# THE TENSION FIELD OF THE ENERGY FUNCTIONAL ON RIEMANNIAN SUBMERSION 

Boo-Yong Choi*


#### Abstract

In this paper, we will study the tension field of the function related to a Riemannain submersion $\pi: N \rightarrow M$ with totally geodesic fibres. In case that the Riemannain submersion $\pi: N \rightarrow M$ particularly has a smooth map $f: M \rightarrow N$ which happens to be a section, we will show that tension field $\tau(f)$ of the energy functional can be decomposed into the horizontal and vertical parts.


## 1. Introduction

Let $M$ and $N$ be complete Riemannian manifolds. Asumme $M$ is compact. A smooth map $f: M \rightarrow N$ is called harmonic if it is a critical point of the energy functional. This critical point of the energy functional is written and characterized in terms of some differential equation (called the Euler-Lagrange equation). And we can now calculate the tension field to obtain the Euler-Lagrange equation. We consider the case when $N$ is a fibre bundle over $M$, and $f: M \rightarrow N$ is a smooth map which happens to be a section of this fibration. We will consider the case when the fibres are totally geodesic compact submanifolds, and hence $N$ is also a compact Riemannian manifold. In this case, the Euler-Lagrange equation for such a section is formulated([1], [2], [3], [4]). In this paper, we will obtain both horizontal and vertical parts of the tension field. In section 2 , we are primarily devoted to a summary of known results of Riemannian submersion([5]). And in section 3, we will set up the tension field of $f: M \rightarrow N$ which happens to be a section of this fibration. In main result, we will decompose $\tau(f)$ as horizontal parts $\tau^{\mathcal{H}}(f)$ and vertical parts $\tau^{\mathcal{V}}(f)$.

[^0]
## 2. Riemannian submersion with totally geodesic fibres

We will study a particular case of Riemannian submersion with totally geodesic fibres, so we need some properties and formulas about Riemannian submersion. We will use the terminology of O'Neill([5]).

Definition 2.1. A Riemannian submersion $\pi: N \rightarrow M$ is a submersion of Riemannian manifolds such that:
(1) The fibre $\pi^{-1}(x), x \in M$, are Riemannian submanifolds of $N$,
(2) $d \pi$ preserves scalar products of vectors normal to fibres.

Given a Riemannian submersion $\pi$ from $N$ to $M$, we denote by $\mathcal{V}$ the vector subbundle of $T N$ defined by the foliation of $N$ by the fibres of $\pi$. $\mathcal{H}$ will denote the complementary distribution of $\mathcal{V}$ in $T N$ determined by the metric on $N$. Following O'Neill([5]), we define the tensor $T$ for arbitrary vector fields $E$ and $F$ by $T_{E} F=\mathcal{H} \nabla_{\mathcal{V} E} \mathcal{V} F+\mathcal{V} \nabla_{\mathcal{V} E} \mathcal{H} F$ where $\mathcal{V} E, \mathcal{H} F$, etc. denote the vertical and horizontal projections of vector field $E$ and $\nabla$ is the covariant derivative of $N$. O'Neill has described the following three properties of the tensor $T$ :

1. $T_{E}$ is a skew-symmetric operator on the tangent space of $N$ reversing horizontal and vertical subspaces.
2. $T_{E}=T_{\mathcal{V} E}$
3. For vertical vector fields $V$ and $W, T$ is symmetric, i.e., $T_{V} W=$ $T_{W} V$.

In fact, along a fibre $T$ is the second fundamental form of the fibre provided we restrict ourselves to vertical vector fields. Next we define the integrability tensor $A$ associated with the submersion. For arbitrary vector fields $E$ and $F$,

$$
A_{E} F=\mathcal{H} \nabla_{\mathcal{H} E} \mathcal{V} F+\mathcal{V} \nabla_{\mathcal{H} E} \mathcal{H} F
$$

$1^{\prime}$. At each point $A_{E}$ is a skew-symmetric operator on $T N$ reversing the horizontal and vertical subspaces.
$2^{\prime} . A_{E}=A_{\mathcal{H E}}$
$3^{\prime}$. For $X, Y$ horizontal $A$ is alternating, i.e., $A_{X} Y=-A_{Y} X$.
We define a vector field $X$ on $N$ to be basic provided $X$ is horizontal and $\pi$-related to a vector field $\tilde{X}$ on $M$. Every vector field $\tilde{X}$ on $M$ has a horizontal lift $X$ to $N$ and $X$ is basic.

We recall the following standard results about Riemannian submersion.

Lemma 2.2. If $X$ and $Y$ are basic vector fields on $N$, then
(a) $g(X, Y)=h(\tilde{X}, \tilde{Y}) \circ \pi$ where $g$ is the metric on $N$ and $h$ the metric on $M$,
(b) $\mathcal{H}[X, Y]$ is basic and is $\pi$-related to $[\tilde{X}, \tilde{Y}]$
(c) $\mathcal{H} \nabla_{X} Y$ is basic and is $\pi$-related to $\tilde{\nabla}_{\tilde{X}} \tilde{Y}$ where $\tilde{\nabla}$ is the Riemannian connection on $M$.

Lemma 2.3. If $X$ and $Y$ are horizontal vector fields, then

$$
A_{X} Y=\frac{1}{2} \mathcal{V}[X, Y] .
$$

Lemma 2.4. Let $X$ and $Y$ be horizontal vector fields, and $V$ and $W$ vertical vector fields. Then
(a) $\nabla_{V} W=T_{V} W+\mathcal{V} \nabla_{V} W$
(b) $\nabla_{V} X=T_{V} X+\mathcal{H} \nabla_{V} X$
(c) $\nabla_{X} V=A_{X} V+\mathcal{V} \nabla_{X} V$
(d) $\nabla_{X} Y=A_{X} Y+\mathcal{H} \nabla_{X} Y$

Furthermore, if $X$ is basic, $\mathcal{H} \nabla_{V} X=A_{X} V$.
Lemma 2.5. Let $X$ be a horizontal vector field and $W$ a vertical vector field. Then
(a) $\left(\nabla_{X} A\right)_{W}=-A_{A_{X} W}$,
(b) $\left(\nabla_{W} T\right)_{X}=-T_{T_{W} X}$.

The next result gives a geometric characterization of the parallelism of the fundamental tensors $T$ and $A$.
(a) If $A$ is parallel, then $A$ is identically zero, i.e., $\nabla_{E} A=0$ implies $A=0$.
(b) If $T$ is parallel, then $T$ is identically zero, i.e., $\nabla_{E} T=0$ implies $T=0$.

Thus Riemannian submersions with parallel integrability tensors $A$ are characterized as those whose horizontal distributions are integrable, and Riemannian submersions with parallel tensors $T$ as those fibres are totally geodesic.

Assume $\pi: N \rightarrow M$ has the structure of a fibred space; as usual, assume $\pi$ is a Riemannian submersion and, in addition, $N$ is complete. Let $\gamma$ be a smooth curve in $M$ with $\gamma(0)=p$ and $\gamma\left(t_{0}\right)=q$. Then the family of unique horizontal lifts of $\gamma$ to $N$ denoted by $\left\{\tilde{\gamma}_{x}\right\}$ with $\tilde{\gamma}_{x}(0)=x \in \pi^{-1}(p)$, we have $F_{\gamma}(x)=\tilde{\gamma}_{x}\left(t_{0}\right)$ and therefore, the mapping $F_{\gamma}$ are diffeomorphisms between the fibres. Moreover, a necessary and sufficient condition for the mapping $F_{\gamma}$ to be isometries is that the fibres are totally geodesic.

## 3. Tension field of sections

Let $M$ and $N$ be complete Riemannian manifolds. Assume $M$ is compact. A smooth map $\pi: N \rightarrow M$ is called a Riemannian submersion if $\pi$ is a submersion and if for each $x \in N$, the horizontal subspace of $T_{x} N$ (orthogonal to the fibre over $\pi(x)$ in $N$ ) is mapped isometrically by $\left.d \pi\right|_{x}$ to $T_{\pi(x)} M$. We denote by $\mathcal{H}, \mathcal{V}$ the horizontal and the vertical distribution, respectively. Then we can decompose the tangent bundle $T N=T N^{\mathcal{H}} \oplus T N^{\mathcal{V}}$, where we denote by $T N^{\mathcal{H}}, T N^{\mathcal{V}}$ the horizontal and the vertical subbundle, respectively.

We now consider a Riemannian submersion with totally geodesic fibre $F$, that is, for each $x$ in $N$ with $p=\pi(x), \pi^{-1}(p)=F_{x}$ is a totally geodesic submanifold of $N$. Then all the fibres are isometric to each other and $\pi$ is a Riemannian fibration. Furthermore, the horizontal distribution defines a connection on this fibre bundle. Let $f: M \rightarrow N$ be a smooth map which happens to be a section. The energy functional of the section $f$ is $E(f)=\int_{M} e(f) d v$, where $e(f)=\frac{1}{2}\|d f\|^{2}$ is the energy density of $f$. The differential map $d f$ is a differential 1-form with values in the pull-back bundle $f^{-1}(T N)$ and hence a section of $T^{*} M \otimes f^{-1}(T N)$. Decompose $f^{-1}(T N)$ as $f^{-1}\left(T N^{\mathcal{H}}\right) \oplus f^{-1}\left(T N^{\mathcal{V}}\right)$, and then we have $d f=d f^{\mathcal{H}}+d f^{\mathcal{V}}$, where $d f^{\mathcal{H}} \in \Gamma\left(T^{*} M \otimes f^{-1}\left(T N^{\mathcal{H}}\right)\right)$, $d f^{\mathcal{V}} \in \Gamma\left(T^{*} M \otimes f^{-1}\left(T N^{\mathcal{V}}\right)\right)$, and $\Gamma(\cdot)$ denotes the set of all smooth sections of the corresponding bundle. Then the energy $E(f)$ is given by

$$
E(f)=E^{\mathcal{H}}(f)+E^{\mathcal{V}}(f)=\frac{1}{2} \int_{M}\left\|d f^{\mathcal{H}}\right\|^{2} d v+\frac{1}{2} \int_{M}\left\|d f^{\mathcal{V}}\right\|^{2} d v
$$

Since $f$ is a section of a Riemannian fibration, the linear map $d f_{p}^{\mathcal{H}}$ : $T_{p} M \rightarrow\left(T_{x} N\right)^{\mathcal{H}}$ is an isometry for each $p=\pi(x)$, and hence we have $E^{\mathcal{H}}(f)=\frac{m}{2} \operatorname{Vol}(M)(\operatorname{dim} M=m)$.

For $f: M \rightarrow N$ we now consider the Euler-Lagrange equation of the energy functional. Let $\nabla$ and $\tilde{\nabla}$ be the Levi-Civita connection on $M$ and $N$, respectively, and let $\nabla$ be the induced connection on the pullback bundle. Then we have $\bar{\nabla}(d f) \in \Gamma\left(\left(S^{2} M \otimes f^{-1}\left(T N^{\mathcal{H}}\right)\right) \oplus\left(S^{2} M \otimes\right.\right.$ $\left.f^{-1}\left(T N^{\mathcal{V}}\right)\right)$ ), where $S^{2} M$ is the space of symmetric covariant 2-tensors. Taking trace of the second fundamental form gives the tension field,

$$
\tau(f)=-\bar{\nabla}^{*}(\bar{\nabla} d f)=\operatorname{Tr}(\bar{\nabla} d f) \in \Gamma\left(f^{-1}(T N)\right) .
$$

## 4. Main result

In this section, we will show both horizontal and vertical parts of the tension field. Let $\pi: N \rightarrow M$ be a Riemannian submersion with totally geodesic fibre and $f: M \rightarrow N$ be a smooth section as section 3 . In section 3,

$$
d f \in \Gamma\left\{\left(T^{*} M \otimes f^{-1}\left(T N^{\mathcal{H}}\right) \oplus\left(T^{*} M \otimes f^{-1}\left(T N^{\mathcal{V}}\right)\right\}\right.\right.
$$

we have $d f=d f^{\mathcal{H}}+d f^{\mathcal{V}}$. And

$$
\bar{\nabla}(d f) \in \Gamma\left(\left(S^{2} M \otimes f^{-1}\left(T N^{\mathcal{H}}\right)\right) \oplus\left(S^{2} M \otimes f^{-1}\left(T N^{\mathcal{V}}\right)\right)\right)
$$

Since $\tau(f)=-\bar{\nabla}^{*}(\bar{\nabla} d f)=\operatorname{Tr}(\bar{\nabla} d f)$, we decompose $\tau(f)$ as $\tau(f)=$ $\tau^{\mathcal{H}}(f)+\tau^{\mathcal{V}}(f)$, where $\tau^{\mathcal{H}}(f) \in \Gamma\left(f^{-1}\left(T N^{\mathcal{H}}\right)\right)$ and $\tau^{\mathcal{V}}(f) \in \Gamma\left(f^{-1}\left(T N^{\mathcal{V}}\right)\right)$. For a vector field $X$ on $M$ let $\tilde{X}$ denote the basic vector field which is a horizontal lift of $X$. Then for a local orthonormal frame field $\left\{e_{i}\right\}$ of $M, d f^{\mathcal{H}}\left(e_{i}\right)=\tilde{e}_{i}$.

$$
\begin{aligned}
\tau(f) & =\operatorname{Tr}(\bar{\nabla} d f) \\
& =\sum_{i}(\bar{\nabla} d f)\left(e_{i}, e_{i}\right)=\sum_{i}\left(\bar{\nabla}_{e_{i}} d f\right)\left(e_{i}\right) \\
& =\sum_{i}\left(\bar{\nabla}_{e_{i}}\left(d f\left(e_{i}\right)\right)-d f\left(\nabla_{e_{i}} e_{i}\right)\right) \\
(4.1) & =\sum_{i}\left\{\bar{\nabla}_{e_{i}}\left(d f^{\mathcal{H}}\left(e_{i}\right)+d f^{\mathcal{V}}\left(e_{i}\right)\right)-\left(d f^{\mathcal{H}}\left(\nabla_{e_{i}} e_{i}\right)+d f^{\mathcal{V}}\left(\nabla_{e_{i}} e_{i}\right)\right)\right\} .
\end{aligned}
$$

Now, we can calculate the horizontal and vertical parts of the tension field. Since $\left(\bar{\nabla}_{e_{i}} d f^{\mathcal{H}}\right)\left(e_{i}\right)=\bar{\nabla}_{e_{i}}\left(d f^{\mathcal{H}}\left(e_{i}\right)\right)-d f^{\mathcal{H}}\left(\nabla_{e_{i}} e_{i}\right)$ and $\tilde{\nabla}_{\tilde{e}_{i}} \tilde{e}_{i}=$ $\widetilde{\left(\overline{\nabla_{e_{i}} e_{i}}\right)}$ by O'Neill's formular in [5].

$$
\begin{align*}
\left(\bar{\nabla}_{e_{i}} d f f^{\mathcal{H}}\right)\left(e_{i}\right) & =\left(\bar{\nabla}_{e_{i}} \tilde{e}_{i}\right)-\left(\widetilde{\nabla_{e_{i}} e_{i}}\right) \\
& =\left(\tilde{\nabla}_{\tilde{e_{i}}} \tilde{e}_{i}+\tilde{\nabla}_{d f^{\mathcal{V}}\left(e_{i}\right)} \tilde{e}_{i}\right)-\left(\widetilde{\nabla_{e_{i}} e_{i}}\right) \\
& =\tilde{\nabla}_{d f^{\mathcal{V}}\left(e_{i}\right)} \tilde{e}_{i} . \tag{4.2}
\end{align*}
$$

And since fibres are totally geodesic, we have $\tilde{\nabla}_{d f}{ }_{\left(e_{i}\right)} \tilde{e}_{i} \in \mathcal{H}$ i.e., $\left(\bar{\nabla}_{e_{i}} d f^{\mathcal{H}}\right)^{\mathcal{V}}\left(e_{i}\right)=0$. For the vertical component, we locally extend $d f^{\mathcal{V}}\left(e_{i}\right)$, a vector field along $f$, to a vertical vector field on $N$ which we also denote by $d f^{\mathcal{V}}\left(e_{i}\right)$. We then have

$$
\begin{aligned}
\left(\bar{\nabla}_{e_{i}} d \mathcal{L}^{\mathcal{L}}\right)\left(e_{i}\right) & =\bar{\nabla}_{e_{i}} d f^{\mathcal{V}}\left(e_{i}\right)-d f^{\mathcal{V}}\left(\nabla_{e_{i}} e_{i}\right) \\
& =\left(\tilde{\nabla}_{\tilde{e}_{i}} d f^{\mathcal{V}}\left(e_{i}\right)+\tilde{\nabla}_{d f^{\mathcal{L}}}\left(e_{i}\right) d f^{\mathcal{V}}\left(e_{i}\right)\right)-d f^{\mathcal{V}}\left(\nabla_{e_{i}} e_{i}\right)
\end{aligned}
$$

where $d f^{\mathcal{V}}\left(\nabla_{e_{i}} e_{i}\right)$ and $\tilde{\nabla}_{d f^{\mathcal{V}}\left(e_{i}\right)} d f^{\mathcal{V}}\left(e_{i}\right)$ are in $\mathcal{V}$ because the fibres are totally geodesic. Furthermore $\left[\tilde{e}_{i}, d f^{\mathcal{V}}\left(e_{i}\right)\right] \in \mathcal{V}$. Therefore

$$
\left[\tilde{e}_{i}, d f^{\mathcal{V}}\left(e_{i}\right)\right]^{\mathcal{H}}=\left(\tilde{\nabla}_{\tilde{e}_{i}} d f^{\mathcal{V}}\left(e_{i}\right)\right)^{\mathcal{H}}-\left(\tilde{\nabla}_{d f} \mathcal{V}\left(e_{i}\right) \tilde{e}_{i}\right)^{\mathcal{H}}=0,
$$

so $\left(\tilde{\nabla}_{\tilde{e}_{i}} d \mathcal{J}^{\mathcal{V}}\left(e_{i}\right)\right)^{\mathcal{H}}=\left(\tilde{\nabla}_{d f^{\mathcal{V}}\left(e_{i}\right)} \tilde{e}_{i}\right)^{\mathcal{H}}$. But since $\tilde{\nabla}_{d f^{\mathcal{V}}\left(e_{i}\right)} \tilde{e}_{i} \in \mathcal{H},\left(\tilde{\nabla}_{d f^{\mathcal{V}}\left(e_{i}\right)} \tilde{e}_{i}\right)^{\mathcal{H}}$ $=\tilde{\nabla}_{d f}{ }^{\nu}\left(e_{i}\right) \tilde{e}_{i}$. Thus we conclude that

$$
\begin{equation*}
\left(\bar{\nabla}_{e_{i}} d f^{\mathcal{V}}\left(e_{i}\right)\right)^{\mathcal{H}}=\left(\tilde{\nabla}_{\tilde{e}_{i}} d f^{\mathcal{V}}\left(e_{i}\right)\right)^{\mathcal{H}}=\tilde{\nabla}_{d f^{\mathcal{V}}\left(e_{i}\right)} \tilde{e}_{i} . \tag{4.3}
\end{equation*}
$$

Now, we can decompose the horizontal and vertical parts of tension field by (4.2) and (4.3).

$$
\begin{aligned}
\tau(f)= & \sum_{i}\left\{\bar{\nabla}_{e_{i}}\left(d f^{\mathcal{H}}\left(e_{i}\right)+d f^{\mathcal{V}}\left(e_{i}\right)\right)-\left(d f^{\mathcal{H}}\left(\nabla_{e_{i}} e_{i}\right)+d f^{\mathcal{V}}\left(\nabla_{e_{i}} e_{i}\right)\right)\right\} \\
= & \sum_{i}\left\{\tilde{\nabla}_{d f^{\mathcal{V}}\left(e_{i}\right)} \tilde{e}_{i}\right\} \\
& +\sum_{i}\left\{\left(\bar{\nabla}_{e_{i}} d f^{\mathcal{V}}\left(e_{i}\right)\right)^{\mathcal{H}}+\left(\bar{\nabla}_{e_{i}} d f^{\mathcal{V}}\left(e_{i}\right)\right)^{\mathcal{V}}-d f^{\mathcal{V}}\left(\nabla_{e_{i}} e_{i}\right)\right\} \\
= & 2 \sum_{i}\left\{\tilde{\nabla}_{d f^{\mathcal{V}}\left(e_{i}\right)} \tilde{e}_{i}\right\}+\sum_{i}\left\{\left(\bar{\nabla}_{e_{i}} d f^{\mathcal{V}}\left(e_{i}\right)\right)^{\mathcal{V}}-d f^{\mathcal{V}}\left(\nabla_{e_{i}} e_{i}\right)\right\} .
\end{aligned}
$$

Therefore we now have the following.
Theorem 4.1. For $f: M \rightarrow N$ be a smooth section as section 3, we can decompose $\tau(f)$ as horizontal and vertical parts i.e.,

$$
\tau(f)=\tau^{\mathcal{H}}(f)+\tau^{\mathcal{V}}(f)
$$

where $\tau^{\mathcal{H}}(f)=2 \sum_{i}\left(\tilde{\nabla}_{d f^{\mathcal{V}}\left(e_{i}\right)} \tilde{e}_{i}\right)$ and $\tau^{\mathcal{V}}(f)=\sum_{i}\left\{\left(\bar{\nabla}_{e_{i}} d f^{\mathcal{V}}\left(e_{i}\right)\right)^{\mathcal{V}}-\right.$ $\left.d f^{\mathcal{V}}\left(\nabla_{e_{i}} e_{i}\right)\right\}$.

## References

[1] B. Y. Choi and J. W. Yim, Distributions on Riemannian Manifolds, which are Harmonic maps, Tohoku Math. J. 55 (2003), 175-188.
[2] T. Ishihara, Harmonic section of tangent bundles, J. Math. Tokushima Univ. 13 (1979), 23-27.
[3] G. Jensen and M. Rigoli, Harmonic Gauss maps, Pacific J. Math. 136 (1989), 261-282.
[4] J. J. Konderak, On harmonic vector field, Publ. Mat. 36 (1992), 217-228.
[5] B. O'Neill, The fundamental equations of a submersion, Michigan Math. J. 13 (1966), 459-469.
*
Department of Mathematics
Air Force Academy
Cheongwon 363-849, Republic of Korea
E-mail: byc8734@hanmail.net


[^0]:    Received February 28, 2011; Accepted May 16, 2011.
    2010 Mathematics Subject Classification: Primary 53C30; Secondary 53C43, 58 E 20.

    Key words and phrases: tension field, Riemannian submersion, section, eulerlagrange equation.

