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GENERALIZED SINGLE INTEGRAL INVOLVING KAMPÉ DE FÉRIET FUNCTION

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Abstract. The aim of this paper is to obtain twenty five Eulerian type single integrals in the form of a general single integral involving Kampé de Fériet function. The results are derived with the help of the generalized classical Watson's theorem obtained earlier by Lavoie, Grondin and Rathie. A few interesting special cases of our main result are also given.

1. Introduction

We recall the definition of generalized Kampé de Fériet function [11]:

$$
\begin{aligned}
&F^{p:q;k}_{l:m;n}\left[\begin{array}{lll l} (a_p) & :(b_q) & ;(c_k) & ; \\ (\alpha_l) & :(\beta_m) & ;(\gamma_n) & ; \end{array} x,\, y\right] \\
&= \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s\ x^r\,y^s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s\ r!\,s!},
\end{aligned}
\tag{1.1}
$$

where, for convergence,

(i) $p+q<l+m+1$, $p+k<l+n+1$, $|x|<\infty$, $|y|<\infty$
or

(ii) $p+q<l+m+1$, $p+k<l+n+1$ and

$$
\begin{cases} |x|^{\frac{1}{p-l}}+|y|^{\frac{1}{p-l}}<1 & ,if\ p>l \\ \max\{|\mathrm{x}|,|\mathrm{y}|\}<1 & ,if\ p\le l. \end{cases}
$$

Although the double hypergeometric sreies defined by (1.1) reduces to the Kampé de Fériet function [1] in the special case : $q=k$ and

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$m = n$,
yet it is usually refereed to in the literature as the Kampé de Fériet series. The Kampé de Fériet function defined in (1.1) can be specialized to be expressed in terms of generalized hypergeometric series, among other things, as following instances:

$$F^{p:0;0}_{q:0;0}\left[\begin{matrix}\alpha_1,\cdots,\alpha_p; \\ \beta_1,\cdots,\beta_q;\end{matrix}\ x,y\right] = {}_pF_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p \\ \beta_1,\cdots,\beta_q\end{matrix};x+y\right]. \tag{1.2}$$

$$\begin{aligned} &F^{0:p;r}_{0:q;s}\left[\begin{matrix}-;\alpha_1,\cdots,\alpha_p;\gamma_1,\cdots,\gamma_r \\ -;\beta_1,\cdots,\beta_q;\delta_1,\cdots,\delta_s\end{matrix}\ x,\,y\right] \\ &= {}_pF_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p \\ \beta_1,\cdots,\beta_q\end{matrix};x\right]{}_rF_s\left[\begin{matrix}\gamma_1,\cdots,\gamma_r \\ \delta_1,\cdots,\delta_s\end{matrix};y\right].\end{aligned} \tag{1.3}$$

$$\begin{aligned} &F^{p:1;1}_{q:0;0}\left[\begin{matrix}\alpha_1,\cdots,\alpha_p;\nu;\sigma \\ \beta_1,\cdots,\beta_q;\ -;\ -\end{matrix}\ x,\,x\right] \\ &= {}_{p+1}F_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p;\nu+\sigma \\ \beta_1,\cdots,\beta_q\end{matrix};x\right].\end{aligned} \tag{1.4}$$

$$\begin{aligned} &F^{p:0;0}_{q:1;1}\left[\begin{matrix}\alpha_1,\cdots,\alpha_p;\ -;\ - \\ \beta_1,\cdots,\beta_q;\ \nu;\ \sigma\end{matrix}\ x,\,x\right] \\ &= {}_{p+2}F_{q+3}\left[\begin{matrix}\alpha_1,\cdots,\alpha_p;\Delta(2;\nu+\sigma-1) \\ \beta_1,\cdots,\beta_q,\ \nu,\sigma,\nu+\sigma-1\end{matrix};4x\right],\end{aligned} \tag{1.5}$$

where, and in what follows, $\Delta(l;\lambda)$ abbreviates the array of l parameters

$$\frac{\lambda}{l},\frac{(\lambda+1)}{l},\cdots,\frac{(\lambda+l-1)}{l},\ \ l=1,2,3,....$$

For more details, see Srivastava and Karlsson [10, pp. 28 - 32].

2. Results required

The following results will be required in our present investigations.

$$
\begin{aligned}
&\int_0^1 x^{c-1}(1-x)^{c+j-1}{}_2F_1\left[\begin{array}{c} a,\ b \\ \frac{1}{2}(a+b+1+i) \end{array}; x\right] dx \\
&= \frac{\Gamma(c)\Gamma(c+j)}{\Gamma(2c+j)}\,{}_3F_2\left[\begin{array}{c} a,\ b,\ c \\ \frac{1}{2}(a+b+1+i),\ 2c+j \end{array}; 1\right]
\end{aligned} \tag{2.1}
$$

provided $\Re(c) > 0, \Re(c+j) > 0$ and $\Re(2c-a-b+i+1) > 0$ for $i, j = 0, \pm 1, \pm 2$.
This result (2.1) is a special case of a general double integral given in Erdelyi et al. [3, pp. 399, Eq.(5)]

In 1992, Lavoie et al. [8] have given the generalization of the Watson's theorem on the sum of a ${}_3F_2$ and obtained the following twenty five results in the form of a single result:
(2.2)

$$
\begin{aligned}
&{}_3F_2\left(\begin{array}{ccc} a, & b, & c \\ a+b+1+i, & 2c+j & \end{array}; 1\right) \\
&= A_{i,j}\, \frac{2^{a+b+i-2}\,\Gamma\left(\frac{a+b+i+1}{2}\right)\Gamma\left(c+[\frac{j}{2}]+\frac{1}{2}\right)\Gamma\left(c-\frac{a+b+|i+j|-j-1}{2}\right)}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)} \\
&\times \left\{ B_{i,j}\, \frac{\Gamma\left(\frac{a}{2}+\frac{1-(-1)^i}{4}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\frac{1}{2}+[\frac{j}{2}]-\frac{(-1)^j 1-(-1)^i}{4}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}+[\frac{j}{2}]\right)} \right. \\
&\left. +C_{i,j}\, \frac{\Gamma\left(\frac{a}{2}+\frac{1-(-1)^i}{4}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j+1}{2}\right]+\frac{(-1)^j(1-(-1)^i)}{4}\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j+1}{2}\right]\right)} \right\}
\end{aligned}
$$

provided $\Re(2c-a-b) > -1-i-2j$ with $i, j = 0, \pm 1, \pm 2$. Here and in what follows, $[x]$ is the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x. The coefficients $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ are given respectively in [8].

3. Main results

The following results for reducibility of Kampé de Fériet function will be established in this section.

(3.1)

$$\int_0^1 x^{c-1}(1-x)^{c+j-1}{}_2F_1\left[\begin{matrix} a,\ b \\ \frac{1}{2}(a+b+1+i)\end{matrix};x\right]$$

$$\times F^{\lambda\,;\,\mu:\mu}_{\nu\,;\,\rho:\rho}\left[\begin{matrix}(\alpha_\lambda) & :(\beta_\mu) & ;(\beta'_\mu) & ; & z_1x(1-x)\\ (\gamma_\nu) & :(\delta_\rho) & ;(\delta'_\rho) & ; & z_2x(1-x)\end{matrix}\right]dx$$

$$= \sum_{p,q=0}^{\infty} A_{p,q}{z_1}^p{z_2}^q\frac{\Gamma(c+p+q)\Gamma(c+p+q+j)}{\Gamma(2c+2p+2q+j)}$$

$$A_{i,j}\frac{2^{a+b+i-2}\Gamma(\frac{a+b+i+1}{2})\Gamma(c+p+q+[\frac{1}{2}j]+\frac{1}{2})\Gamma(c+p+q-\frac{a+b+|i+j|-j-1}{2})}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)}$$

$$\times\left\{B_{i,j}\frac{\Gamma(\frac{a}{2}+\frac{1-(-1)^i}{4})\Gamma(\frac{b}{2})}{\Gamma(c+p+q-\frac{a}{2}+\frac{1}{2}+[\frac{j}{2}]-\frac{(-1)^j(1-(-1)^i)}{4})\Gamma(c+p+q-\frac{b}{2}+\frac{1}{2}+[\frac{j}{2}])}\right.$$

$$\left.+C_{i,j}\frac{\Gamma(\frac{a}{2}+\frac{1-(-1)^i}{4})\Gamma(\frac{b+1}{2})}{\Gamma(c+p+q-\frac{a}{2}+\frac{1}{2}+[\frac{j+1}{2}]-\frac{(-1)^j(1-(-1)^i)}{4})\Gamma(c+p+q-\frac{b}{2}+[\frac{j+1}{2}])}\right\}$$

provided $\Re(c)>0$, for $j=-1,-2$; $\Re(c+j)>0$ for $j=0,1,2$. Also,

(i) $\lambda+\mu<\nu+\rho+1$, $|z_1|<\infty$, $|z_2|<\infty$
or
(ii) $\lambda+\mu=\nu+\rho+1$ and

$$\begin{cases}|z_1|^{\frac{1}{\lambda-\nu}}+|z_2|^{\frac{1}{\lambda-\nu}}<1 & ,if\ \lambda>\nu\\ \max\{|z_1|,|z_2|\}<1 & ,if\ \lambda\leq\nu.\end{cases}$$

Also, the coefficients $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ can obtain from the tables given in [8] by replacing c by $c+p+q$,

$$A_{p,q}=\frac{\prod_{j=1}^{\lambda}(\alpha_j)_{p+q}\prod_{j=1}^{\mu}(\beta_j)_p\prod_{j=1}^{\mu}(\beta'_j)_q}{\prod_{j=1}^{\nu}(\gamma_j)_{p+q}\prod_{j=1}^{\rho}(\delta_j)_p\prod_{j=1}^{\rho}(\delta'_j)_q\ \ p!\,q!}.$$

4. Proof of (3.1)

To prove (3.1), we proceed as follows : Let

$$I=\int_0^1 x^{c-1}(1-x)^{c+j-1}{}_2F_1\left[\begin{matrix} a,\ b \\ \frac{1}{2}(a+b+1+i)\end{matrix};x\right] \tag{4.1}$$

$$\times F^{\lambda\,;\,\mu:\mu}_{\nu\,;\,\rho:\rho}\left[\begin{matrix}(\alpha_\lambda) & :(\beta_\mu) & ;(\beta'_\mu) & ; & z_1x(1-x)\\ (\gamma_\nu) & :(\delta_\rho) & ;(\delta'_\rho) & ; & z_2x(1-x)\end{matrix}\right]dx.$$

Expressing Kampé de Fériet function as a double series, we have

$$
\begin{aligned}
I = \int_0^1 & x^{c-1}(1-x)^{c+j-1} {}_2F_1 \left[\begin{matrix} a, \;\; b \\ \frac{1}{2}(a+b+1+i) \end{matrix} ; x \right] \\
& \times \sum_{p,q=0}^{\infty} A_{p,q} {z_1}^p {z_2}^q x^p (1-x)^p x^q (1-x)^q dx,
\end{aligned} \tag{4.2}
$$

where

$$
A_{p,q} = \frac{\prod_{j=1}^{\lambda} (\alpha_j)_{p+q} \prod_{j=1}^{\mu} (\beta_j)_p \prod_{j=1}^{\mu} (\beta_j^{'})_q}{\prod_{j=1}^{\nu} (\gamma_j)_{p+q} \prod_{j=1}^{\rho} (\delta_j)_p \prod_{j=1}^{\rho} (\delta_j^{'})_q \;\; p!\, q!}.
$$

Changing the order of integration and summatiom which is justified due to the uniformly convergence of the series, we derive

$$
\begin{aligned}
I = & \sum_{p,q=0}^{\infty} A_{p,q} {z_1}^p {z_2}^q \int_0^1 x^{c+p+q-1}(1-x)^{c+p+q+j-1} \\
& \times {}_2F_1 \left[\begin{matrix} a, \;\; b \\ \frac{1}{2}(a+b+1+i) \end{matrix} ; x \right] dx
\end{aligned} \tag{4.3}
$$

which, upon using (2.1), becomes

$$
\begin{aligned}
I = & \sum_{p,q=0}^{\infty} A_{p,q} {z_1}^p {z_2}^q \frac{\Gamma(c+p+q)\Gamma(c+p+q+j)}{\Gamma(2c+2p+2q+j)} \\
& \times {}_3F_2 \left[\begin{matrix} a, \;\; b, \;\; c+p+q \\ \frac{1}{2}(a+b+1+i), \;\; 2(c+p+q)+j \end{matrix} ; 1 \right].
\end{aligned} \tag{4.4}
$$

By making use of (2.2) and replacing c by $c+p+q$, we finally arrive at the right-hand side of (3.1). This completes the proof of (3.1).

5. Special cases

In this section, we shall mention some of the interesting special cases of main result (3.1).

(i) In (3.1), if we take $i = j = 0$, then, we have, after a little simplification, the following transformation formula:

$$\begin{aligned}
&\int_0^1 x^{c-1}(1-x)^{c-1}{}_2F_1\left[\begin{matrix} a, \ b \\ \frac{1}{2}(a+b+1) \end{matrix}; x\right] \\
&\times F^{\lambda\,;\,\mu:\mu}_{\nu\,;\,\rho:\rho}\left[\begin{matrix} (\alpha_\lambda) & : (\beta_\mu) & ; (\beta'_\mu) & ; & z_1 x(1-x) \\ (\gamma_\nu) & : (\delta_\rho) & ; (\delta'_\rho) & ; & z_2 x(1-x) \end{matrix}\right] dx \\
&= \frac{2^{a+b+2c-1}\Gamma(\frac{a+b+1}{2})\Gamma(\frac{a}{2})\Gamma(\frac{b}{2})\Gamma(c)\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2})}{\Gamma(a)\Gamma(b)\Gamma(c-\frac{a}{2}+\frac{1}{2})\Gamma(c-\frac{b}{2}+\frac{1}{2})} \\
&\times F^{\lambda+2\,;\,\mu:\mu}_{\nu+2\,;\,\rho:\rho}\left[\begin{matrix} (\alpha_\lambda), c, c+\frac{1-a-b}{2} & : (\beta_\mu) & ; (\beta'_\mu) & ; & \frac{z_1}{4}, \frac{z_2}{4} \\ (\gamma_\nu), c+\frac{1-a}{2}, c+\frac{1-b}{2} & : (\delta_\rho) & ; (\delta'_\rho) & ; & \end{matrix}\right]
\end{aligned} \tag{5.1}$$

provided that the conditions easily obtainable from (3.1) are satisfied.

(ii) In (3.1), if we take $i = 0$, $j = -1$, then, we have, after a little simplification, the following transformation formula:

$$\begin{aligned}
&\int_0^1 x^{c-1}(1-x)^{c-2}{}_2F_1\left[\begin{matrix} a, \ b \\ \frac{1}{2}(a+b+1) \end{matrix}; x\right] \\
&\times F^{\lambda\,;\,\mu:\mu}_{\nu\,;\,\rho:\rho}\left[\begin{matrix} (\alpha_\lambda) & : (\beta_\mu) & ; (\beta'_\mu) & ; & z_1 x(1-x) \\ (\gamma_\nu) & : (\delta_\rho) & ; (\delta'_\rho) & ; & z_2 x(1-x) \end{matrix}\right] dx \\
&= \frac{2^{a+b+2c}\Gamma(\frac{a+b+1}{2})\Gamma(\frac{a}{2})\Gamma(\frac{b}{2})\Gamma(c-1)\Gamma(c-\frac{a}{2}-\frac{b}{2}-\frac{1}{2})}{\Gamma(a)\Gamma(b)\Gamma(c-\frac{a}{2}-\frac{1}{2})\Gamma(c-\frac{b}{2}-\frac{1}{2})} \\
&\times F^{\lambda+2\,;\,\mu:\mu}_{\nu+2\,;\,\rho:\rho}\left[\begin{matrix} (\alpha_\lambda), c-1, c-\frac{1+a+b}{2} & : (\beta_\mu) & ; (\beta'_\mu) & ; & \frac{z_1}{4}, \frac{z_2}{4} \\ (\gamma_\nu), c-\frac{1+a}{2}, c-\frac{1+b}{2} & : (\delta_\rho) & ; (\delta'_\rho) & ; & \end{matrix}\right] \\
&+ \frac{2^{a+b-2c}\Gamma(\frac{a+b+1}{2})\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(\frac{b}{2}+\frac{1}{2})\Gamma(c-1)\Gamma(c-\frac{a}{2}-\frac{b}{2}-\frac{1}{2})}{\Gamma(a)\Gamma(b)\Gamma(c-\frac{a}{2})\Gamma(c-\frac{b}{2})} \\
&\times F^{\lambda+2\,;\,\mu:\mu}_{\nu+2\,;\,\rho:\rho}\left[\begin{matrix} (\alpha_\lambda), c-1, c-\frac{1+a+b}{2} & : (\beta_\mu) & ; (\beta'_\mu) & ; & \frac{z_1}{4}, \frac{z_2}{4} \\ (\gamma_\nu), c-\frac{a}{2}, c-\frac{b}{2} & : (\delta_\rho) & ; (\delta'_\rho) & ; & \end{matrix}\right]
\end{aligned} \tag{5.2}$$

provided that the conditions easily obtainable from (3.1) are satisfied.

(iii) In (3.1), if we take $i = 0$, $j = 1$, then, we have, after a little simplification, the following transformation formula:

(5.3)

$$\int_0^1 x^{c-1}(1-x)^c {}_2F_1\left[\begin{matrix} a, \ b \\ \frac{1}{2}(a+b+1) \end{matrix} ; x\right]$$

$$\times F^{\lambda\,;\,\mu:\mu}_{\nu\,;\,\rho:\rho}\left[\begin{matrix} (\alpha_\lambda) & : (\beta_\mu) & ; (\beta'_\mu) & ; & z_1 x(1-x) \\ (\gamma_\nu) & : (\delta_\rho) & ; (\delta'_\rho) & ; & z_2 x(1-x) \end{matrix}\right] dx$$

$$= \frac{2^{a+b-2c-2}\Gamma(\frac{a+b+1}{2})\Gamma(\frac{a}{2})\Gamma(\frac{b}{2})\Gamma(c)\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2})}{\Gamma(a)\Gamma(b)\Gamma(c-\frac{a}{2}+\frac{1}{2})\Gamma(c-\frac{b}{2}+\frac{1}{2})}$$

$$\times F^{\lambda+2\,;\,\mu:\mu}_{\nu+2\,;\,\rho:\rho}\left[\begin{matrix} (\alpha_\lambda), c, c+\frac{1-a-b}{2} & : (\beta_\mu) & ; (\beta'_\mu) & ; & \frac{z_1}{4}, \frac{z_2}{4} \\ (\gamma_\nu), c+\frac{1-a}{2}, c+\frac{1-b}{2} & : (\delta_\rho) & ; (\delta'_\rho) & ; & \end{matrix}\right]$$

$$- \frac{2^{a+b-2c-2}\Gamma(\frac{a+b+1}{2})\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(\frac{b}{2}+\frac{1}{2})\Gamma(c)\Gamma(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2})}{\Gamma(a)\Gamma(b)\Gamma(c-\frac{a+1}{2})\Gamma(c-\frac{b}{2}+1)}$$

$$\times F^{\lambda+2\,;\,\mu:\mu}_{\nu+2\,;\,\rho:\rho}\left[\begin{matrix} (\alpha_\lambda), c, \ c+\frac{1-a-b}{2} & : (\beta_\mu) & ; (\beta'_\mu) & ; & \frac{z_1}{4}, \frac{z_2}{4} \\ (\gamma_\nu), c-\frac{a}{2}+1, c-\frac{b}{2}+1 & : (\delta_\rho) & ; (\delta'_\rho) & ; & \end{matrix}\right]$$

provided that the conditions easily obtainable from (3.1) are satisfied.

Similarly other results can also obtain.

References

[1] P. Appell et J. Kampé de Fériet(1926), *Fonctions Hypergéométriques et Hypersphériques Polynomes d'Hermite*, Gauthier-Villars, Paris.

[2] J. Choi, A. K. Rathie, and H. Harsh, *Remarks on a summation formula for three variables hypergeometric function X_8 and certain hypergeometric transformations*, East Asian Math. J. **25** (2009), no. 4, 481–486.

[3] A. Erdélyi et al., *Tables of integral transforms, Vol. II*, McGraw-Hill, New York, 1954.

[4] H. Exton, *Hypergeometric functions of three variables*, J. Indian acad. Math. **4** (1982), 113–119.

[5] Y. S. Kim, J. Choi, and A. K. Rathie, *Remark on two results by Padmanabham for Exton's triple hypergeometric series X_8*, Honam Math. J. **27** (2005), no. 4, 603–608.

[6] Y. S. Kim and A. K. Rathie, *On an extension formula for the triple hypergeometric series X_8 due to Exton*, Bull. Korean Math. Soc. **44** (2007), no. 4, 743–751.

[7] Y. S. Kim and A. K. Rathie, *Another method for Padmanabham's transformation formula for Exton's triple hypergeometric series X_8*, Commun. Korean Math. Soc. **24** (2009), no. 4, 517–521.

[8] J. L. Lavoie, F. Grondin and A. K. Rathie, *Generalizations of Watson's theorem on the sum of a* ${}_3F_2$, Indian J. Math. **32** (1992), no. 1, 23–32.
[9] S. W. Lee and Y. S. Kim, *An extension of the triple hypergeometric series by Exton*, Honam Math. J. **32** (2010), no. 1, 61–71.
[10] H. M. Srivastava and P. W. Karlsson(1985), *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester); Wiley, New York, Chichester, Brisbane, and Toronto.
[11] J. Yang, $U(n+1)$ *extensions of some basic hypergeometric series identities*, Adv. Stud. Contem. Math. **18** (2009), no. 2, 201–218.
[12] C. Zhang and Z. Zhang, *Extension of two q- series identities*, Adv. Stud. Contem. Math. **13** (2006), no. 1, 81–85.

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