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FUNCTIONAL RELATIONS INVOLVING SRIVASTAVA'S HYPERGEOMETRIC FUNCTIONS H_B AND $F^{(3)}$

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Abstract. B. C. Carlson [Some extensions of Lardner's relations between ${}_0F_3$ and Bessel functions, *SIAM J. Math. Anal.* **1**(2) (1970), 232–242] presented several useful relations between Bessel and generalized hypergeometric functions that generalize some earlier results. Here, by simply splitting Srivastava's hypergeometric function H_B into eight parts, we show how some useful and generalized relations between Srivastava's hypergeometric functions H_B and $F^{(3)}$ can be obtained. These main results are shown to be specialized to yield certain relations between functions ${}_0F_1$, ${}_1F_1$, ${}_0F_3$, Ψ_2, and their products including different combinations with different values of parameters and signs of variables. We also consider some other interesting relations between the Humbert Ψ_2 function and Kampé de Fériet function, and between the product of exponential and Bessel functions with Kampé de Fériet functions.

1. Introduction and preliminaries

Investigation of multiple hypergeometric functions is essentially motivated by the fact that the solutions of many applied problems involving the thermal conductivity and dynamics, electromagnetic oscillation and aerodynamics, quantum mechanics and potential theory are obtainable with the help of such hypergeometric (higher and special or transcendent) functions (see [7, 11, 25, 27]). Functions of such kind are often referred to as special functions of mathematical physics. They mainly appear in the solution of partial differential equations which are dealt

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with harmonic analysis method (see [9]). In view of various applications, it is interesting in itself and seems to be very important to conduct a continuous research of multiple hypergeometric functions. For instance, in [33], a comprehensive list of hypergeometric functions of three variables as many as 205 is recorded, together with their regions of convergence. It is noted that Riemann's functions and the fundamental solutions of the degenerate second-order partial differential equations are expressible by means of hypergeometric functions of several variables (see [2, 4, 5, 6, 12, 13, 14, 15, 16, 17, 26, 29, 36, 37, 38]). Therefore, an investigation of the boundary-value problems for these partial differential equations, we need to investigate the properties of aforementioned hypergeometric functions of several variables (see [18, 19, 20, 21, 22, 28, 34]).

Lardner [23] gave some connections between Bessel functions and hypergeometric ${}_0F_3$-series, in particular,

$$ {}_0F_3\left(\frac{1}{2},\frac{1}{2},1;z\right)=\frac{1}{2}\left[J_0\left(4z^{\frac{1}{4}}\right)+I_0\left(4z^{\frac{1}{4}}\right)\right] \tag{1.1}$$

and

$$\begin{aligned} &\operatorname{ber}(x)={}_0F_3\left(\frac{1}{2},\frac{1}{2},1;-\frac{x^4}{256}\right)\\ &\text{and}\quad \operatorname{bei}(x)=\frac{x^2}{4}\,{}_0F_3\left(\frac{3}{2},\frac{3}{2},1;-\frac{x^4}{256}\right), \end{aligned} \tag{1.2}$$

where J_ν and I_ν denote a Bessel function and a modified Bessel function of order ν (see [1]; also [35]) defined by

$$\begin{aligned} J_\nu(z)=&\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(-\,;\,\nu+1\,;\,-\frac{z^2}{4}\right),\\ I_\nu(z)=&\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(-\,;\,\nu+1\,;\,\frac{z^2}{4}\right), \end{aligned} \tag{1.3}$$

and ber(x) and bei(x) (x real) denote the Kelvin's functions (see [10, p. 6]) defined by

$$\operatorname{ber}(x)+i\operatorname{bei}(x)=J_0\left(x\,e^{i\,\frac{3}{4}\pi}\right)=I_0\left(x\,e^{i\,\frac{1}{4}\pi}\right). \tag{1.4}$$

Carlson [8] generalized these results for arbitrary parameters to give the following results

$$(1.5)\quad \begin{aligned} &{}_0F_3\left(\frac{1}{2}, c, c+\frac{1}{2}; z\right) \\ &= \frac{1}{2}\Gamma(2c)\left(2z^{\frac{1}{4}}\right)^{1-2c}\left[I_{2c-1}\left(4z^{\frac{1}{4}}\right) + J_{2c-1}\left(4z^{\frac{1}{4}}\right)\right] \end{aligned}$$

and

$$(1.6)\quad \begin{aligned} &{}_0F_3\left(\frac{3}{2}, c, c+\frac{1}{2}; z\right) \\ &= \frac{1}{2}\Gamma(2c)\left(2z^{\frac{1}{4}}\right)^{-2c}\left[I_{2c-2}\left(4z^{\frac{1}{4}}\right) - J_{2c-2}\left(4z^{\frac{1}{4}}\right)\right]. \end{aligned}$$

Srivastava (see [30, 31, 33]) discovered the existence of three additional complete triple hypergeometric functions H_A, H_B and H_C of the second order. One of them is presented as follows:

$$(1.7)\quad \begin{aligned} &H_B(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) \\ &= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p}(a_2)_{m+n}(a_3)_{n+p}}{(c_1)_m (c_2)_n (c_3)_p \, m!n!p!} x^m y^n z^p \end{aligned}$$

where $\mathbb{C}$ and $\mathbb{Z}_0^-$ denote the set of complex numbers and the set of nonpositive integers, respectively, and $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [32]):

$$\begin{aligned} (\lambda)_n &:= \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1), \ldots, \lambda(\lambda+n-1) & (n \in \mathbb{N} := \{1,2,3,\ldots\}) \end{cases} \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \qquad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-), \end{aligned}$$

$\Gamma(x)$ being the well-known Gamma function. The three-dimensional region of convergence of (1.7) is given by Srivastava and Karlsson [33]: $\left(r+s+t+2\sqrt{rst} < 1, |x| := r, \ |y| := s, \ |z| := t\right)$, where the positive quantities r, s and t are associated with radii convergence.

2. Relationships between Srivastava's hypergeometric functions H_B and $F^{(3)}$

Here we establish some interesting and useful functional relations between Srivastava's functions H_B and $F^{(3)}$. For this purpose we separate the triple series in (1.7) into eight parts by considering $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$

the set of nonnegative integers as the disjoint union of the set of even numbers and the set of odd numbers in each of summation indices m, n, and p. In fact, for any complex c_1, c_2 and $c_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and for any finite complex x, y, and z, it is seen that the series H_B converges absolutely so that it can be rearranged into 8 series as follows:

$$
\begin{aligned}
&H_B\left(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z\right) \\
&\quad = \sum_{m,n,p=0}^{1} \left\{ \sum_{i,j,k=0}^{\infty} \frac{(a_1)_{2(i+k)+m+p}}{(c_1)_{2i+m}\,(c_2)_{2j+n}} \right. \\
&\quad \left. \cdot \frac{(a_2)_{2(i+j)+m+n}\,(a_3)_{2(j+k)+n+p}}{(c_3)_{2k+p}\,(2i+m)!\,(2j+n)!\,(2k+p)!}\, x^{2i+m} y^{2j+n} z^{2k+p} \right\}.
\end{aligned} \tag{2.1}
$$

In view of the following easily-derivable identities

$$(a)_{2l+k} = (a)_k\, 4^l \left(\frac{a+k}{2}\right)_l \left(\frac{a+k+1}{2}\right)_l \qquad (k,\, l \in \mathbb{N}_0)$$

and the Pochhammer symbol [23, 24], after a little simplification, we have

Theorem 2.1. *H_B can be expressed in terms of the following eight $F^{(3)}$ functions.*

(2.2)

$$
\begin{aligned}
&H_B\left(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z\right) = \sum_{m,n,p=0}^{1} \frac{(a_1)_{m+p}\,(a_2)_{m+n}\,(a_3)_{n+p}}{(c_1)_m\,(c_2)_n\,(c_3)_p\, m!n!p!} x^m y^n z^p \\
&\cdot F^{(3)}\left[\begin{array}{l} -:: \frac{a_1+m+p}{2}, \frac{a_1+m+p+1}{2};\ \frac{a_2+m+n}{2}, \frac{a_2+m+n+1}{2};\ \frac{a_3+n+p}{2}, \frac{a_3+n+p+1}{2}; \\ -:: -;\ -;\ -; \end{array}\right. \\
&\qquad 1;\ \frac{c_1+m}{2}, \frac{c_1+m+1}{2}, \frac{m+1}{2}, \frac{m+2}{2};\ 1;\ \frac{c_2+n}{2}, \frac{c_2+n+1}{2}, \frac{n+1}{2}, \frac{n+2}{2}; \\
&\qquad\qquad \left. 1;\ \frac{c_3+p}{2}, \frac{c_3+p+1}{2}, \frac{p+1}{2}, \frac{p+2}{2};\ x^2,\, y^2,\, z^2 \right],
\end{aligned}
$$

*where Srivastava's generalized hypergeometric function $F^{(3)}$ of the fourth order is given as follows (*see [33]*):*

$$F^{(3)}\left[\begin{matrix} - :: & b_1, b_2; & b_1', b_2'; & b_1'', b_2''; & c_1; & c_1'; \\ - :: & -; & -; & -; & h_1, h_2, h_3, h_4; & h_1', h_2', h_3', h_4'; \end{matrix}\right.$$

$$\left.\begin{matrix} c_1''; \\ h_1'', h_2'', h_3'', h_4''; \end{matrix}\; x,\, y,\, z\right] = \sum_{i,j,k=0}^{\infty} \frac{(b_1)_{i+k}\,(b_2)_{i+k}\left(b_1'\right)_{i+j}\left(b_2'\right)_{i+j}}{(h_1)_i\,(h_2)_i\,(h_3)_i\,(h_4)_i\left(h_1'\right)_j}$$

$$\cdot \frac{\left(b_1''\right)_{j+k}\left(b_2''\right)_{j+k}(c_1)_i\left(c_1'\right)_j\left(c_1''\right)_k}{\left(h_2'\right)_j\left(h_3'\right)_j\left(h_4'\right)_j\left(h_1''\right)_k\left(h_2''\right)_k\left(h_3''\right)_k\left(h_4''\right)_k\, i!j!k!}\, x^i y^j z^k.$$

Conversely, by considering various signs of variables x, y and z in (2.2), $F^{(3)}$ can be expressed in terms of the functions H_B.

THEOREM 2.2. *Each of the following functional relations holds true.*

(2.3)

$$8F^{(3)}\left[\begin{matrix} - :: & \frac{a_1}{2}, \frac{a_2+1}{2}; & \frac{a_2}{2}, \frac{a_2+1}{2}; & \frac{a_3}{2}, \frac{a_3+1}{2}; & -; & -; \\ - :: & -; & -; & -; & \frac{c_1}{2}, \frac{c_1+1}{2}, \frac{1}{2}; & \frac{c_2}{2}, \frac{c_2+1}{2}, \frac{1}{2}; \end{matrix}\right.$$

$$\left.\begin{matrix} -; \\ \frac{c_3}{2}, \frac{c_3+1}{2}, \frac{1}{2}; \end{matrix}\; x^2,\, y^2,\, z^2\right] = \sum_{m,n,p=0}^{1} H_B\left[(-1)^m x, (-1)^n y, (-1)^p z\right];$$

(2.4)

$$\frac{8\,a_1\,a_2\,x}{c_1}\,F^{(3)}\left[\begin{matrix} - :: & \frac{a_1+1}{2}, \frac{a_1+2}{2}; & \frac{a_2+1}{2}, \frac{a_2+2}{2}; & \frac{a_3}{2}, \frac{a_3+1}{2}; & -; \\ - :: & -; & -; & -; & \frac{c_1+1}{2}, \frac{c_1+2}{2}, \frac{3}{2}; \end{matrix}\right.$$

$$\left.\begin{matrix} -; & -; \\ \frac{c_2}{2}, \frac{c_2+1}{2}, \frac{1}{2}; & \frac{c_3}{2}, \frac{c_3+1}{2}, \frac{1}{2}; \end{matrix}\; x^2,\, y^2,\, z^2\right]$$

$$= \sum_{m,n,p=0}^{1} (-1)^m\, H_B\left[(-1)^m x, (-1)^n y, (-1)^p z\right];$$

(2.5)

$$\frac{8a_2a_3y}{c_2}\,F^{(3)}\left[\begin{matrix} - :: & \frac{a_1}{2}, \frac{a_1+1}{2}; & \frac{a_2+1}{2}, \frac{a_2+2}{2}; & \frac{a_3+1}{2}, \frac{a_3+2}{2}; & -; \\ - :: & -; & -; & -; & \frac{c_1}{2}, \frac{c_1+1}{2}, \frac{1}{2}; \end{matrix}\right.$$

$$\left.\begin{matrix} -; & -; \\ \frac{c_2+1}{2}, \frac{c_2+2}{2}, \frac{3}{2}; & \frac{c_3}{2}, \frac{c_3+1}{2}, \frac{1}{2}; \end{matrix}\; x^2,\, y^2,\, z^2\right]$$

$$= \sum_{m,n,p=0}^{1} (-1)^n\, H_B\left[(-1)^m x, (-1)^n y, (-1)^p z\right];$$

(2.6)

$$\frac{8a_1a_3z}{c_3}\,F^{(3)}\left[\begin{array}{l} -::\ \frac{a_1+1}{2},\ \frac{a_1+2}{2};\ \ \frac{a_2}{2},\ \frac{a_2+1}{2};\ \ \frac{a_3+1}{2},\ \frac{a_3+2}{2};\ \ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \ \frac{c_1}{2},\ \frac{c_1+1}{2},\ \frac{1}{2};\ \ \frac{c_2}{2},\ \frac{c_2+1}{2},\ \frac{1}{2};\ \ \frac{c_3+1}{2},\ \frac{c_3+2}{2},\ \frac{3}{2}; \end{array}\ x^2,\ y^2,\ z^2\right]$$

$$= \sum_{m,n,p=0}^{1} (-1)^p\ H_B\left[(-1)^m x, (-1)^n y, (-1)^p z\right];$$

(2.7)

$$\frac{8a_1a_2a_3\,(a_2+1)\,xy}{c_1c_2}$$

$$\cdot F^{(3)}\left[\begin{array}{l} -::\ \frac{a_1+1}{2},\ \frac{a_1+2}{2};\ \ \frac{a_2+2}{2},\ \frac{a_2+3}{2};\ \ \frac{a_3+1}{2},\ \frac{a_3+2}{2};\ \ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \ \frac{c_1+1}{2},\ \frac{c_1+2}{2},\ \frac{3}{2};\ \ \frac{c_2+1}{2},\ \frac{c_2+2}{2},\ \frac{3}{2};\ \ \frac{c_3}{2},\ \frac{c_3+1}{2},\ \frac{1}{2}; \end{array}\ x^2,\ y^2,\ z^2\right]$$

$$= \sum_{m,n,p=0}^{1} (-1)^{m+n}\ H_B\left[(-1)^m x, (-1)^n y, (-1)^p z\right];$$

(2.8)

$$\frac{8a_1a_2a_3\,(a_1+1)\,xz}{c_1c_3}$$

$$\cdot F^{(3)}\left[\begin{array}{l} -::\ \frac{a_1+2}{2},\ \frac{a_1+3}{2};\ \ \frac{a_2+1}{2},\ \frac{a_2+2}{2};\ \ \frac{a_3+1}{2},\ \frac{a_3+2}{2};\ \ -;\ -;\ -; \\ -::\ -;\ -;\ -;\ \ \frac{c_1+1}{2},\ \frac{c_1+2}{2},\ \frac{3}{2};\ \ \frac{c_2}{2},\ \frac{c_2+1}{2},\ \frac{1}{2};\ \ \frac{c_3+1}{2},\ \frac{c_3+2}{2},\ \frac{3}{2}; \end{array}\ x^2,\ y^2,\ z^2\right]$$

$$= \sum_{m,n,p=0}^{1} (-1)^{m+p}\, H_B\left[(-1)^m x, (-1)^n y, (-1)^p z\right];$$

(2.9)

$$\frac{8a_1a_2a_3\,(a_3+1)\,yz}{c_2c_3} \cdot F^{(3)}\left[\begin{array}{l} -::\ \frac{a_1+1}{2},\ \frac{a_1+2}{2};\ \frac{a_2+1}{2},\ \frac{a_2+2}{2};\ \frac{a_3+2}{2},\ \frac{a_3+3}{2};\ -;\ -;\ -;\ x^2,\ y^2,\ z^2 \\ -::\ -;\ -;\ -;\ \frac{c_1}{2},\ \frac{c_1+1}{2},\ \frac{1}{2};\ \frac{c_2+1}{2},\ \frac{c_2+2}{2},\ \frac{3}{2};\ \frac{c_3+1}{2},\ \frac{c_3+2}{2},\ \frac{3}{2}; \end{array}\right]$$

$$= \sum_{m,n,p=0}^{1} (-1)^{n+p}\, H_B\left[(-1)^m\, x, (-1)^n\, y, (-1)^p\, z\right];$$

(2.10)

$$\frac{8a_1a_2a_3\,(a_1+1)\,(a_2+1)\,(a_3+1)\,xyz}{c_1c_2c_3} \cdot F^{(3)}\left[\begin{array}{l} -::\ \frac{a_1+2}{2},\ \frac{a_1+3}{2};\ \frac{a_2+2}{2},\ \frac{a_2+3}{2};\ \frac{a_3+2}{2},\ \frac{a_3+3}{2};\ -;\ -;\ -;\ x^2,\ y^2,\ z^2 \\ -::\ -;\ -;\ -;\ \frac{c_1+1}{2},\ \frac{c_1+2}{2},\ \frac{3}{2};\ \frac{c_2+1}{2},\ \frac{c_2+2}{2},\ \frac{3}{2};\ \frac{c_3+1}{2},\ \frac{c_3+2}{2},\ \frac{3}{2}; \end{array}\right]$$

$$= \sum_{m,n,p=0}^{1} (-1)^{m+n+p}\, H_B\left[(-1)^m\, x, (-1)^n\, y, (-1)^p\, z\right],$$

where $H_B(x, y, z)$ is given in (1.7).

3. Limiting cases

Here we express the triple hypergeometric functions in terms of simpler hypergeometric functions. For this purpose we try to find some additional decompositions related to Srivastava's hypergeometric functions. It is noted that these results are potentially useful for the researchers who are involved in dealing with various kinds of hypergeometric functions.

Here we use the method suggested in [8]. By making use of following transformations $a_1 \sim 1/\varepsilon$, $x \sim \varepsilon x$, $z \sim \varepsilon z$ in the functional relations (2.1) and (2.4) to (2.11) and taking the limit as $\varepsilon \to 0$ in each of the resulting identities, we find

Corollary 3.1. *Each of the following functional relations holds true.*

$$\begin{aligned}
&{}_1H_B\left(a_2,a_3;c_1,c_2,c_3;x,y,z\right)\\
&=\sum_{m,n,p=0}^{1}\frac{(a_2)_{m+n}\,(a_3)_{n+p}}{(c_1)_m\,(c_2)_n\,(c_3)_p\,m!n!p!}x^m y^n z^p\\
&\cdot F^{(3)}\left[\begin{array}{l}-::\;-;\;\frac{a_2+m+n}{2},\frac{a_2+m+n+1}{2};\;\frac{a_3+n+p}{2},\frac{a_3+n+p+1}{2};\\ -::\;-;\;-;\;-;\end{array}\right.\\
&\quad\begin{array}{l}1;\;1;\\ \frac{c_1+m}{2},\frac{c_1+m+1}{2},\frac{m+1}{2},\frac{m+2}{2};\;\frac{c_2+n}{2},\frac{c_2+n+1}{2},\frac{n+1}{2},\frac{n+2}{2};\end{array}\\
&\quad\left.\begin{array}{l}1;\\ \frac{c_3+p}{2},\frac{c_3+p+1}{2},\frac{p+1}{2},\frac{p+2}{2};\end{array}\frac{x^2}{4},y^2,\frac{z^2}{4}\right]
\end{aligned}\tag{3.1}$$

$$\begin{aligned}
&8F^{(3)}\left[\begin{array}{l}-::\;-;\;\frac{a_2}{2},\frac{a_2+1}{2};\;\frac{a_3}{2},\frac{a_3+1}{2};\;-;\;-;\\ -::\;-;\;-;\;-;\;\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\;\frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\end{array}\right.\\
&\left.\begin{array}{l}-;\\ \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2};\end{array}\frac{x^2}{4},y^2,\frac{z^2}{4}\right]=\sum_{m,n,p=0}^{1}{}_1H_B\left[(-1)^m x,(-1)^n y,(-1)^p z\right];
\end{aligned}\tag{3.2}$$

$$\begin{aligned}
&\frac{8a_2x}{c_1}F^{(3)}\left[\begin{array}{l}-::\;-;\;\frac{a_2+1}{2},\frac{a_2+2}{2};\;\frac{a_3}{2},\frac{a_3+1}{2};\;-;\\ -::\;-;\;-;\;-;\;\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\end{array}\right.\\
&\quad\left.\begin{array}{l}-;\;-;\\ \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\;\frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2};\end{array}\frac{x^2}{4},y^2,\frac{z^2}{4}\right]\\
&=\sum_{m,n,p=0}^{1}(-1)^m\,{}_1H_B\left[(-1)^m x,(-1)^n y,(-1)^p z\right];
\end{aligned}\tag{3.3}$$

$$\begin{aligned}
&\frac{8a_2a_3y}{c_2}F^{(3)}\left[\begin{array}{l}-::\;-;\;\frac{a_2+1}{2},\frac{a_2+2}{2};\;\frac{a_3+1}{2},\frac{a_3+2}{2};\;-;\\ -::\;-;\;-;\;-;\;\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\end{array}\right.\\
&\quad\left.\begin{array}{l}-;\;-;\\ \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\;\frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2};\end{array}\frac{x^2}{4},y^2,\frac{z^2}{4}\right]\\
&=\sum_{m,n,p=0}^{1}(-1)^n\,{}_1H_B\left[(-1)^m x,(-1)^n y,(-1)^p z\right];
\end{aligned}\tag{3.4}$$

(3.5)
$$\frac{8a_3 z}{c_3} F^{(3)}\left[\begin{matrix} - :: & -; & \frac{a_2}{2}, \frac{a_2+1}{2}; & \frac{a_3+1}{2}, \frac{a_3+2}{2}; & -; \\ - :: & -; & -; & -; & \frac{c_1}{2}, \frac{c_1+1}{2}, \frac{1}{2}; \end{matrix}\right.$$
$$\left.\begin{matrix} -; & -; & \\ \frac{c_2}{2}, \frac{c_2+1}{2}, \frac{1}{2}; & \frac{c_3+1}{2}, \frac{c_3+2}{2}, \frac{3}{2}; & \frac{x^2}{4}, y^2, \frac{z^2}{4} \end{matrix}\right]$$
$$= \sum_{m,n,p=0}^{1} (-1)^p \, {}_1H_B\left[(-1)^m x, (-1)^n y, (-1)^p z\right];$$

(3.6)
$$\frac{8a_2 a_3 (a_2+1)\, xy}{c_1 c_2}$$
$$\cdot F^{(3)}\left[\begin{matrix} - :: & -; & \frac{a_2+2}{2}, \frac{a_2+3}{2}; & \frac{a_3+1}{2}, \frac{a_3+2}{2}; & -; \\ - :: & -; & -; & -; & \frac{c_1+1}{2}, \frac{c_1+2}{2}, \frac{3}{2}; \end{matrix}\right.$$
$$\left.\begin{matrix} -; & -; & \\ \frac{c_2+1}{2}, \frac{c_2+2}{2}, \frac{3}{2}; & \frac{c_3}{2}, \frac{c_3+1}{2}, \frac{1}{2}; & \frac{x^2}{4}, y^2, \frac{z^2}{4} \end{matrix}\right]$$
$$= \sum_{m,n,p=0}^{1} (-1)^{m+n} \, {}_1H_B\left[(-1)^m x, (-1)^n y, (-1)^p z\right];$$

(3.7)
$$\frac{8a_2 a_3 (a_1+1)\, xz}{c_1 c_3}$$
$$\cdot F^{(3)}\left[\begin{matrix} - :: & -; & \frac{a_2+1}{2}, \frac{a_2+2}{2}; & \frac{a_3+1}{2}, \frac{a_3+2}{2}; & -; \\ - :: & -; & -; & -; & \frac{c_1+1}{2}, \frac{c_1+2}{2}, \frac{3}{2}; \end{matrix}\right.$$
$$\left.\begin{matrix} -; & -; & \\ \frac{c_2}{2}, \frac{c_2+1}{2}, \frac{1}{2}; & \frac{c_3+1}{2}, \frac{c_3+2}{2}, \frac{3}{2}; & \frac{x^2}{4}, y^2, \frac{z^2}{4} \end{matrix}\right]$$
$$= \sum_{m,n,p=0}^{1} (-1)^{m+p} \, {}_1H_B\left[(-1)^m x, (-1)^n y, (-1)^p z\right];$$

(3.8)
$$\frac{8a_2 a_3 (a_3+1)\, yz}{c_2 c_3}$$
$$\cdot F^{(3)}\left[\begin{matrix} - :: & -; & \frac{a_2+1}{2}, \frac{a_2+2}{2}; & \frac{a_3+2}{2}, \frac{a_3+3}{2}; & -; \\ - :: & -; & -; & -; & \frac{c_1}{2}, \frac{c_1+1}{2}, \frac{1}{2}; \end{matrix}\right.$$
$$\left.\begin{matrix} -; & -; & \\ \frac{c_2+1}{2}, \frac{c_2+2}{2}, \frac{3}{2}; & \frac{c_3+1}{2}, \frac{c_3+2}{2}, \frac{3}{2}; & \frac{x^2}{4}, y^2, \frac{z^2}{4} \end{matrix}\right]$$
$$= \sum_{m,n,p=0}^{1} (-1)^{n+p} \, {}_1H_B\left[(-1)^m x, (-1)^n y, (-1)^p z\right];$$

$$
\begin{aligned}
&\frac{8a_2a_3\,(a_2+1)\,(a_3+1)\,xyz}{c_1c_2c_3}\\
&\cdot F^{(3)}\left[\begin{matrix} -::\ -;\ \frac{a_2+2}{2},\frac{a_2+3}{2};\ \frac{a_3+2}{2},\frac{a_3+3}{2};\ -;\\ -::\ -;\ -;\ -;\ \frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\end{matrix}\right.\\
(3.9)\qquad&\qquad\left.\begin{matrix} -;\ -;\\ \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\ \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2};\end{matrix}\ \frac{x^2}{4},y^2,\frac{z^2}{4}\right]\\
&\qquad=\sum_{m,n,p=0}^{1}(-1)^{m+n+p}\,{}_1H_B\left[(-1)^m\,x,(-1)^n\,y,(-1)^p\,z\right],
\end{aligned}
$$

where

$$
\begin{aligned}
{}_1H_B(x,y,z)&={}_1H_B\,(a_2,a_3;c_1,c_2,c_3;x,y,z)\\
(3.10)\qquad&:=\lim_{\varepsilon\to 0}H_B\left(\frac{1}{\varepsilon},a_2,a_3;c_1,c_2,c_3;\varepsilon x,y,\varepsilon z\right)\\
&=\sum_{m,n,p=0}^{\infty}\frac{(a_2)_{m+n}\,(a_3)_{n+p}}{(c_1)_m\,(c_2)_n\,(c_3)_p\,m!n!p!}x^my^nz^p.
\end{aligned}
$$

Further setting $a_2 \sim 1/\varepsilon,\ x \sim \varepsilon x,\ \ y \sim \varepsilon y$ in (3.10) and taking the limit as $\varepsilon \to 0$ in the resulting identity, we have

$$
\begin{aligned}
(3.11)\qquad&\lim_{\varepsilon\to 0}{}_1H_B\left(\frac{1}{\varepsilon},a_3;c_1,c_2,c_3;\varepsilon x,\varepsilon y,z\right)\\
&={}_0F_1\,(c_1;x)\,\Psi_2\,(a_3;c_2,c_3;y,z)\,,
\end{aligned}
$$

where Ψ_2 is a known function defined by

$$
\Psi_2\,(a_3;c_2,c_3;y,z)=\sum_{n,p=0}^{\infty}\frac{(a_3)_{n+p}}{(c_2)_n\,(c_3)_p\,n!p!}y^nz^p.
$$

Using these replacements in (3.1) to (3.9) and taking the limit as $\varepsilon \to 0$ in the resulting identities, we find

COROLLARY 3.2. *Each of the following functional relations holds true.*

$$\begin{aligned}
&{}_0F_1\left(c_1;x\right)\Psi_2\left(a_3;c_2,c_3;y,z\right)\\
&=\sum_{m,n,p=0}^{1}\frac{\left(a_3\right)_{n+p}}{\left(c_1\right)_m\left(c_2\right)_n\left(c_3\right)_p m!n!p!}x^m y^n z^p\\
&\quad\cdot{}_1F_4\left(1;\frac{c_1+m}{2},\frac{c_1+m+1}{2},\frac{m+1}{2},\frac{m+2}{2};\frac{x^2}{16}\right)\\
&\quad\cdot F^{2:1;1;}_{0:4;4;}\left[\begin{array}{l}\frac{a_3+n+p}{2},\frac{a_3+n+p+1}{2}:\quad 1;\\ -:\ \frac{c_2+n}{2},\frac{c_2+n+1}{2},\frac{n+1}{2},\frac{n+2}{2};\end{array}\begin{array}{l}1;\\ \frac{c_3+p}{2},\frac{c_3+p+1}{2},\frac{p+1}{2},\frac{p+2}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right];
\end{aligned}\tag{3.12}$$

$$\begin{aligned}
&8{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right)\\
&\cdot F^{2:0;0;}_{0:3;3;}\left[\begin{array}{l}\frac{a_3}{2},\frac{a_3+1}{2}: \qquad -; \qquad -;\\ -:\ \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\ \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]\\
&=\left[{}_0F_1\left(c_1;x\right)+{}_0F_1\left(c_1;-x\right)\right]\sum_{n,p=0}^{1}\Psi_2\left[a_3;c_2,c_3;(-1)^n y,(-1)^p z\right];
\end{aligned}\tag{3.13}$$

$$\begin{aligned}
&\frac{8x}{c_1}{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right)\\
&\cdot F^{2:0;0;}_{0:3;3;}\left[\begin{array}{l}\frac{a_3}{2},\frac{a_3+1}{2}: \qquad -; \qquad -;\\ -;\ \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2};\ \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]\\
&=\left[{}_0F_1\left(c_1;x\right)-{}_0F_1\left(c_1;-x\right)\right]\sum_{n,p=0}^{1}\Psi_2\left[a_3;c_2,c_3;(-1)^n y,(-1)^p z\right];
\end{aligned}\tag{3.14}$$

$$\begin{aligned}
&\frac{8a_3 y}{c_2}{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right)\\
&\cdot F^{2:0;0;}_{0:3;3;}\left[\begin{array}{l}\frac{a_3+1}{2},\frac{a_3+2}{2}: \qquad -; \qquad -;\\ -;\ \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2};\ \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]\\
&=\left[{}_0F_1\left(c_1;x\right)+{}_0F_1\left(c_1;-x\right)\right]\sum_{n,p=0}^{1}(-1)^n\Psi_2\left[a_3;c_2,c_3;(-1)^n y,(-1)^p z\right];
\end{aligned}\tag{3.15}$$

(3.16)

$$\frac{8a_3z}{c_3}{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right)$$
$$\cdot F^{2:0;0;}_{0:3;3;}\left[\begin{array}{l}\frac{a_3+1}{2},\frac{a_3+2}{2}: \qquad -; \qquad -; \\ \qquad - : \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2}; \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]$$
$$=\left[{}_0F_1\left(c_1;x\right)+{}_0F_1\left(c_1;-x\right)\right]\sum_{n,p=0}^{1}\left(-1\right)^p\Psi_2\left[a_3;c_2,c_3;\left(-1\right)^n y,\left(-1\right)^p z\right];$$

(3.17)

$$\frac{8a_3xy}{c_1c_2}{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right)$$
$$\cdot F^{2:0;0;}_{0:3;3;}\left[\begin{array}{l}\frac{a_3+1}{2},\frac{a_3+2}{2}: \qquad -; \qquad -; \\ \qquad - : \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2}; \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]$$
$$=\left[{}_0F_1\left(c_1;x\right)-{}_0F_1\left(c_1;-x\right)\right]\sum_{n,p=0}^{1}\left(-1\right)^n\Psi_2\left[a_3;c_2,c_3;\left(-1\right)^n y,\left(-1\right)^p z\right];$$

(3.18)

$$\frac{8a_3\left(a_1+1\right)xz}{c_1c_3}{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right)$$
$$\cdot F^{2:0;0;}_{0:3;3;}\left[\begin{array}{l}\frac{a_3+1}{2},\frac{a_3+2}{2}: \qquad -; \qquad -; \\ \qquad - : \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2}; \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]$$
$$=\left[{}_0F_1\left(c_1;x\right)-{}_0F_1\left(c_1;-x\right)\right]\sum_{n,p=0}^{1}\left(-1\right)^p\Psi_2\left[a_3;c_2,c_3;\left(-1\right)^n y,\left(-1\right)^p z\right];$$

(3.19)

$$\frac{8a_3\left(a_3+1\right)yz}{c_2c_3}{}_0F_3\left(\frac{c_1}{2},\frac{c_1+1}{2},\frac{1}{2};\frac{x^2}{16}\right)$$
$$\cdot F^{2:0;0;}_{0:3;3;}\left[\begin{array}{l}\frac{a_3+2}{2},\frac{a_3+3}{2}: \qquad -; \qquad -; \\ \qquad - : \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2}; \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]$$
$$=\left[{}_0F_1\left(c_1;x\right)+{}_0F_1\left(c_1;-x\right)\right]\sum_{n,p=0}^{1}\left(-1\right)^{n+p}\Psi_2\left[a_3;c_2,c_3;\left(-1\right)^n y,\left(-1\right)^p z\right];$$

(3.20)

$$\frac{8a_3(a_3+1)xyz}{c_1c_2c_3}\,{}_0F_3\left(\frac{c_1+1}{2},\frac{c_1+2}{2},\frac{3}{2};\frac{x^2}{16}\right)$$

$$\cdot F^{2:0;0;}_{0:3;3;}\left[\begin{array}{l}\frac{a_3+2}{2},\frac{a_3+3}{2}: \quad -; \quad -;\\ -: \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2}; \frac{c_3+1}{2},\frac{c_3+2}{2},\frac{3}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]$$

$$=\left[{}_0F_1(c_1;x)-{}_0F_1(c_1;-x)\right]\sum_{n,p=0}^{1}(-1)^{n+p}\,\Psi_2\left[a_3;c_2,c_3;(-1)^n y,(-1)^p z\right],$$

where $F^{2:0;0}_{0:3;3}$ *is a hypergeometric Kampé de Fériet function (see* [33, 3]*) and* ${}_pF_q$ *is the generalized Gauss hypergeometric function (*see [33]*).*

4. Special cases

Setting $x=0$ in (3.12) to (3.20), we obtain the following interesting relations between Humbert and Kampé de Fériet functions.

Corollary 4.1. Each of the following functional relations holds true.

(4.1)

$$\Psi_2(a_3;c_2,c_3;y,z)$$

$$=\sum_{n,p=0}^{1}\frac{(a_3)_{n+p}}{(c_2)_n(c_3)_p\,n!p!}y^n z^p\,F^{2:1;1}_{0:4;4}\left[\begin{array}{l}\frac{a_3+n+p}{2},\frac{a_3+n+p+1}{2}: \quad 1; \quad 1;\\ -: \frac{c_2+n}{2},\frac{c_2+n+1}{2},\frac{n+1}{2},\frac{n+2}{2}; \frac{c_3+p}{2},\frac{c_3+p+1}{2},\frac{p+1}{2},\frac{p+2}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right];$$

(4.2)

$$4F^{2:0;0}_{0:3;3}\left[\begin{array}{l}\frac{a_3}{2},\frac{a_3+1}{2}: \quad -; \quad -;\\ -: \frac{c_2}{2},\frac{c_2+1}{2},\frac{1}{2}; \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]$$

$$=\sum_{n,p=0}^{1}\Psi_2\left[a_3;c_2,c_3;(-1)^n y,(-1)^p z\right];$$

(4.3)

$$\frac{4a_3y}{c_2}F^{2:0;0}_{0:3;3}\left[\begin{array}{l}\frac{a_3+1}{2},\frac{a_3+2}{2}: \quad -; \quad -;\\ -: \frac{c_2+1}{2},\frac{c_2+2}{2},\frac{3}{2}; \frac{c_3}{2},\frac{c_3+1}{2},\frac{1}{2};\end{array}\frac{y^2}{4},\frac{z^2}{4}\right]$$

$$=\sum_{n,p=0}^{1}(-1)^n\,\Psi_2\left[a_3;c_2,c_3;(-1)^n y,(-1)^p z\right];$$

$$\frac{4a_3 z}{c_3} F^{2:0;0}_{0:3;3}\left[\begin{array}{l} \frac{a_3+1}{2}, \frac{a_3+2}{2} : \qquad -; \qquad -; \\ \qquad - : \frac{c_2}{2}, \frac{c_2+1}{2}, \frac{1}{2}; \quad \frac{c_3+1}{2}, \frac{c_3+2}{2}, \frac{3}{2}; \end{array} \frac{y^2}{4}, \frac{z^2}{4}\right] = \sum_{n,p=0}^{1} (-1)^p \, \Psi_2 \left[a_3; c_2, c_3; (-1)^n y, (-1)^p z\right]; \tag{4.4}$$

$$\frac{4a_3 (a_3+1) yz}{c_2 c_3} \cdot F^{2:0;0}_{0:3;3}\left[\begin{array}{l} \frac{a_3+2}{2}, \frac{a_3+3}{2} : \qquad -; \qquad -; \\ \qquad - : \frac{c_2+1}{2}, \frac{c_2+2}{2}, \frac{3}{2}; \quad \frac{c_3+1}{2}, \frac{c_3+2}{2}, \frac{3}{2}; \end{array} \frac{y^2}{4}, \frac{z^2}{4}\right] = \sum_{n,p=0}^{1} (-1)^{n+p} \, \Psi_2 \left[a_3; c_2, c_3; (-1)^n y, (-1)^p z\right]. \tag{4.5}$$

Moreover, using the transformation $c \sim 2c$ in (4.1) to (4.5) and considering the identity:

$$\Psi_2 (c; c, c; x, y) = e^{x+y} {}_0F_1 (c; xy), \tag{4.6}$$

we find

Corollary 4.2. Each of the following functional relations holds true.

$$e^{y+z} {}_0F_1 (2c; yz) = \sum_{n,p=0}^{1} \frac{(2c)_{n+p}}{(2c)_n (2c)_p n! p!} y^n z^p \cdot F^{2:1;1}_{0:4;4}\left[\begin{array}{l} c+\frac{n+p}{2}, c+\frac{n+p+1}{2} : \qquad 1; \\ \qquad - : c+\frac{n}{2}, c+\frac{n+1}{2}, \frac{n+1}{2}, \frac{n+2}{2}; \end{array} \begin{array}{l} 1; \\ c+\frac{p}{2}, c+\frac{p+1}{2}, \frac{p+1}{2}, \frac{p+2}{2}; \end{array} \frac{y^2}{4}, \frac{z^2}{4}\right]; \tag{4.7}$$

$$4F^{2:0;0}_{0:3;3}\left[\begin{array}{l} c, c+\frac{1}{2} : \qquad -; \qquad -; \\ \qquad - : c, c+\frac{1}{2}, \frac{1}{2}; \quad c, c+\frac{1}{2}, \frac{1}{2}; \end{array} \frac{y^2}{4}, \frac{z^2}{4}\right] = \left(e^{y+z} + e^{-y-z}\right) {}_0F_1 (2c; yz) + \left(e^{-y+z} + e^{y-z}\right) {}_0F_1 (2c; -yz); \tag{4.8}$$

(4.9)
$$4yF_{0:3;3}^{2:0;0}\left[\begin{array}{l} c+\frac{1}{2}, c+1: \quad -; \quad -; \quad \frac{y^2}{4}, \frac{z^2}{4} \\ \quad -: \quad c+\frac{1}{2}, c+1, \frac{3}{2}; \quad c, c+\frac{1}{2}, \frac{1}{2}; \end{array}\right] = \left(e^{y+z}-e^{-y-z}\right){}_0F_1\left(2c; yz\right) - \left(e^{-y+z}-e^{y-z}\right){}_0F_1\left(2c; -yz\right);$$

(4.10)
$$4zF_{0:3;3}^{2:0;0}\left[\begin{array}{l} c+\frac{1}{2}, c+1: \quad -; \quad -; \quad \frac{y^2}{4}, \frac{z^2}{4} \\ \quad -: \quad c, c+\frac{1}{2}, \frac{1}{2}; \quad c+\frac{1}{2}, c+1, \frac{3}{2}; \end{array}\right] = \left(e^{y+z}-e^{-y-z}\right){}_0F_1\left(2c; yz\right) + \left(e^{-y+z}-e^{y-z}\right){}_0F_1\left(2c; -yz\right);$$

(4.11)
$$\frac{2\left(2c+1\right)yz}{c} \cdot F_{0:3;3}^{2:0;0}\left[\begin{array}{l} c+1, c+\frac{3}{2}: \quad -; \quad -; \quad \frac{y^2}{4}, \frac{z^2}{4} \\ \quad -: \quad c+\frac{1}{2}, c+1, \frac{3}{2}; \quad c+\frac{1}{2}, c+1, \frac{3}{2}; \end{array}\right] = \left(e^{y+z}+e^{-y-z}\right){}_0F_1\left(2c; yz\right) - \left(e^{-y+z}+e^{y-z}\right){}_0F_1\left(2c; -yz\right).$$

5. Concluding remarks

Considering the following known functional relations

(5.1)
$$H_B\left(a_1, a_2, c_3; c_1, c_3, c_3; x, y, z\right) = \left(1-y\right)^{-a_2}\left(1-z\right)^{-a_1} F_4\left(a_1, a_2; c_1, c_3; \frac{x}{\left(1-y\right)\left(1-z\right)}, \frac{yz}{\left(1-y\right)\left(1-z\right)}\right);$$

(5.2)
$$H_B\left(c_1, a_2, c_1; c_1, c_1, c_1; x, y, z\right) = \left(1-z\right)^{a_2-c_1}\left(1-x-y-z\right)^{-a_2} F\left(\frac{c_1}{2}, \frac{c_1+1}{2}; c_1; \frac{4xyz}{\left(1-x-y-z\right)^2}\right);$$

(5.3)
$$H_B\left(1, a_2, 1; 1, 1, 1; x, y, z\right) = \frac{\left(1-z\right)^{a_2-1}\left(1-x-y-z\right)^{1-a_2}}{\sqrt{\left(1-x-y-z\right)^2-4xyz}};$$

$$\begin{aligned} &\Psi_2\left(a_3; c_2, c_3; x, x\right) \\ &= {}_3F_3\left(a_3, \frac{1}{2}\left(c_2+c_3\right), \frac{1}{2}\left(c_2+c_3-1\right); c_2, c_3, c_2+c_3-1; 4x\right); \end{aligned} \tag{5.4}$$

$$\Psi_2\left(c; c, c; x, x\right) = {}_1F_1\left(c-\frac{1}{2}; 2c-1; 4x\right), \tag{5.5}$$

it is seen to derive a variety of interesting and (potentially) useful functional relations among above-suggested well-known hypergeometric functions from those identities presented here. This work remains to be done by interested researchers who may be intriguing in themselves and need such functional relations.

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