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LINEAR MAPPINGS IN BANACH MODULES OVER A UNITAL C^*–ALGEBRA

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Abstract. We prove the Hyers-Ulam stability of generalized Jensen's equations in Banach modules over a unital C^*-algebra. It is applied to show the stability of generalized Jensen's equations in a Hilbert module over a unital C^*-algebra. Moreover, we prove the stability of linear operators in a Hilbert module over a unital C^*-algebra.

1. Generalized Jensen's equations

Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*-algebra $C^*(G,\omega)$. $C^*(\mathbb{Z}^m,\omega)$ is said to be a *noncommutative torus of rank* m and denoted by A_ω. The multiplier ω determines a subgroup S_ω of G, called its *symmetry group*, and the multiplier is called *totally skew* if the symmetry group S_ω is trivial. And A_ω is called *completely irrational* if ω is totally skew (see [2]). It was shown in [2] that if G is a locally compact abelian group and ω is a totally skew multiplier on G, then $C^*(G,\omega)$ is a simple C^*-algebra. It was shown in [7, Theorem 1.5] that if A_ω is a completely irrational noncommutative torus, then A_ω has stable rank 1, where *"stable rank* 1" means that the set of invertible elements is dense in the given C^*-algebra.

Let E_1 and E_2 be Banach spaces. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0,1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

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for all $x, y \in E_1$. Th.M. Rassias [14] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in E_1$.

Lemma A. *Let V and W be vector spaces, and let d, s, t be positive integers. A mapping $f : V \to W$ with $f(0) = 0$ is a solution of the equation*

$$\text{(A)} \qquad df(\frac{sx + ty}{d}) = sf(x) + tf(y)$$

for all $x, y \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x + y) = f(x) + f(y)$ for all $x, y \in V$.

Proof. Assume that $f : V \to W$ satisfies the equation (A). Then

$$\begin{aligned} df(\frac{s}{d}x) &= df(\frac{sx + t \cdot 0}{d}) = sf(x) + tf(0) = sf(x), \\ df(\frac{t}{d}x) &= df(\frac{s \cdot 0 + tx}{d}) = sf(0) + tf(x) = tf(x) \end{aligned}$$

for all $x \in V$. So

$$f(\frac{s}{d}x) = \frac{s}{d}f(x) \quad \& \quad f(\frac{t}{d}x) = \frac{t}{d}f(x)$$

for all $x \in V$. And

$$\begin{aligned} f(x) &= f(\frac{s}{d} \cdot \frac{d}{s}x) = \frac{s}{d}f(\frac{d}{s}x), \\ f(x) &= f(\frac{t}{d} \cdot \frac{d}{t}x) = \frac{t}{d}f(\frac{d}{t}x) \end{aligned}$$

for all $x \in V$. So

$$f(\frac{d}{s}x) = \frac{d}{s}f(x) \quad \& \quad f(\frac{d}{t}x) = \frac{d}{t}f(x)$$

for all $x \in V$. Thus

$$\begin{aligned} f(x + y) &= \frac{1}{d} \cdot df(\frac{s}{d} \cdot \frac{d}{s}x + \frac{t}{d} \cdot \frac{d}{t}y) = \frac{1}{d}(sf(\frac{d}{s}x) + tf(\frac{d}{t}y)) \\ &= \frac{1}{d}(s \cdot \frac{d}{s}f(x) + t \cdot \frac{d}{t}f(y)) = f(x) + f(y) \end{aligned}$$

for all $x, y \in V$.

Conversely, assume that $f : V \to W$ satisfies the additive Cauchy equation. Since $f(sx) = df(\frac{s}{d}x)$ for all $x \in V$,

$$f(\frac{s}{d}x) = \frac{1}{d}f(sx) = \frac{s}{d}f(x)$$

for all $x \in V$. Similarly, one can show that

$$f(\frac{t}{d}y) = \frac{1}{d}f(ty) = \frac{t}{d}f(y)$$

for all $y \in V$. So $f(\frac{sx+ty}{d}) = f(\frac{s}{d}x) + f(\frac{t}{d}y) = \frac{s}{d}f(x) + \frac{t}{d}f(y)$ for all $x, y \in V$. Thus

$$df(\frac{sx+ty}{d}) = sf(x) + tf(y)$$

for all $x, y \in V$. □

Throughout this paper, let A be a unital C^*-algebra with norm $|\cdot|$, and let ${}_A\mathcal{B}$ and ${}_A\mathcal{D}$ be left Banach A-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively. Let s, t be different positive integers, d a positive integer, and $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ a function such that

$$\text{(i)} \qquad \widetilde{\varphi}(x, y) := \sum_{k=0}^{\infty} (\frac{t}{s})^{2k} \varphi((\frac{s}{t})^{2k}x, (\frac{s}{t})^{2k}y) < \infty$$

for all $x, y \in {}_A\mathcal{B}$. Let $A_{\frac{s}{d}} = \{a \in A \mid \ |a| = \frac{s}{d}\}$, A_{in} the set of invertible elements in A, $A^+_{\frac{s}{d}}$ the set of positive elements in $A_{\frac{s}{d}}$, $\mathcal{U}(A)$ the set of unitary elements in A, $\frac{s}{d}\mathcal{U}(A) = \{\frac{s}{d}u \mid \ u \in \mathcal{U}(A)\}$, and ${}_A\mathcal{H}$ a left Hilbert A-module with norm $\|\cdot\|$.

Kadison and Pedersen [12] showed the following.

Lemma B [12, Theorem 1]. *Let $a \in A$ and $|a| < 1 - \frac{2}{m}$ for some integer m greater than 2. Then there are m unitary elements $u_1, u_2, \cdots, u_m \in A$ such that $ma = u_1 + u_2 + \cdots + u_m$.*

In this paper, we prove the Hyers-Ulam stability of the functional equation (A) in Banach modules over a unital C^*-algebra.

2. Stability of generalized Jensen's equations in Banach modules over a C^*-algebra

THEOREM 2.1. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{D}$ be a mapping with $f(0) = 0$ such that*

$$\|df(\frac{sx+ty}{d}) - sf(x) - tf(y)\| \le \varphi(x,y) \tag{ii}$$

for all $x, y \in {}_A\mathcal{B}$. Then there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ such that

$$\|f(x) - T(x)\| \le \frac{1}{s}\widetilde{\varphi}(x, -\frac{s}{t}x) + \frac{t}{s^2}\widetilde{\varphi}(-\frac{s}{t}x, (\frac{s}{t})^2 x) \tag{iii}$$

for all $x \in {}_A\mathcal{B}$. Furthermore, if $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$, then the additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ is $\mathbb{R}$-linear.

Proof. Let $x \in {}_A\mathcal{B}$. For $y = -\frac{s}{t}x$, the inequality (ii) implies

$$\|sf(x) + tf(-\frac{s}{t}x)\| \le \varphi(x, -\frac{s}{t}x). \tag{1}$$

Replacing x by $-\frac{s}{t}x$ and y by $(\frac{s}{t})^2 x$, the inequality (ii) implies

$$\|sf(-\frac{s}{t}x) + tf((\frac{s}{t})^2 x)\| \le \varphi(-\frac{s}{t}x, (\frac{s}{t})^2 x). \tag{2}$$

From (1) and (2), we get

$$\|f(x) - (\frac{t}{s})^2 f((\frac{s}{t})^2 x)\| \le \frac{1}{s}\varphi(x, -\frac{s}{t}x) + \frac{t}{s^2}\varphi(-\frac{s}{t}x, (\frac{s}{t})^2 x).$$

Applying the induction argument on n, we obtain

$$\begin{aligned}\|f(x) - (\frac{t}{s})^{2n} f((\frac{s}{t})^{2n} x)\| \le \sum_{k=0}^{n-1} (&\frac{1}{s}(\frac{t}{s})^{2k}\varphi((\frac{s}{t})^{2k}x, -(\frac{s}{t})^{2k+1}x) \\ &+ \frac{1}{s}(\frac{t}{s})^{2k+1}\varphi(-(\frac{s}{t})^{2k+1}x, (\frac{s}{t})^{2k+2}x)).\end{aligned} \tag{3}$$

We claim that the sequence $\{(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)\}$ is a Cauchy sequence. Indeed, for $n > m$, we have

$$\begin{aligned}&\|(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x) - (\frac{t}{s})^{2m} f((\frac{s}{t})^{2m}x)\| \\ &\le \sum_{k=m}^{n-1} \|(\frac{t}{s})^{2k+2} f((\frac{s}{t})^{2k+2}x) - (\frac{t}{s})^{2k} f((\frac{s}{t})^{2k}x)\| \\ &\le \sum_{k=m}^{n-1} (\frac{1}{s}(\frac{t}{s})^{2k}\varphi((\frac{s}{t})^{2k}x, -(\frac{s}{t})^{2k+1}x) \\ &\qquad + \frac{1}{s}(\frac{t}{s})^{2k+1}\varphi(-(\frac{s}{t})^{2k+1}x, (\frac{s}{t})^{2k+2}x)).\end{aligned}$$

From (i), it follows that

$$\lim_{m\to\infty}\sum_{k=m}^{n-1}(\frac{1}{s}(\frac{t}{s})^{2k}\varphi((\frac{s}{t})^{2k}x,-(\frac{s}{t})^{2k+1}x)$$
$$+\frac{1}{s}(\frac{t}{s})^{2k+1}\varphi(-(\frac{s}{t})^{2k+1}x,(\frac{s}{t})^{2k+2}x))=0.$$

Since ${}_A\mathcal{D}$ is a Banach space, the sequence $\{(\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x)\}$ converges. Define

$$T(x)=\lim_{n\to\infty}(\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x)$$

for all $x\in {}_A\mathcal{B}$. Taking the limit in (3) as $n\to\infty$, we obtain

$$\|f(x)-T(x)\|\le\frac{1}{s}\widetilde{\varphi}(x,-\frac{s}{t}x)+\frac{t}{s^2}\widetilde{\varphi}(-\frac{s}{t}x,(\frac{s}{t})^2x)$$

for all $x\in {}_A\mathcal{B}$. This completes the proof of the inequality (iii). From the definition of T, we get

$$(\frac{s}{t})^{2n}T(x)=T((\frac{s}{t})^{2n}x)\text{ and }T(0)=0. \tag{4}$$

From (i), (ii), and the definition of T,

$$\|dT(\frac{sx+ty}{d})-sT(x)-tT(y)\|$$
$$=\lim_{n\to\infty}(\frac{t}{s})^{2n}\|df((\frac{s}{t})^{2n}\frac{sx+ty}{d})-sf((\frac{s}{t})^{2n}x)-tf((\frac{s}{t})^{2n}y)\|$$
$$\le\lim_{n\to\infty}(\frac{t}{s})^{2n}\varphi((\frac{s}{t})^{2n}x,(\frac{s}{t})^{2n}y)=0$$

for all $x,y\in {}_A\mathcal{B}$. So

$$dT(\frac{sx+ty}{d})=sT(x)+tT(y)$$

for all $x,y\in {}_A\mathcal{B}$. By Lemma A, T is additive.

If $F:{}_A\mathcal{B}\to {}_A\mathcal{D}$ is another additive mapping satisfying (iii), then it follows from (iii), (4) and the proof of Lemma A that

$$\|T(x)-F(x)\|=\|(\frac{t}{s})^{2n}T((\frac{s}{t})^{2n}x)-(\frac{t}{s})^{2n}F((\frac{s}{t})^{2n}x)\|$$
$$\le\|(\frac{t}{s})^{2n}T((\frac{s}{t})^{2n}x)-(\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x)\|$$
$$+\|(\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x)-(\frac{t}{s})^{2n}F((\frac{s}{t})^{2n}x)\|$$
$$\le 2(\frac{t}{s})^{2n}(\frac{1}{s}\widetilde{\varphi}((\frac{s}{t})^{2n}x,(\frac{s}{t})^{2n}(-\frac{s}{t})x)+\frac{t}{s^2}\widetilde{\varphi}((\frac{s}{t})^{2n}(-\frac{s}{t})x),(\frac{s}{t})^{2n}(\frac{s}{t})^2x)),$$

which tends to zero as $n \to \infty$ by (i). Thus we conclude that

$$T(x) = F(x)$$

for all $x \in {}_A\mathcal{B}$. This completes the uniqueness of T.

Assume that $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$. By the assumption, $\frac{s}{t}$ is a rational number which is not an integer. The additive mapping T given above is similar to the additive mapping T given in the proof of [14, Theorem]. By the same reasoning as the proof of [14, Theorem], the additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ is $\mathbb{R}$-linear. □

Corollary 2.2. *Let $0 < p < 1$ and $t < s$. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{D}$ be a mapping with $f(0) = 0$ such that*

$$\|df(\frac{sx+ty}{d}) - sf(x) - tf(y)\| \le \|x\|^p + \|y\|^p$$

for all $x, y \in {}_A\mathcal{B}$. Then there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ such that

$$\|f(x) - T(x)\| \le \frac{s^{2(1-p)}}{s^{2(1-p)} - t^{2(1-p)}}(\frac{1}{s} + \frac{1}{t^p s^{1-p}} + \frac{t^{1-p}}{s^{2-p}} + \frac{t^{1-2p}}{s^{2-2p}})||x||^p$$

for all $x \in {}_A\mathcal{B}$.

Proof. Define $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ by $\varphi(x, y) = ||x||^p + ||y||^p$, and apply Theorem 2.1. □

Corollary 2.3. *Let $p > 1$ and $t > s$. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{D}$ be a mapping with $f(0) = 0$ such that*

$$\|df(\frac{sx+ty}{d}) - sf(x) - tf(y)\| \le \|x\|^p + \|y\|^p$$

for all $x, y \in {}_A\mathcal{B}$. Then there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ such that

$$\|f(x) - T(x)\| \le \frac{t^{2(p-1)}}{t^{2(p-1)} - s^{2(p-1)}}(\frac{1}{s} + \frac{s^{p-1}}{t^p} + \frac{s^{p-1}}{t^{p-1}} + \frac{s^{2-2p}}{t^{2p-1}})||x||^p$$

for all $x \in {}_A\mathcal{B}$.

Proof. Define $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ by $\varphi(x, y) = ||x||^p + ||y||^p$, and apply Theorem 2.1. □

THEOREM 2.4. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{D}$ be a continuous mapping with $f(0) = 0$ such that*

$$\|df(\frac{sax + tay}{d}) - saf(x) - taf(y)\| \le \varphi(x, y)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{B}$. If the sequence $(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ satisfying (iii).

Proof. Let $a = 1 \in A_1^+$. By Theorem 2.1, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ satisfying (iii). By the continuity and the uniform convergence, the $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ is continuous.

By the assumption, for each $a \in A_1^+ \cup \{i\}$,

$$\|df(\frac{(s+t)a}{d}(\frac{s}{t})^{2n}x) - (s+t)af((\frac{s}{t})^{2n}x)\| \le \varphi((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}x)$$

for all $x \in {}_A\mathcal{B}$. Using the fact that for each $b \in A$ and each $z \in {}_A\mathcal{D}$ $\|bz\| \le K|b| \cdot \|z\|$ for some $K > 0$, one can show that

$$\begin{aligned}\|daf(\frac{s+t}{d}(\frac{s}{t})^{2n}x) &- (s+t)af((\frac{s}{t})^{2n}x)\| \\ &\le K|a| \cdot \|df(\frac{s+t}{d}(\frac{s}{t})^{2n}x) - (s+t)f((\frac{s}{t})^{2n}x)\| \\ &\le K\varphi((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}x)\end{aligned}$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. So

$$\begin{aligned}\|df(\frac{(s+t)a}{d}(\frac{s}{t})^{2n}x) &- daf(\frac{s+t}{d}(\frac{s}{t})^{2n}x)\| \\ &\le \|df(\frac{(s+t)a}{d}(\frac{s}{t})^{2n}x) - (s+t)af((\frac{s}{t})^{2n}x)\| \\ &\quad + \|daf(\frac{s+t}{d}(\frac{s}{t})^{2n}x) - (s+t)af((\frac{s}{t})^{2n}x)\| \\ &\le \varphi((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}x) + K\varphi((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}x)\end{aligned}$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. Thus

$$(\frac{t}{s})^{2n}\|df(\frac{(s+t)a}{d}(\frac{s}{t})^{2n}x) - daf(\frac{s+t}{d}(\frac{s}{t})^{2n}x)\| \to 0$$

as $n \to \infty$ for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. Hence

$$\begin{aligned} dT(\frac{s+t}{d}ax) &= \lim_{n\to\infty}(\frac{t}{s})^{2n} df(\frac{(s+t)a}{d}(\frac{s}{t})^{2n}x) = \lim_{n\to\infty} daf(\frac{s+t}{d}(\frac{s}{t})^{2n}x) \\ &= daT(\frac{s+t}{d}x) \end{aligned}$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. So

$$T(ax) = \frac{d}{s+t}T(\frac{s+t}{d}ax) = \frac{d}{s+t}aT(\frac{s+t}{d}x) = aT(x)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$.

For any element $a \in A$, $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$, and $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = (\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-$, where $(\frac{a+a^*}{2})^+$, $(\frac{a+a^*}{2})^-$, $(\frac{a-a^*}{2i})^+$, and $(\frac{a-a^*}{2i})^-$ are positive elements (see [8, Lemma 38.8]). So

$$\begin{aligned} T(ax) =& T((\frac{a+a^*}{2})^+x - (\frac{a+a^*}{2})^-x + i(\frac{a-a^*}{2i})^+x - i(\frac{a-a^*}{2i})^-x) \\ =& (\frac{a+a^*}{2})^+T(x) + (\frac{a+a^*}{2})^-T(-x) \\ &+ (\frac{a-a^*}{2i})^+T(ix) + (\frac{a-a^*}{2i})^-T(-ix) \\ =& (\frac{a+a^*}{2})^+T(x) - (\frac{a+a^*}{2})^-T(x) \\ &+ i(\frac{a-a^*}{2i})^+T(x) - i(\frac{a-a^*}{2i})^-T(x) \\ =& ((\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-)T(x) \\ =& aT(x) \end{aligned}$$

for all $a \in A$ and all $x \in {}_A\mathcal{B}$. Hence

$$T(ax+by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{B}$, as desired. □

THEOREM 2.5. *Let A be a unital C^*-algebra of stable rank 1. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{D}$ be a continuous mapping with $f(0) = 0$ such that*

$$\|df(\frac{sax+tay}{d}) - saf(x) - taf(y)\| \le \varphi(x,y)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ *and all* $x, y \in {}_A\mathcal{B}$. *If the sequence* $(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n} x)$ *converges uniformly on* ${}_A\mathcal{B}$, *then there exists a unique continuous* A*-linear mapping* $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ *satisfying* (iii).

Proof. By the same reasoning as the proof of Theorem 2.4, there exists a unique continuous $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ satisfying (iii), and

$$T(ax) = aT(x) \tag{1}$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x \in {}_A\mathcal{B}$.

Let $b \in A_1^+ \setminus A_{in}$. Since A_{in} is dense in A, there exists a sequence $\{b_m\}$ in A_{in} such that $b_m \to b$ as $m \to \infty$. Put $c_m = \frac{1}{|b_m|} b_m$. Then $c_m \to \frac{1}{|b|} b = b$ as $m \to \infty$. Put $a_m = \sqrt{c_m^* c_m}$. Then $a_m \to b$ as $m \to \infty$ and $a_m \in A_1^+ \cap A_{in}$. Thus there exists a sequence $\{a_m\}$ in $A_1^+ \cap A_{in}$ such that $a_m \to b$ as $m \to \infty$, and by the continuity of T

$$\lim_{m\to\infty} T(a_m x) = T(\lim_{m\to\infty} a_m x) = T(bx) \tag{2}$$

for all $x \in {}_A\mathcal{B}$. By (1),

$$\|T(a_m x) - bT(x)\| = \|a_m T(x) - bT(x)\| \to \|bT(x) - bT(x)\| = 0 \tag{3}$$

as $m \to \infty$. By (2) and (3),

$$\begin{aligned} \|T(bx) - bT(x)\| \leq& \|T(bx) - T(a_m x)\| + \|T(a_m x) - bT(x)\| \\ \to& 0 \text{ as } m \to \infty \end{aligned} \tag{4}$$

for all $x \in {}_A\mathcal{B}$. By (1) and (4), $T(ax) = aT(x)$ for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is similar to the proof of Theorem 2.4. □

THEOREM 2.6. *Let* $f : {}_A\mathcal{B} \to {}_A\mathcal{D}$ *be a mapping with* $f(0) = 0$ *such that*

$$\|df(\frac{sax + tay}{d}) - saf(x) - taf(y)\| \leq \varphi(x, y)$$

for all $a \in A_1^+ \cup \{i\}$ *and all* $x, y \in {}_A\mathcal{B}$. *If* $f(\lambda x)$ *is continuous in* $\lambda \in \mathbb{R}$ *for each fixed* $x \in {}_A\mathcal{B}$, *then there exists a unique* A*-linear mapping* $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ *satisfying* (iii).

Proof. Let $a = 1 \in A_1^+$. By Theorem 2.1, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ satisfying (iii).

The rest of the proof is similar to the proof of Theorem 2.4. □

Theorem 2.7. *Let A be a unital C^*-algebra of stable rank 1. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{D}$ be a mapping with $f(0) = 0$ such that*

$$\|df(\frac{sax + tay}{d}) - saf(x) - taf(y)\| \le \varphi(x, y)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, y \in {}_A\mathcal{B}$. Assume that $f(ax)$ is continuous in $a \in A_1 \cup \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$, and that the sequence $(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n} ax)$ converges uniformly on A_1 for each fixed $x \in {}_A\mathcal{B}$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ satisfying (iii).

Proof. By the same reasoning as the proof of Theorem 2.4, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ satisfying (iii), and

$$T(ax) = aT(x)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. By the continuity and the uniform convergence, one can show that $T(ax)$ is continuous in $a \in A_1$ for each $x \in {}_A\mathcal{B}$.

The rest of the proof is similar to the proof of Theorem 2.5. □

3. Stability of generalized Jensen's equations in a Hilbert module over a C^*-algebra

In this section, let $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a mapping with $h(0) = 0$ such that $h((\frac{s}{t})^{2n} x) = (\frac{s}{t})^{2n} h(x)$ for all positive integers n and all $x \in {}_A\mathcal{H}$.

We are going to prove the Hyers-Ulam stability of generalized Jensen's equations in a Hilbert module over a unital C^*-algebra.

Theorem 3.1. *Let $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a continuous mapping such that*

$$\|dh(\frac{sax + tay}{d}) - sah(x) - tah(y)\| \le \varphi(x, y)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{H}$. Then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a bounded A-linear operator. Furthermore,

(1) *if the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequality*

$$\|h(x) - h^*(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$*, then the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a self-adjoint operator,*

(2) *if the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfies the inequality*

$$\|h \circ h^*(x) - h^* \circ h(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$*, then the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a normal operator,*

(3) *if the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfies the inequalities*

$$\|h \circ h^*(x) - x\| \le \varphi(x, x),$$
$$\|h^* \circ h(x) - x\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$*, then the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a unitary operator, and*

(4) *if the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfies the inequalities*

$$\|h \circ h(x) - h(x)\| \le \varphi(x, x),$$
$$\|h^*(x) - h(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$*, then the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a projection.*

Proof. The sequence $(\frac{t}{s})^{2n} h((\frac{s}{t})^{2n} x)$ converges uniformly on ${}_A\mathcal{H}$. By Theorem 2.4, there exists a unique continuous A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (iii). By the assumption,

$$T(x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} h((\frac{s}{t})^{2n} x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} (\frac{s}{t})^{2n} h(x) = h(x)$$

for all $x \in {}_A\mathcal{H}$, where the mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is given in the proof of Theorem 2.1. Hence the A-linear operator T is the mapping h. So the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a continuous A-linear operator. Thus the A-linear operator $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is bounded (see [9, Proposition II.1.1]).

(1) By the assumption,

$$\|h((\frac{s}{t})^{2n} x) - h^*((\frac{s}{t})^{2n} x)\| \le \varphi((\frac{s}{t})^{2n} x, (\frac{s}{t})^{2n} x)$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus

$$(\frac{t}{s})^{2n}\|h((\frac{s}{t})^{2n}x) - h^*((\frac{s}{t})^{2n}x)\| \to 0$$

as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$h(x) = \lim_{n\to\infty}(\frac{t}{s})^{2n}h((\frac{s}{t})^{2n}x) = \lim_{n\to\infty}(\frac{t}{s})^{2n}h^*((\frac{s}{t})^{2n}x) = h^*(x)$$

for all $x \in {}_A\mathcal{H}$. So the mapping h is a self-adjoint operator.

The proofs of the others are similar to the proof of (1). □

THEOREM 3.2. *Let A be a unital C^*-algebra of stable rank 1. Let $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a continuous mapping such that*

$$\|dh(\frac{sax + tay}{d}) - sah(x) - tah(y)\| \le \varphi(x, y)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, y \in {}_A\mathcal{H}$. Then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a bounded A-linear operator. Furthermore, the properties, given in the statement of Theorem 3.1, hold.

Proof. The sequence $(\frac{t}{s})^{2n}h((\frac{s}{t})^{2n}x)$ converges uniformly on ${}_A\mathcal{H}$. By the same reasoning as the proof of Theorem 2.5, there exists a unique continuous $\mathbb{R}$-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (iii), and

$$T(ax) = aT(x)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x \in {}_A\mathcal{H}$. By the same method as the proof of Theorem 2.5, one can show that

$$T(ax) = aT(x)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{H}$. By the same reasoning as the proof of Theorem 2.4, the mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is A-linear.

The rest of the proof is the same as the proof of Theorem 3.1. So the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a bounded A-linear operator, and the properties, given in the statement of Theorem 3.1, hold. □

4. Stability of generalized Jensen's equations in Banach modules over a C^*-algebra and unitary elements

In this section, we are going to prove the Hyers-Ulam stability of generalized Jensen's equations in Banach modules over a unital C^*-algebra associated with unitary elements.

THEOREM 4.1. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{D}$ be a continuous mapping with $f(0) = 0$ such that*

$$\|df(ax + \frac{t}{d}y) - daf(x) - tf(y)\| \le \varphi(x, y)$$

for all $a \in A^+_{\frac{s}{d}} \cup \{i\}$ and all $x, y \in {}_A\mathcal{B}$. If the sequence $(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n} x)$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ satisfying (iii).

Proof. Let $a = \frac{s}{d} \in A^+_{\frac{s}{d}}$. By Theorem 2.1, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ satisfying (iii). By the continuity and the uniform convergence, the $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ is continuous.

Since $a = \frac{s}{d} \in A^+_{\frac{s}{d}}$,

$$\|df(\frac{s}{d}x + \frac{t}{d}y) - sf(x) - tf(y)\| \le \varphi(x, y)$$

for all $x, y \in {}_A\mathcal{B}$. For each $a \in A^+_{\frac{s}{d}} \cup \{i\}$,

$$\|df(\frac{s}{d}\frac{d}{s}ax + \frac{t}{d}x) - sf(\frac{d}{s}ax) - tf(x)\| \le \varphi(\frac{d}{s}ax, x)$$

for all $x \in {}_A\mathcal{B}$. So

$$\begin{aligned}\|sf(\frac{d}{s}ax) - daf(x)\| \le & \|df(ax + \frac{t}{d}x) - daf(x) - tf(x)\| \\ & + \|df(\frac{s}{d}\frac{d}{s}ax + \frac{t}{d}x) - sf(\frac{d}{s}ax) - tf(x)\| \\ \le & \varphi(x, x) + \varphi(\frac{d}{s}ax, x)\end{aligned}$$

for all $a \in A^+_{\frac{s}{d}} \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. Thus $(\frac{t}{s})^{2n}\|sf(\frac{d}{s}(\frac{s}{t})^{2n}ax) - daf((\frac{s}{t})^{2n}x)\| \to 0$ as $n \to \infty$ for all $a \in A^+_{\frac{s}{d}} \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. Hence

$$sT(\frac{d}{s}ax) = \lim_{n\to\infty} (\frac{t}{s})^{2n} sf(\frac{d}{s}(\frac{s}{t})^{2n}ax) = \lim_{n\to\infty} (\frac{t}{s})^{2n} daf((\frac{s}{t})^{2n}x) = daT(x)$$

for all $a \in A^+_{\frac{s}{d}} \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. But $T(dx) = dT(x)$ since T is additive, and so $T(\frac{1}{s}x) = \frac{1}{s}T(x)$. So $T(ax) = aT(x)$ for all $a \in A^+_{\frac{s}{d}} \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. Since T is $\mathbb{R}$-linear and $T(ax) = aT(x)$ for each $a \in A^+_{\frac{s}{d}} \cup \{i\}$,

$$T(ax) = T(\frac{d}{s}|a| \cdot \frac{sa}{d|a|}x) = \frac{d}{s}|a| \cdot T(\frac{sa}{d|a|}x) = \frac{d}{s}|a| \cdot \frac{sa}{d|a|} \cdot T(x) = aT(x)$$

for all positive elements $a \in A \setminus \{0\}$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is the same as the proof of Theorem 2.4. □

Combining the trick of the proof of Theorem 2.5 and the trick of the proof of Theorem 4.1 yields the following.

THEOREM 4.2. *Let A be a unital C^*-algebra of stable rank 1. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{D}$ be a continuous mapping with $f(0) = 0$ such that*

$$\|df(ax + \frac{t}{d}y) - daf(x) - tf(y)\| \le \varphi(x, y)$$

for all $a \in (A^+_{\frac{s}{d}} \cap A_{in}) \cup \{i\}$ and all $x, y \in {}_A\mathcal{B}$. If the sequence $(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ satisfying (iii).

Similarly, one can obtain similar results to Theorem 2.6 and Theorem 2.7.

THEOREM 4.3. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{D}$ be a continuous mapping with $f(0) = 0$ such that*

$$\|df(ax + \frac{t}{d}y) - daf(x) - tf(y)\| \le \varphi(x, y)$$

for all $a \in \frac{s}{d}\mathcal{U}(A)$ and all $x, y \in {}_A\mathcal{B}$. If the sequence $(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ satisfying (iii).

Proof. Let $a = \frac{s}{d} \in \frac{s}{d}\mathcal{U}(A)$. By Theorem 2.1, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ satisfying (iii). By the continuity and the uniform convergence, the $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ is continuous.

Since $a = \frac{s}{d} \in \frac{s}{d}\mathcal{U}(A)$,

$$\|df(\frac{s}{d}x + \frac{t}{d}y) - sf(x) - tf(y)\| \le \varphi(x, y)$$

for all $x, y \in {}_A\mathcal{B}$. For each $a \in \frac{s}{d}\mathcal{U}(A)$,

$$\|df(\frac{s}{d}\frac{d}{s}ax + \frac{t}{d}x) - sf(\frac{d}{s}ax) - tf(x)\| \leq \varphi(\frac{d}{s}ax, x)$$

for all $x \in {}_A\mathcal{B}$. So

$$\begin{aligned}\|sf(\frac{d}{s}ax) - daf(x)\| \leq & \|df(ax + \frac{t}{d}x) - daf(x) - tf(x)\| \\ & + \|df(\frac{s}{d}\frac{d}{s}ax + \frac{t}{d}x) - sf(\frac{d}{s}ax) - tf(x)\| \\ \leq & \varphi(x, x) + \varphi(\frac{d}{s}ax, x)\end{aligned}$$

for all $a \in \frac{s}{d}\mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Thus $(\frac{t}{s})^{2n}\|sf(\frac{d}{s}(\frac{s}{t})^{2n}ax) - daf((\frac{s}{t})^{2n}x)\| \to 0$ as $n \to \infty$ for all $a \in \frac{s}{d}\mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$sT(\frac{d}{s}ax) = \lim_{n\to\infty}(\frac{t}{s})^{2n}sf(\frac{d}{s}(\frac{s}{t})^{2n}ax) = \lim_{n\to\infty}(\frac{t}{s})^{2n}daf((\frac{s}{t})^{2n}x) = daT(x)$$

for all $a \in \frac{s}{d}\mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. But $T(\frac{d}{s}x) = \frac{d}{s}T(x)$ since T is additive. So

$$T(ax) = aT(x)$$

for all $a \in \frac{s}{d}\mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

By Lemma B, for each element $a \in A_{\frac{s}{d}}$, there are unitary elements $v_1, v_2, \cdots, v_m \in A$ such that $m\frac{d}{2s}a = v_1 + v_2 + \cdots + v_m$ for some positive integer m. Hence

$$\begin{aligned}T(\frac{m}{2}ax) &= T(\frac{s}{d}v_1x + \frac{s}{d}v_2x + \cdots + \frac{s}{d}v_mx) \\ &= (\frac{s}{d}v_1 + \frac{s}{d}v_2 + \cdots + \frac{s}{d}v_m)T(x) = \frac{m}{2}aT(x)\end{aligned}$$

for all $a \in A_{\frac{s}{d}}$ and all $x \in {}_A\mathcal{B}$. But $T(\frac{m}{2}ax) = \frac{m}{2}T(ax)$. So $T(ax) = aT(x)$ for all $a \in A_{\frac{s}{d}}$ and all $x \in {}_A\mathcal{B}$. Hence

$$T(ax) = T(\frac{d}{s}|a| \cdot \frac{sa}{d|a|}x) = \frac{d}{s}|a| \cdot T(\frac{sa}{d|a|}x) = \frac{d}{s}|a| \cdot \frac{sa}{d|a|} \cdot T(x) = aT(x)$$

for all $a \in A \setminus \{0\}$ and all $x \in {}_A\mathcal{B}$. Thus the $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ is a continuous A-linear mapping satisfying (iii), as desired. □

THEOREM 4.4. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{D}$ be a mapping with $f(0) = 0$ such that*

$$\|df(ax + \frac{t}{d}y) - daf(x) - tf(y)\| \le \varphi(x, y)$$

for all $a \in \frac{s}{d}\mathcal{U}(A)$ and all $x, y \in {}_A\mathcal{B}$. If $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$, then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{D}$ satisfying (iii).

Proof. Let $a = \frac{s}{d} \in \frac{s}{d}\mathcal{U}(A)$. By Theorem 2.1, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{B}$ satisfying (iii). By the same reasoning as the proof of Theorem 4.3, the $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{B}$ is an A-linear mapping, as desired. □

5. Stability of generalized Jensen's equations in a Hilbert module over a C^*-algebra and unitary elements

In this section, let $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a mapping with $h(0) = 0$ such that $h((\frac{s}{t})^{2n}x) = (\frac{s}{t})^{2n}h(x)$ for all positive integers n and all $x \in {}_A\mathcal{H}$.

We are going to prove the Hyers-Ulam stability of generalized Jensen's equations in a Hilbert module over a unital C^*-algebra associated with unitary elements.

Combining the trick of the proof of Theorem 3.1 and the trick of the proof of Theorem 4.1 yields the following.

THEOREM 5.1. *Let $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a continuous mapping such that*

$$\|dh(ax + \frac{t}{d}y) - dah(x) - th(y)\| \le \varphi(x, y)$$

for all $a \in A^+_{\frac{s}{d}} \cup \{i\}$ and all $x, y \in {}_A\mathcal{H}$. Then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a bounded A-linear operator. Furthermore, the properties, given in the statement of Theorem 3.1, hold.

Combining the trick of the proof of Theorem 3.2 and the trick of the proof of Theorem 4.2 yields the following.

THEOREM 5.2. *Let A be a unital C^*-algebra of stable rank 1. Let $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a continuous mapping such that*

$$\|dh(ax + \frac{t}{d}y) - dah(x) - th(y)\| \le \varphi(x, y)$$

for all $a \in (A^+_{\frac{s}{d}} \cap A_{in}) \cup \{i\}$ *and all* $x, y \in {}_A\mathcal{H}$. *Then the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a bounded A-linear operator. Furthermore, the properties, given in the statement of Theorem 3.1, hold.*

Combining the trick of the proof of Theorem 4.3 and the trick of the proof of Theorem 3.1 yields the following.

THEOREM 5.3. *Let* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *be a continuous mapping such that*

$$\|dh(ax + \frac{t}{d}y) - dah(x) - th(y)\| \le \varphi(x, y)$$

for all $a \in \frac{s}{d}\mathcal{U}(A)$ *and all* $x, y \in {}_A\mathcal{H}$. *Then the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a bounded A-linear operator. Furthermore, the properties, given in the statement of Theorem 3.1, hold.*

So the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is an element of the C^*-algebra $\mathcal{L}({}_A\mathcal{H})$ of all bounded A-linear operators on ${}_A\mathcal{H}$.

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