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GOLDEN RATIO RIESZ-NÁGY-TAKÁCS DISTRIBUTION

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Abstract. We study some properties of the Riemann-Stieltjes integrals with respect to the Riesz-Nágy-Takács distribution $H_{a,p}$ and its inverse $H_{p,a}$ on the unit interval satisfying the equation $1-a=a^2$ and $p=1-a$. Using the properties of the dual distributions $H_{a,p}$ and $H_{p,a}$, we compare the Riemann-Stieltjes integrals of $H_{a,p}$ over some essential intervals with that of its inverse $H_{p,a}$ over the related intervals.

1. Introduction

Golden ratio $\phi = \frac{\sqrt{5}+1}{2}$ is the popular number related to the Fibonacci sequence. Recently we([2, 5]) also studied the moments and Riemann-Stieltjes integral of the Riesz-Nágy-Takács distribution([12]) which is a strictly increasing singular function. Further we([3, 4]) also discussed the Riemann-Stieltjes integrals with respect to the Riesz-Nágy-Takács distribution $H_{a,p}$ and $H_{a',p'}$ satisfying the equations $1-a=a^m$ and $a'=(1-a')^m$ where m is a positive integer over some fundamental intervals. In this paper, we study $H_{a,p}$ and its dual distribution or its inverse $H_{p,a}$ where $a=\frac{1}{\phi}$ with $\phi=\frac{\sqrt{5}+1}{2}$ and $p=1-a$. We note that $1-a=a^2$ where $a=\frac{1}{\phi}$, so we can apply our results([2, 3, 4, 5]) to this peculiar example $H_{p,a}$, which we will call the golden ratio Riesz-Nágy-Takács distributions.

In this paer, we find some solutions of area comparison equations related to the Riemann-Stieltjes integrals with respect to the Riesz-Nágy-Takács distribution $H_{a,p}$ and its inverse $H_{p,a}$. This will give some insight for the golden ratio Riesz-Nágy-Takács distributions.

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2. Preliminaries

Let $\mathbb{N}$ be the set of the positive integers. Consider $a \in (0,1)$ and $p \in (0,1)$. We([2]) recall the Riesz-Nágy-Takács distribution

$$H_{a,p}(x) = \Sigma_{j=1}^{\infty} \frac{(1-p)^{j-1}}{p^{j-1}} p^{a_j}$$

for

$$x = \Sigma_{j=1}^{\infty} \frac{(1-a)^{j-1}}{a^{j-1}} a^{a_j} \in (0,1]$$

with integers $1 \leq a_1 < a_2 < \cdots < a_j < \cdots$ and $H_{a,p}(0) = 0$.

We define an n-th fundamental interval $I_{i_1 \cdots i_n} = f_{i_1} \circ \cdots \circ f_{i_n}(I)$ for $H_{a,p}$ where $f_0(x) = ax$ and $f_1(x) = (1-a)x + a$ on $I = [0,1]$, $i_j \in \{0,1\}$ and $1 \leq j \leq n$. Clearly there are 2^n members of the n-th fundamental intervals in $[0,1]$. We note that $[0,1]$ is the self-similar set by the iteration function system $\{f_0, f_1\}$([8]) satisfying the open set condition.

We give some integral equations for the Riesz-Nágy-Takács distributions $H_{a,p}$ over the fundamental intervals.

Proposition 2.1. ([5]) *For the Riesz-Nágy-Takács distributions $F = H_{a,p}$, we have*

$$\int_{\gamma}^{\gamma + a^{n-k}(1-a)^k} \phi(t) dF(t) = p^{n-k}(1-p)^k \int_0^1 \phi(a^{n-k}(1-a)^k t + \gamma) dF(t)$$

where $[\gamma, \gamma + a^{n-k}(1-a)^k]$ is an n-th fundamental interval where $k = 0, 1, ..., n-1, n$.

From now on, we fix $F(x) = H_{a,p}(x)$ on the unit interval where $1 - a = a^2$ and $p = 1 - a$. Clearly we see that $a = \frac{1}{\phi}$ with $\phi = \frac{\sqrt{5}+1}{2}$. We call F *the golden ratio Riesz-Nágy-Takács distribution*. We also note that $F^{-1}(x) = H_{p,a}(x)$ on the unit interval, that is the inverse function on the unit interval of the golden ratio Riesz-Nágy-Takács distribution F is the dual distribution([4]) of F. From now on, we define $[0,a], [0,p], [a,1], [p,1]$ to be *the essential intervals*.

We also give some integral equations for dual Riesz-Nágy-Takács distributions F, F^{-1} over some essential intervals, which also can be derived from the above Proposition.

Proposition 2.2. ([3, 5])

$$\int_0^p \phi(t) dF(t) = p^2 \int_0^1 \phi(pt) dF(t).$$

Proposition 2.3. ([4, 5])

$$\int_a^1 \phi(t)dF^{-1}(t) = p^2 \int_0^1 \phi(pt + (1-p))dF^{-1}(t).$$

3. Main results

We give the value $F(x)$ for $x \in [0,1]$ using the β-expansion([13]) of x.

Theorem 3.1. *For every* $x \in [0,1]$, $x = \Sigma_{j=1}^{\infty} \frac{1}{\phi^{k_j}}$ *for some integers*

$$1 \le k_1 < k_1 + 1 \le k_2 < k_2 + 1 \le \cdots \le k_j < k_j + 1 \le \cdots ,$$

where ϕ *is the golden ratio* $\frac{\sqrt{5}+1}{2}$. *Further*

$$F(\Sigma_{j=1}^{\infty} \frac{1}{\phi^{k_j}}) = \Sigma_{j=1}^{\infty} \frac{1}{\phi^{2k_j - 3j + 3}}.$$

Proof. For every $x \in (0,1]$, there are integers $1 \le a_1 < a_2 < \cdots < a_j < \cdots$ such that $x = \Sigma_{j=1}^{\infty} \frac{(1-a)^{j-1}}{a^{j-1}} a^{a_j}$([12]). Noting $\frac{1-a}{a} = a = \frac{1}{\phi}$, we have $\frac{(1-a)^{j-1}}{a^{j-1}} a^{a_j} = \frac{1}{\phi^{k_j}}$ where $k_j = a_j + j - 1$. This gives the first argument. The fact that $\frac{1-p}{p} = \phi$ and $p = \frac{1}{\phi^2}$ gives $\frac{(1-p)^{j-1}}{p^{j-1}} p^{a_j} = \frac{1}{\phi^{2a_j - j + 1}}$. The second argument follows from $a_j = k_j - j + 1$. □

We compute some Riemann-Stieltjes integrals with respect to the golden ratio Riesz-Nágy-Takács distribution F and its inverse F^{-1} on the essential intervals $[0,p], [0,a]$.

Theorem 3.2. *On the essential intervals* $[0,p], [0,a]$, *we have*

(1) $\int_0^p tdF(t) = \frac{13-21a}{3-4a}$,
(2) $\int_0^p tdF^{-1}(t) = \frac{13a-8}{3-4a}$,
(3) $\int_0^a tdF(t) = \frac{5a-3}{3-4a}$,
(4) $\int_0^a tdF^{-1}(t) = \frac{18-29a}{3-4a}$.

Proof. (1) follows from Proposition 2.2 and Proposition 2.1. Noting $F^{-1} = H_{p,a}$, we have (2) from Proposition 2.1. (3) follows from Proposition 2.1. Noting

$$\int_0^a tdF^{-1}(t) = \int_0^1 tdF^{-1}(t) - \int_a^1 tdF^{-1}(t),$$

we have (4) from Proposition 2.3 and Proposition 2.1. □

REMARK 3.3. In the above Theorem, $[0,a]$ is a fundamental interval for F and $[0,p]$ is a fundamental interval for F^{-1}. Further $[0,p]$ is a fundamental interval for F whereas $[0,a]$ is not a fundamental interval for F^{-1}.

The next Corollary is a comparison of the Riemann-Stieltjes integrals of an identity function with respect to F and F^{-1} over a fundamental interval.

COROLLARY 3.4. *For the value* $\int_0^p tdF^{-1}(t)$, *we have*
(1) $\int_0^p tdF(t) = a\int_0^p tdF^{-1}(t)$,
(2) $\int_0^a tdF(t) = \frac{1}{a^2}\int_0^p tdF^{-1}(t)$.

Proof. It follows from the above Theorem. □

COROLLARY 3.5. *We have a reciprocal relation between* F *and* F^{-1} *such that*

$$\int_0^p tdF(t) + \int_0^a tdF(t) = \int_0^p tdF^{-1}(t) + \int_0^a tdF^{-1}(t).$$

Proof. It follows from the above Theorem. □

REMARK 3.6. A simple geometric comparison gives the inequality

$$\int_0^p tdF^{-1}(t) < \frac{ap}{2} < \int_0^a tdF(t).$$

The intermediate value theorem leads us to arrive at the finding of the solution between p and a of the equation

$$g(x) = \frac{\int_0^x tdF(t)}{\int_0^p tdF^{-1}(t)} = 1$$

since $\int_0^p tdF(t) = a\int_0^p tdF^{-1}(t) < \int_0^p tdF^{-1}(t)$ and g is a continuous function on the unit interval. More precisely,

$$g(p) = a < 1 < g(a) = \frac{1}{a^2} < g(1) = \frac{1}{a^5}.$$

THEOREM 3.7. $\int_0^x tdF(t) = \int_0^p tdF^{-1}(t)$ *for* $x = \frac{p}{1-p^2} = \frac{3-\sqrt{5}}{3\sqrt{5}-5} \in (p,a)$.

Proof. Let

$$g(x) = \frac{\int_0^x tdF(t)}{\int_0^p tdF^{-1}(t)}.$$

We find $g(p+\Sigma_{n=1}^{k} a^{2n+2}(1-a)^{n-1}) < 1$ for each $k \in \mathbb{N}$, whereas $g(p+\Sigma_{n=1}^{k} a^{2n+1}(1-a)^{n-1}) > 1$ for each $k \in \mathbb{N}$. This follows from the iterated application of Proposition 2.1. This gives our conclusion that

$$p+\Sigma_{n=1}^{\infty} a^{2n+2}(1-a)^{n-1} = p+\Sigma_{n=1}^{\infty} p^{n+1}p^{n} = \frac{p}{1-p^2}$$

is the solution of $g(x)=1$. □

Remark 3.8. Consider a continuous function

$$h(x) = \frac{\int_0^x t dF(t)}{\int_0^x t dF^{-1}(t)}$$

on $(0,1]$. The same arguments of the above Corollaries give the inequality

$$h(p) = a < 1 < h(a) = \frac{1}{3a-1} < h(1) = \frac{1}{a^2}.$$

The intermediate value theorem leads us to arrive at the finding of the solution between p and a of the equation $h(x)=1$. However we note that

$$h\left(\frac{p}{1-p^2}\right) = \frac{\int_0^{\frac{p}{1-p^2}} t dF(t)}{\int_0^{\frac{p}{1-p^2}} t dF^{-1}(t)} = \frac{\int_0^p t dF^{-1}(t)}{\int_0^{\frac{p}{1-p^2}} t dF^{-1}(t)} < \frac{\int_0^p t dF^{-1}(t)}{\int_0^p t dF^{-1}(t)} = 1.$$

There must be a solution $x \in (\frac{p}{1-p^2}, a)$ such that $h(x)=1$ from the intermediate value theorem. It would be interesting to find such a solution.

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