a_1}{b_4}, \frac{a_4}{b_1}\right)$, but need not to be a generalized trapezoidal fuzzy set.

4. Examples

EXAMPLE 4.1. *For two generalized trapezoidal sets,* $A = (1, 2, \frac{1}{2}, 3, 6)$ *and* $B = (2, 4, \frac{7}{10}, 5, 8)$, *we have the followings.*

$$\mu_{A(+)B}(z) = \begin{cases} 0, & z < 3, 14 \le z \\ \frac{7(z-3)}{34}, & 3 \le z < \frac{38}{7} \\ \frac{1}{2}, & \frac{38}{7} \le z < \frac{62}{7} \\ \frac{7(14-z)}{72}, & \frac{62}{7} \le z < 14 \end{cases}$$

$$\mu_{A(-)B}(z) = \begin{cases} 0, & z < -7, 4 \leq z \\ \frac{7(7+z)}{44}, & -7 \leq z < -\frac{27}{7} \\ \frac{1}{2}, & -\frac{27}{7} \leq z < -\frac{3}{7} \\ \frac{7(4-z)}{62}, & -\frac{3}{7} \leq z < 4 \end{cases}$$

$$\mu_{A(\cdot)B}(z) = \begin{cases} 0, & z < 2, 48 \leq z \\ \frac{-12+\sqrt{4+70z}}{20}, & 2 \leq z < \frac{48}{7} \\ \frac{1}{2}, & \frac{48}{7} \leq z < \frac{123}{7} \\ \frac{43-\sqrt{169+35z}}{30}, & \frac{123}{7} \leq z < 48 \end{cases}$$

$$\mu_{A(/)B}(z) = \begin{cases} 0, & z < \frac{1}{8}, 3 \leq z \\ \frac{7(-1+8z)}{2(7+15z)}, & \frac{1}{8} \leq z < \frac{14}{41} \\ \frac{1}{2}, & \frac{4}{3} \leq z < \frac{20}{9} \\ \frac{7(3-z)}{21+10z}, & \frac{7}{8} \leq z < 3 \end{cases}$$

Example 4.2. *Let us consider now two triangular fuzzy numbers, $A = (2, 4, 7)$ and $B = (3, 6, 11)$. These may be identified with $A = (2, 4, 1, 4, 7)$ and $B = (3, 6, 1, 6, 11)$. Then*

$$\mu_{A(+)B}(z) = \begin{cases} 0, & z < 5, 18 \leq z \\ \frac{-5+z}{5}, & 5 \leq z < 10 \\ \frac{18-z}{8}, & 10 \leq z < 18 \end{cases}$$

$$\mu_{A(-)B}(z) = \begin{cases} 0, & z < -9, 4 \leq z \\ \frac{9+z}{7}, & -9 \leq z < -2 \\ \frac{4-z}{6}, & -2 \leq z < 4 \end{cases}$$

$$\mu_{A(\cdot)B}(z) = \begin{cases} 0, & z < 6, 77 \leq z \\ \frac{-6+\sqrt{6z}}{6}, & 6 \leq z < 24 \\ \frac{34-\sqrt{1+15z}}{15}, & 24 \leq z < 77 \end{cases}$$

$$\mu_{A(/)B}(z) = \begin{cases} 0, & z < \frac{2}{11}, \frac{7}{3} \leq z \\ \frac{11z-2}{5z+2}, & \frac{2}{11} \leq z < \frac{2}{3} \\ \frac{7-3z}{3+3z}, & \frac{2}{3} \leq z < \frac{7}{3} \end{cases}$$

Example 4.3. *For two fuzzy sets, $A = \big((3, \frac{4}{5}, 9)\big)$ and $B = \big((2, \frac{6}{7}, 7)\big)$, we identify these with $A = (3, 6, \frac{4}{5}, 6, 9)$ and $B = (2, \frac{9}{2}, \frac{6}{7}, \frac{9}{2}, 7)$. Then*

$$\mu_{A(+)B}(z) = \begin{cases} 0, & z < 5, 16 \leq z \\ \frac{3(z-5)}{20}, & 5 \leq z < \frac{31}{3} \\ \frac{4}{5}, & \frac{31}{3} \leq z < \frac{32}{3} \\ \frac{3(16-z)}{20}, & \frac{32}{3} \leq z < 16 \end{cases}$$

$$\mu_{A(-)B}(z) = \begin{cases} 0, & z < -4, 7 \leq z \\ \frac{2(4+z)}{20}, & -4 \leq z < \frac{4}{3} \\ \frac{4}{5}, & \frac{4}{3} \leq z < \frac{5}{3} \\ \frac{3(7-z)}{20}, & \frac{5}{3} \leq z < 7 \end{cases}$$

$$\mu_{A(\cdot)B}(z) = \begin{cases} 0, & z < 6, 63 \leq z \\ \frac{2(-13+\sqrt{28+z})}{35}, & 6 \leq z < 26 \\ \frac{4}{5}, & 26 \leq z < 28 \\ \frac{4(21-\sqrt{7z})}{35}, & 28 \leq z < 63 \end{cases}$$

$$\mu_{A(/)B}(z) = \begin{cases} 0, & z < \frac{3}{7}, \frac{9}{2} \leq z \\ \frac{12(-3+7z)}{5(9+7z)}, & \frac{3}{7} \leq z < \frac{9}{7} \\ \frac{4}{5}, & \frac{9}{7} \leq z < \frac{18}{13} \\ \frac{12(9-2z)}{5(9+7z)}, & \frac{18}{13} \leq z < \frac{9}{2} \end{cases}$$

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Korea Institute for Curriculum and Evaluation
Seoul 100-784, Republic of Korea
E-mail: yibongju@kice.re.kr

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Department of Mathematics and Research Institute for Basic Sciences
Jeju National University
Jeju 690-756, Republic of Korea
E-mail: yunys@jejunu.ac.kr