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EXISTENCE OF NASH EQUILIBRIUM IN A NON-COMPACT ACYCLIC STRATEGIC GAME WITH INFINITE PLAYERS

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Abstract. In this paper, we will prove an equilibrium existence theorem of a non-compact acyclic strategic game with affine constraint correspondences which is comparable with equilibrium existence theorems due to Debreu, Nash, Kim-Kum, and Lu in several aspects.

1. Introduction

In mathematical economics, showing the existence of equilibrium is the main problem of investigating various kind of economic models. In general economic models, convexity assumptions are essential and basic to apply the well-known fixed point theorems as in [1,4,6-8]. Until now, there have been a number of generalized convex conditions investigated by several authors, and using those concepts, there have been numerous equilibrium existence theorems in generalized games as in [4,8].

On the other hand, in some economic models, neither convexity nor quasiconvexity assumptions can be guaranteed, e.g., the best response correspondences in the pure strategy spaces of auctions, political contests, models of imperfect competition are not convex-valued in general (e.g., see [1]). Hence we shall need some general concepts for removing the convexity assumptions of the strategy spaces and the payoff functions. For these purposes, some homological conditions are introduced in general. In this direction, Kim and Kum [5] recently investigated some general existence of pure-strategy Nash equilibria with acyclic values by using Begle's fixed point theorem.

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In this paper, following Debreu's method [3], we will prove an equilibrium existence theorem in a non-compact acyclic strategic game with affine constraint correspondences which is comparable with equilibrium existence theorems due to Debreu [3], Nash [7], Kim-Kum [5], and Lu [6] in several aspects.

2. Preliminaries

Let I be a (possibly uncountable) set of players, and let X_i be a nonempty topological space as an action space for each $i \in I$, and denote $X_{-i} := \prod_{j \in I \setminus \{i\}} X_j$. For an action profile $x = (x_i)_{i \in I} \in X = \Pi_{i \in I} X_i$, we shall write $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots) \in X_{-i}$; and if $x_i \in X_i$, $x_{-i} \in X_{-i}$, we simply write a typical strategy profile

$$x = (x_{-i}, x_i) := (x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots) \in X_{-i} \times X_i.$$

We now introduce some general notions and terminologies in generalized non-cooperative strategic games. A *generalized Nash game* (or *social system*) is an ordered triples $\Gamma = (X_i; T_i, f_i)_{i \in I}$ where for each player $i \in I$, the nonempty set X_i is a player's pure strategy space, $T_i : X \to 2^{X_i}$ is a player's constraint correspondence, and $f_i : X \to \mathbb{R}$ is a player's payoff (or utility) function. The set X, *joint strategy space*, is the Cartesian product of the individual strategy spaces, and the element of X_i is called a *strategy*. Then, a strategy tuples $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ is called the *Nash equilibrium* (or *social equilibrium*) for the game Γ if the following system of inequalities holds: for each $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}), \text{ and } f_i(\bar{x}_{-i}, \bar{x}_i) \geq f_i(\bar{x}_{-i}, x_i) \text{ for each } \quad x_i \in T_i(\bar{x}).$$

In [2], Begle introduced a common general notion of an lc space which contains both a compact convex subset in a locally convex topological vector space, and an absolute neighborhood retract(ANR), and for the definition and properties of lc space, see [2].

The following Begle fixed point theorem, which is a generalization of the Eilenberg-Montgomery fixed point theorem into an lc space, is essential in general acyclic settings:

LEMMA 2.1. [2] *Let X be a nonempty compact acyclic lc space and $T : X \to 2^X$ have closed graph in $X \times X$ with nonempty acyclic values. Then T has a fixed point.*

If X is a compact convex subset in a locally convex Hausdorff topological vector space, then X is a compact acyclic lc space so that Fan-Glicksberg's fixed point theorem is a consequence of Lemma 2.1.

Throughout this paper, all topological spaces are assumed to be Hausdorff, and for the other standard notations and terminologies, we shall refer to [4,5].

3. Nash equilibrium in a non-compact acyclic strategic game

In a generalized Nash game $\Gamma = (X_i; T_i, f_i)_{i\in I}$, we recall that the set of all real-valued payoff functions $\{f_i \mid i \in I\}$ satisfy the *unconditional summability* [5] if any rearrangement $\sum_{j\in I} f_j(x_{-j}, x_j)$ of the infinite sum $\sum_{i\in I} f_i(x_{-i}, x_i)$ converges to the same real value. Indeed, the unconditional summability should be needed for a generalized Nash game with infinite players, and it should be noted that $\{f_i \mid i \in I\}$ is unconditionally summable if the sum $\sum_{i\in I} f_i(x_{-i}, x_i)$ converges absolutely. From now on, we may assume that the set of all real-valued payoff functions $\{f_i \mid i \in I\}$ satisfies the unconditional summablity in a generalized Nash game Γ.

For the existence of Nash equilibrium in a generalized Nash game Γ, let us define the total sum of payoff function $H : X \times X \to \mathbb{R}$ associated with the strategic game Γ as follows:

$$H(y, x) := \sum_{i\in I} f_i(x_{-i}, y_i) \quad \text{for each} \quad x,\, y \in X = \prod_{i\in I} X_i.$$

Now we can obtain the following equivalences which generalizes the theorems due to Kim-Kum [5] and Nikaido-Isoda [8]:

LEMMA 3.1. *Let $\Gamma = (X_i; T_i, f_i)_{i\in I}$ be a generalized Nash game where I be a (possibly uncountable) set of players. Then the followings are equivalent:*

(1) $\bar{x} \in X = \prod_{i\in I} X_i$ is a Nash equilibrium for Γ;
(2) for each $i \in I$, $\bar{x}_i \in T_i(\bar{x})$, and
$\quad f_i(\bar{x}_{-i}, \bar{x}_i) \geq f_i(\bar{x}_{-i}, x_i)$ *for all* $x_i \in T_i(\bar{x})$;
(3) for each $i \in I$, $\bar{x}_i \in T_i(\bar{x})$, and
$\quad H(\bar{x}, \bar{x}) \geq H(y, \bar{x})$ *for all* $y \in \Pi_{i\in I} T_i(\bar{x})$.

Proof. The equivalence of (1) and (2) follows immediately from the definition of a Nash equilibrium. The implication (2) $\Rightarrow$ (3) is obtained by adding both sides of the inequalities $f_i(\bar{x}_{-i}, \bar{x}_i) \geq f_i(\bar{x}_{-i}, x_i)$ for all $i \in I$. To prove (3) implies (2), we first fix i, and take $y = (\bar{x}_{-i}, y_i)$ where $y_i \in T_i(\bar{x})$. Then the inequality $H(\bar{x}, \bar{x}) \geq H(y, \bar{x})$ may be written as

$$f_i(\bar{x}_{-i}, \bar{x}_i) - f_i(\bar{x}_{-i}, y_i) + \sum_{j\neq i} \big(f_j(\bar{x}_{-j}, \bar{x}_j) - f_j(\bar{x}_{-j}, y_j)\big) \geq 0.$$

Since $y_j = \bar{x}_j$ whenever $j \neq i$, we have that for all $i \in I$, $f_i(\bar{x}_{-i}, \bar{x}_i) - f_i(\bar{x}_{-i}, y_i) \geq 0$, which proves (2). □

REMARK 3.2. Lemma 3.1 generalizes the previous results due to Kim-Kum [5], and Nikaido-Isoda [8] in the following aspects:
(i) the set I of players need not be a finite set;
(ii) the inequalities on $H(y,x)$ need not satisfy in the whole strategy set X_i, but on the i-th player's constraint set $T_i(\bar{x})$.

For simplicity of notations, we denote the fixed point set $\mathcal{F} \subseteq X$ of a correspondence $T = \Pi_{i\in I} T_i : X \to 2^X$ by

$$\mathcal{F} = \{x \in X \mid x_i \in T_i(x) \quad \text{for all } \ i \in I\},$$

and the range of T by $\mathcal{R}(T)$ in a generalized Nash game $\Gamma = (X_i\,; T_i, f_i)_{i\in I}$.

Applying the Begle fixed point theorem, we now prove a new existence theorem of Nash equilibrium for a non-compact acyclic strategic game with affine constraint correspondences as follows:

THEOREM 3.3. *Let* $\Gamma = (X_i\,; T_i, f_i)_{i\in I}$ *be a generalized Nash game, where* I *be a (possibly uncountable) set of players, such that for each* $i \in I$, *the strategy set* X_i *is a convex subset in a topological vector space, and* D_i *is a nonempty compact subset of* X_i. *Let* $X = \prod_{i\in I} X_i$, *and* $D = \prod_{i\in I} D_i$. *For each* $i \in I$, $f_i : X \to \mathbb{R}$ *is a player's payoff function, and* $T_i : X \to 2^{D_i}$ *is upper semicontinuous such that each* $T_i(x)$ *is a nonempty closed subset of* D_i. *For each* $i \in I$, T_i *satisfies the affine condition such that for each* $\lambda \in [0,1]$,

$$\lambda T_i(x) + (1-\lambda)T_i(y) \subseteq T_i(\lambda x + (1-\lambda)y) \qquad \text{for all } \ x, y \in X.$$

Furthermore, assume that D *and* $\mathcal{F}$ *are acyclic lc spaces, and each* $T(x)$ *is acyclic, and* $H : X \times X \to \mathbb{R}$ *satisfy the following:*

(1) $(x,y) \mapsto H(y,x) - H(x,x)$ *is lower semicontinuous in* $X \times X$;
(2) for each $x \in \mathcal{F}$, $\{y \in \mathcal{R}(T) \mid H(x,x) < H(y,x)\}$ *is convex;*
(3) for each $y \in \mathcal{R}(T)$, $\{x \in \mathcal{F} \mid H(y,x) \leq H(x,x)\}$ *is nonempty acyclic.*

Then Γ *has a Nash equilibrium* $\bar{x} \in X$, *i.e., for each* $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}), \quad \text{and} \quad f_i(\bar{x}_{-i}, \bar{x}_i) \geq f_i(\bar{x}_{-i}, x_i) \quad \text{for each} \quad x_i \in T_i(\bar{x}).$$

Proof. Suppose the contrary; then, by Lemma 3.1 (3), for each $x \in X$, either $x_i \notin T_i(x)$ for some $i \in I$, or there exists $y \in T(x)$ such that $H(x,x) < H(y,x)$. $(*)$

Since each T_i is upper semicontinuous having nonempty compact values, the restriction $T|_D : D \to 2^D$, defined by $T|_D(x) := \Pi_{i\in I}T_i(x)$ for each $x \in D$, is also upper semicontinuous having nonempty compact acyclic values. Therefore, by Lemma 2.1, there exists a fixed point $\hat{x} \in D$ for T, i.e., $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$. Then the fixed point set $\mathcal{F} \subseteq D$ of the correspondence T is a nonempty closed subset of D by the upper semicontinuity of T. Moreover, by the affine assumption on T_i, $\mathcal{F}$ is a convex set. Indeed, if $x_1, x_2 \in \mathcal{F}$ and $\lambda \in [0,1]$ are arbitrarily given. Then, for each $i \in I$, $(x_1)_i \in T_i(x_1)$ and $(x_2)_i \in T_i(x_2)$. Let $x = \lambda x_1 + (1-\lambda)x_2 \in X$, then for each $i \in I$,

$$\begin{aligned} x_i = \lambda(x_1)_i + (1-\lambda)(x_2)_i &\in \lambda T_i(x_1) + (1-\lambda)T_i(x_2) \\ &\subseteq T_i(\lambda x_1 + (1-\lambda)x_2) = T_i(x); \end{aligned}$$

so that $\mathcal{F}$ is a convex subset of D and hence $\mathcal{F}$ is an acyclic subset of D. Similarly, we can also have that $\mathcal{R}(T)$ is a convex subset of D.

For each $x \in \mathcal{F}$, $x_i \in T_i(x)$ for all $i \in I$, and hence there must exist $y \in T(x) \subset \mathcal{R}(T)$ such that $H(x,x) < H(y,x)$ in the equation $(*)$ of the reduction ad absurdam. For each $y \in \mathcal{R}(T)$, we let

$$N(y) := \{x \in \mathcal{F} \mid H(x,x) < H(y,x)\}.$$

By the assumption (1), each $N(y)$ is (possibly empty) open in $\mathcal{F}$. If $x \in \mathcal{F}$, then $x \in N(y)$ for some $y \in \mathcal{R}(T)$ so that we have $\bigcup_{y\in\mathcal{R}(T)} N(y) = \mathcal{F}$. Since $\mathcal{F}$ is compact, there exists a finite number of points $\{y_1, \ldots, y_n\} \subset \mathcal{R}(T)$, and nonempty open sets $N(y_1), \ldots, N(y_n)$ such that $\bigcup_{i=1}^n N(y_i) = \mathcal{F}$. Let $\{\alpha_i \mid i = 1, \ldots, n\}$ be the continuous partition of unity subordinated to the open covering $\{N(y_i) \mid i = 1, \ldots, n\}$ of the compact set $\mathcal{F}$.

We now define a mapping $\phi : \mathcal{F} \to \mathcal{R}(T)$ by

$$\phi(x) := \sum_{i=1}^n \alpha_i(x)\, y_i \qquad \text{for each} \quad x \in \mathcal{F}.$$

Then, ϕ is a continuous mapping since each α_i is continuous. By the assumption (2), for fixed $x \in \mathcal{F}$, the set $\{y \in \mathcal{R}(T) \mid H(x,x) < H(y,x)\}$ is convex in $\mathcal{R}(T)$ so that we have

$$\begin{aligned} \phi(x) &\in co\,\{y_i \in \mathcal{R}(T) \mid \alpha_i(x) \neq 0\} \\ &\subseteq \{y \in \mathcal{R}(T) \mid H(x,x) < H(y,x)\}; \end{aligned} \tag{$\dagger$}$$

and hence we also see that ϕ maps $\mathcal{F}$ into $\mathcal{R}(T)$.

Next, we define a correspondence $S : \mathcal{R}(T) \to 2^{\mathcal{F}}$ by

$$S(y) := \{x \in \mathcal{F} \mid H(y,x) \leq H(x,x)\} \qquad \text{for each} \quad y \in \mathcal{R}(T).$$

Then, for each $y \in \mathcal{R}(T)$, by the assumption (3), $S(y)$ is a nonempty acyclic subset in $\mathcal{F}$. By the assumption (1) again, the set $\{(x,y) \in \mathcal{F} \times \mathcal{F} \mid H(y,x) - H(x,x) \leq 0\}$ is nonempty closed in $\mathcal{F} \times \mathcal{F}$ and hence it is compact. Therefore, for each $y \in \mathcal{R}(T)$, $S(y)$ is the projection of a nonempty compact set in $\mathcal{F} \times \mathcal{F}$ so that $S(y)$ is compact. Next, it is easy to see that S has a closed graph in $\mathcal{R}(T) \times \mathcal{F}$. In fact, for any nets $(x_\alpha) \to x_o$, $(y_\alpha) \to y_o$, $y_\alpha \in S(x_\alpha)$, we have $H(y_\alpha, x_\alpha) \leq H(x_\alpha, x_\alpha)$. Since the mapping $(x,y) \mapsto H(y,x) - H(x,x)$ is lower semicontinuous, we have $H(y_o, x_o) \leq H(x_o, x_o)$. Hence $y_o \in S(x_o)$ and S has a closed graph in $\mathcal{R}(T) \times \mathcal{F}$.

Finally, we define a correspondence $\Phi : \mathcal{F} \to 2^{\mathcal{F}}$ by

$$\Phi(x) = (S \circ \phi)(x) \qquad \text{for each} \quad x \in \mathcal{F}.$$

Since S has a closed graph and ϕ is continuous, Φ has a closed graph in a compact set $\mathcal{F} \times \mathcal{F}$, and each $\Phi(x)$ is nonempty compact acyclic. Therefore, by Lemma 2.1 again, there exists a fixed point $\bar{x} \in \mathcal{F}$ for Φ such that $\bar{x} \in \Phi(\bar{x}) = S(\phi(\bar{x}))$. Let $x^* = \phi(\bar{x}) \in \mathcal{R}(T)$; then

$$\bar{x} \in S(x^*) = \{x \in \mathcal{F} \mid H(x^*, x) \leq H(x,x)\}$$

so that $H(x^*, \bar{x}) \leq H(\bar{x}, \bar{x})$. On the other hand, since $x^* = \phi(\bar{x})$, by the inclusion (†), we have

$$x^* = \phi(\bar{x}) \in \{y \in \mathcal{R}(T) \mid H(\bar{x}, \bar{x}) < H(y, \bar{x})\}$$

so that $H(x^*, \bar{x}) > H(\bar{x}, \bar{x})$, which is a contradiction. □

REMARK 3.4. (1) Theorem 3.3 is a new equilibrium existence theorem which is comparable with the previous equilibrium existence theorems due to Becker-Damianov [1], Debreu [3], Nash [7], Kim-Kum [5], Nikaido-Isoda [8], and Lu [6] in the following aspects:

(i) the set I of players need not be a finite set;

(ii) every payoff function f_i need not be (quasi)concave nor continuous on X. Indeed, when $T_i(x) = X_i = D_i$ for each $i \in I$, then we have $\mathcal{R}(T) = \mathcal{F} = X$ so that if each $f_i : X \to \mathbb{R}$ is continuous on X and the function $y_i \mapsto f_i(x_{-i}, y_i)$ is quasiconcave on X_i as in [6-8], then the assumptions (1) and (2) of Theorem 3.3 are satisfied.

(2) In Theorem 3.2 of Kim-Kum [5], X_i need not be convex but X is assumed to be a nonempty compact acyclic subset. Furthermore, their assumptions (2) and (3) are stronger than the corresponding assumptions in Theorem 3.3.

As remarked before, when $X_i = D_i$ is compact convex, and $T_i(x) := X_i$ for each $x \in X$ and $i \in I$ in Theorem 3.3, then we have $\mathcal{R}(T) =$

$\mathcal{F} = X$ and T_i automatically satisfies the affine condition so that we can obtain the following:

THEOREM 3.5. *Let* $\Gamma = (X_i\,; f_i)_{i\in I}$ *be a generalized Nash game, where* I *be a set of players, such that for each* $i \in I$*, the strategy set* X_i *is a convex subset in a topological vector space, and* $X = \prod_{i\in I} X_i$*, and* $f_i : X \to \mathbb{R}$ *is a player's payoff function. Assume that* X *is a nonempty compact acyclic lc space, and* $H : X \times X \to \mathbb{R}$ *satisfy the following:*

(1) $(x, y) \mapsto H(y, x) - H(x, x)$ *is lower semicontinuous in* $X \times X$*;*
(2) for each $x \in X$*,* $\{y \in X \mid H(x, x) < H(y, x)\}$ *is convex in* X*;*
(3) for each $y \in X$*,* $\{x \in X \mid H(y, x) \leq H(x, x)\}$ *is acyclic.*

Then Γ *has a Nash equilibrium* $\bar{x} \in X$

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