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THE NONEXISTENCE OF WARPING FUNCTIONS ON RIEMANNIAN WARPED PRODUCT MANIFOLDS

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Abstract. In this paper, when N is a compact Riemannian manifold of class (C), we consider the nonexistence of some warping functions on Riemannian warped product manifolds $M = [a, \infty) \times_f N$ with prescribed scalar curvatures.

1. Introduction

One of the basic problems in the differential geometry is to study the set of curvature function over a given manifold.

The well-known problem in differential geometry is whether a given metric on a compact Riemannian manifold is necessarily pointwise conformal to some metric with constant scalar curvature or not.

In a recent study ([6]), Jung and Kim have studied the problem of scalar curvature functions on Lorentzian warped product manifolds and obtaind partial results about the existence and nonexistence of Lorentzian warped metric with some prescribed scalar curvature function. In this paper, we study also the existence and nonexistence of Riemannian warped metric with prescribed scalar curvature functions on some Riemannian warped product manifolds.

By the results of Kazdan and Warner ([7, 8, 9]), if N is a compact Riemannian $n-$manifold without boundary $n \geq 3$, then N belongs to one of the following three catagories:

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.

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(B) A Smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is either identically zero or strictly negative somewhere.

(C) Any smooth function on N is the scalar curvature of some Riemannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [7], [8] and [9], Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

For noncompact Riemannian manifolds, many important works have been done on the question how to determine which smooth functions are scalar curvatures of complete Riemannian metrics on an open manifold. Results of Gromov and Lawson ([5]) show that some open manifolds cannot carry complete Riemannian metrics of positive scalar curvature, for example, weakly enlargeable manifolds.

Furthermore, they show that some open manifolds cannot even admit complete Riemannian metrics with scalar curvatures uniformly positive outside a compact set and with Ricci curvatures bounded ([5], [12, p.322]).

On the other hand, it is well known that each open manifold of dimension bigger than 2 admits a complete Riemannian metric of constant negative scalar curvature ([2]). It follows from the results of Aviles and McOwen ([1]) that any bounded negative function on an open manifold of dimension bigger than 2 is the scalar curvature of a complete Riemannian metric.

In [10] and [11], the author considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric.

In this paper, when N is a compact Riemannian manifold, we consider the nonexistence of warping functions on a warped product manifold $M = [a, \infty) \times_f N$ with specific scalar curvatures, where a is a positive constant. That is, it is shown that if the fiber manifold N belongs to class

(C) then M does not admit a Riemannian metric with some positive scalar curvature near the end outside a compact set.

2. Main results

Let (N, g) be a Riemannian manifold of dimension n and let $f : [a, \infty) \to R^+$ be a smooth function, where a is a positive number. A Riemannian warped product of N and $[a, \infty)$ with warping function f is defined to be the product manifold $([a, \infty) \times_f N, g^{'})$ with

$$g^{'} = dt^2 + f^2(t)g. \tag{2.1}$$

Let $R(g)$ be the scalar curvature of (N, g). Then the scalar curvature $R(t, x)$ of $g^{'}$ is given by the equation

$$R(t,x) = \frac{1}{f^2(t)}[R(g)(x) - 2nf(t)f^{''}(t) - n(n-1)|f^{'}(t)|^2] \tag{2.2}$$

for $t \in [a, \infty)$ and $x \in\ N$ (For details, [3] or [5]).

If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (2.2) can be changed into

$$\frac{4n}{n+1}u^{''}(t) + R(t,x)u(t) - R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{2.3}$$

In this paper, we assume that the fiber manifold N is nonempty, connected and a compact Riemannian $n-$manifold without boundary.

If N admits a Riemannian metric of negative or zero scalar curvature, then we let $u(t) = t^{\alpha}$ in (2.3), where $\alpha > 1$ is a constant. We have

$$R(t,x) \le \frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2} < 0, \quad t > \alpha.$$

Then, by Theorem 3.1, Theorem 3.5 and Theorem 3.7 in [4], we have the following theorem.

Theorem 2.1. *For $n \geq 3$, let $M = [a, \infty) \times_f N$ be the Riemannian warped product $(n+1)$-manifold with N compact n-manifold. Suppose that N is in class (A) or (B), then on M there is a geodesically complete Riemannian metric of negative scalar curvature outside a compact set.*

Here the following lemma plays an important role in this paper, whose proof is similar to that of Lemma 1.8 in [11].

Lemma 2.2. *On $[a, \infty)$, there does not exist a positive solution $u(t)$ such that*

$$t^2 u''(t) + \frac{c}{4} u(t) \leq 0 \quad \text{for} \quad t \geq t_0, \tag{2.4}$$

where $c > 1$ and $t_0 > a$ are constants.

Proof. Assume that $u(t)$ satisfies

$$t^2 u''(t) + \frac{c}{4} u(t) \leq 0 \quad \text{for} \quad t \geq t_0,$$

with $c > 1$. Let

$$u(t) = t^\alpha v(t), \quad t \geq t_0$$

where $\alpha > 0$ is a constant and $v(t) > 0$ is a smooth function. Then we have

$$u''(t) = \alpha(\alpha - 1) t^{\alpha-2} v(t) + 2\alpha t^{\alpha-1} v'(t) + t^\alpha v''(t).$$

And we obtain

$$t^\alpha v(t) \left[\alpha(\alpha - 1) + \frac{c}{4}\right] + 2\alpha t^{\alpha+1} v'(t) + t^{\alpha+2} v''(t) \leq 0. \tag{2.5}$$

Let δ be a positive constant such that $\delta^2 = \dfrac{c-1}{4}$. Then we have

$$\alpha(\alpha - 1) + \frac{c}{4} = \left(\alpha - \frac{1}{2}\right)^2 + \frac{c-1}{4} \geq \delta^2.$$

Then δ is a constant independent on α. Equation (2.5) gives

$$2\alpha t v'(t) + t^2 v''(t) \leq -\delta^2 v(t). \tag{2.6}$$

Let $\beta = 2\alpha$ and we choose $\alpha > 0$ such that $\beta < 1$, that is, $\alpha < \frac{1}{2}$. Then equation (2.6) becomes

$$(t^{\beta}v^{'}(t))^{'} \leq -\frac{\delta^2 v(t)}{t^{2-\beta}}.$$

Upon integration we have

$$t^{\beta}v^{'}(t) - \tau^{\beta}v^{'}(\tau) \leq -\int_{\tau}^{t} \frac{\delta^2 v(s)}{s^{2-\beta}} ds, \quad t > \tau > t_0. \tag{2.7}$$

Here we have two following cases :

[case1] If $v^{'}(\tau) \leq 0$ for some $\tau > t_0$, then (2.7) implies that

$$t^{\beta}v^{'}(t) \leq -C$$

for some positive constant C. We have

$$v(t) \leq v(\tau) - \int_{\tau}^{t} \frac{C}{s^{\beta}} ds = v(\tau) - C\frac{s^{1-\beta}}{1-\beta} \mid_{\tau}^{t} \rightarrow -\infty,$$

as $\beta < 1$. Hence $v(t) < 0$ for some t, contradicting that $v(t) > 0$ for all $t \geq t_0$.

[case2] We have $v^{'}(\tau) > 0$ for all $\tau > t_0$. Equation (2.7) implies that

$$\tau^{\beta}v^{'}(\tau) - \int_{\tau}^{t} \frac{\delta^2 v(s)}{s^{2-\beta}} ds \geq 0$$

for all $t > \tau > t_0$. As $v^{'}(t) > 0$ for all $t > t_0$, we have

$$\tau^{\beta}v^{'}(\tau) \geq v(\tau)\int_{\tau}^{t} \frac{\delta^2}{s^{2-\beta}} ds = v(\tau)[\frac{1}{s^{1-\beta}}[-\frac{\delta^2}{1-\beta}]] \mid_{\tau}^{t} .$$

Let $t \rightarrow \infty$ we have

$$\tau^{\beta}v^{'}(\tau) \geq \frac{v(\tau)}{\tau^{1-\beta}}\frac{\delta^2}{1-\beta}.$$

Or after changing the parameter we have

$$\frac{v'(t)}{v(t)} \geq \frac{1}{t}\frac{\delta^2}{1-\beta}, \quad t > t_0.$$

Choose $\alpha < \frac{1}{2}$ close to $\frac{1}{2}$ so that $\beta < 1$ is close to 1. Using the fact that δ is independent on α or β, we have

$$\frac{v'(t)}{v(t)} \geq \frac{N}{t}$$

for a big integer $N > 2$. This gives

$$v(t) \geq Ct^N, \quad t > t_0$$

where C is a positive constant. The inequality (2.7) implies that

$$t^\beta v'(t) \leq \tau^\beta v'(\tau) - \int_\tau^t \frac{C\delta^2 s^N}{s^{2-\beta}} ds \to -\infty \quad \text{as} \quad t \to \infty.$$

Thus $v'(t) < 0$ for t large, which is also a contradiction. Hence there is no solution to equation (2.4). □

From now on, we assume that $R(t, x)$ is the function of only t-variable. Then we have the following theorems.

THEOREM 2.3. *If N belongs to class (A) or (B), that is, $R(g) \leq 0$, then there is no positive solution to equation* (2.3) *with*

$$R(t) \geq \frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad t \geq t_0,$$

where $c > 1$ and $t_0 > a$ are constants.

Proof. Assume that

$$R(t) \geq \frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad t \geq t_0,$$

with $c > 1$. Equation (2.3) gives

$$t^2 u''(t) + \frac{c}{4}u(t) \leq 0.$$

By Lemma 2.2, we complete the proof. □

If N belongs to (A), then a negative constant function on N is the scalar curvature of some Riemannian metric. So we can take a Riemannian metric g_1 on N with scalar curvature $R(g_1) = -\frac{4n}{n+1}k^2$, where k is a positive constant. Then equation (2.3) becomes

$$\frac{4n}{n+1}u''(t) + \frac{4n}{n+1}k^2u(t)^{1-\frac{4}{n+1}} + R(t,x)u(t) = 0. \tag{2.8}$$

In order to prove the nonexistence of some warped product metric, we need the following lemma.

LEMMA 2.4. *Let $u(t)$ be a positive smooth function on $[a,\infty)$. If $u(t)$ satisfies*

$$\frac{u''(t)}{u(t)} \leq \frac{C}{t^2}$$

for some constant $C > 0$, then there exists $t_0 > a$ such that for all $t > t_0$.

$$u(t) \leq C_0 t^\epsilon$$

for some positive constant C_0 and $\epsilon > 1$.

Proof. Since $C > 0$, we can choose $\epsilon > 1$ such that $\epsilon(\epsilon - 1) = C$. Then from the hypothesis, we have

$$t^\epsilon u''(t) \leq \epsilon(\epsilon-1)t^{\epsilon-2}u(t).$$

Upon integration from $t_1(\geq a)$ to $t(> t_1 \geq a)$, and using twice integration by parts, we obtain

$$t^\epsilon u'(t) - \epsilon t^{\epsilon-1}u(t) - t_1^\epsilon u'(t_1) + \epsilon {t_1}^{\epsilon-1}u(t_1) + \epsilon(\epsilon-1)\int_{t_1}^t s^{\epsilon-2}u(s)ds$$

$$\leq C\int_{t_1}^t s^{\epsilon-2}u(s)ds.$$

Therefore we have

$$t^\epsilon u'(t) - \epsilon t^{\epsilon-1}u(t) \leq t_1^\epsilon u'(t_1) - \epsilon t_1^{\epsilon-1}u(t_1). \tag{2.9}$$

We consider two following cases :

[Case 1] There exists $t_1 \geq a$ such that $u'(t_1) \leq 0$.

If there is a number $t_1 \geq a$ such that $u'(t_1) \leq 0$, then we have

$$t^{\epsilon}u'(t) - \epsilon t^{\epsilon-1}u(t) \leq 0.$$

This gives

$$(\ln u(t))' \leq \epsilon(\ln t)'.$$

Hence

$$u(t) \leq c_1 t^{\epsilon}$$

for all $t > t_1$, where c_1 is a positive constant.

[Case 2] There does not exist $t_1 \geq a$ such that $u'(t_1) \leq 0$.

In other words, if $u'(t) > 0$ for all $t \geq a$, then $u(t) \geq c'$ for some positive constant c'. Let c_2 be a positive constant such that

$$t_1^{\epsilon}u'(t_1) - \epsilon t_1^{\epsilon-1}u(t_1) \leq c_2,$$

then equation (2.9) gives

$$t^{\epsilon}u'(t) - \epsilon t^{\epsilon-1}u(t) \leq c_2$$

for all $t > t_1$. Thus

$$\frac{u'(t)}{u(t)} \leq \frac{\epsilon}{t} + \frac{c_2}{u(t)t^{\epsilon}} \leq \frac{\epsilon}{t} + \frac{c_2}{c't^{\epsilon}}.$$

Integrating from t_1 to t we have

$$\ln\frac{u(t)}{u(t_1)} \leq \epsilon\ln\left(\frac{t}{t_1}\right) + \frac{c_2}{(\epsilon-1)c't_1^{\epsilon-1}} \leq \epsilon\ln\left(\frac{c_3 t}{t_1}\right)$$

as $\epsilon > 1$. Here c_3 is a positive constant such that $\ln c_3 \geq \dfrac{c_2}{\epsilon(\epsilon-1)c't_1^{\epsilon-1}}$. Hence we again obtain the inequality

$$u(t) \leq bt^{\epsilon}$$

for some positive constant b and for all $t \geq t_1$.

Thus from two cases we always find $t_0 > a$ and a constant $C_0 > 0$ such that

$$u(t) \leq C_0 t^{\epsilon}$$

for all $t \geq t_0$. □

Using the above lemma, we can prove the following theorem about the nonexistence of warping function, whose proof is similar to that of Lemma 3.3 in [11].

Theorem 2.5. *Suppose that N belongs to class (A). Let g be a Riemannian metric on N of dimension $n (\geq 3)$. We may assume that $R(g) = -\frac{4n}{n+1}k^2$, where k is a positive constant. On $[a, \infty) \times N$, there does not exist a Riemannian warped product metric*

$$g' = dt^2 + f^2(t) g$$

with scalar curvature

$$R(t) \geq -\frac{n(n-1)}{t^2}$$

for all $x \in N$ and $t > t_0 > a$, where t_0 and a are positive constants.

Proof. Assume that we can find a warped product metric on $[a, \infty) \times N$ with

$$R(t) \geq -\frac{n(n-1)}{t^2}$$

for all $x \in N$ and $t > t_0 > a$. In equation (2.3), we have

$$\frac{4n}{n+1}\left[\frac{u''(t)}{u(t)} + \frac{k^2}{u(t)^{\frac{4}{n+1}}}\right] = -R(t) \leq \frac{n(n-1)}{t^2} \tag{2.10}$$

and

$$\frac{u''(t)}{u(t)} \leq \frac{\frac{(n-1)(n+1)}{4}}{t^2}. \tag{2.11}$$

In equation (2.11), we can apply Lemma 2.4 and take $\epsilon = \frac{n+1}{2}$. Hence we have $t_0 > a$ such that

$$u(t) \leq c_0 t^{\frac{n+1}{2}}$$

for some positive constants c_0 and all $t > t_0$.

Then

$$\frac{k^2}{u(t)^{\frac{4}{n+1}}} \geq \frac{c^{'}}{t^2}$$

where $0 < c^{'} \leq \frac{k^2}{c_0^{\frac{4}{n+1}}}$ is a positive constant. Hence equation (2.10) gives

$$\frac{u^{''}(t)}{u(t)} \leq \frac{(n+1)(n-1) - \delta}{4t^2},$$

where $4c^{'} \geq \delta \geq 0$ is a constant. We can choose $\delta^{'} > 0$ such that

$$\frac{(n+1)(n-1) - \delta}{4} = \left(\frac{n+1}{2} - \delta^{'}\right)\left(\frac{n-1}{2} - \delta^{'}\right)$$

for small positive δ. Applying the Lemma 2.4 again, we have $t_1 > a$ such that

$$u(t) \leq c_1 t^{\frac{n+1}{2} - \delta^{'}}$$

for some $c_1 > 0$ and all $t > t_1$. And

$$\frac{k^2}{u(t)^{\frac{4}{n+1}}} \geq \frac{c^{''}}{t^{2-\epsilon}}, \tag{2.12}$$

where $\epsilon = \dfrac{4}{n+1}\delta^{'}$ and $0 < c^{''} \leq \dfrac{k^2}{c_1^{\frac{4}{n+1}}}$. Thus equation (2.11) and (2.12) give

$$\frac{u^{''}(t)}{u(t)} \leq \frac{(n-1)(n+1)}{4t^2} - \frac{c^{''}}{t^{2-\epsilon}},$$

which implies that

$$u^{''}(t) \leq 0$$

for t large. Hence $u(t) \leq c_2 t$ for some constant $c_2 > 0$ and large t. From equation (2.10) we have

$$\frac{u^{''}(t)}{u(t)} \leq \frac{-c_3}{t^{\frac{4}{n+1}}} + \frac{(n+1)(n-1)}{4t^2} \leq -\frac{c_3}{t}$$

for t large enough, as $n \geq 3$. Here c_3 is a positive constant. Multiplying $u(t)$ and integrating from $t^{'}$ to t, we have

$$u^{'}(t) - u^{'}(t^{'}) \leq -c_3 \int_{t^{'}}^{t} \frac{u(s)}{s} ds, \quad t > t^{'}.$$

We consider two following cases :

[Case 1] There exists $t^{'} \geq max\{t_0, t_1\}$ such that $u^{'}(t^{'}) \leq 0$.

If $u^{'}(t^{'}) \leq 0$ for some $t^{'}$, then $u^{'}(t) \leq -c_4$ for some positive constant c_4. Hence $u(t) \leq 0$ for t large enough, contradicting the fact that u is positive.

[Case 2] There does not exist $t^{'} \geq max\{t_0, t_1\}$ such that $u^{'}(t^{'}) \leq 0$. In order words, if $u^{'}(t) > 0$ for all t large, then $u(t)$ is increasing, hence

$$\int_{t^{'}}^{t} \frac{u(s)}{s} ds \geq u(t^{'}) \int_{t^{'}}^{t} \frac{1}{s} ds \to \infty.$$

Thus $u^{'}(t)$ has to be negative for some t large, which is a contradiction to the hypothesis. Therefore there does not exist such warped product metric.

□

If N belongs to class (C), then by the results of Kazdan and Warner ([7, 8, 9]), some positive constant function on N is the scalar curvature of some Riemannian metric. So we can take a Riemannian metric g_1 on N with scalar curvature $R(g_1) = \frac{4n}{n+1} k^2$, where k is a positive constant. Then equation (2.3) becomes

$$\frac{4n}{n+1} u^{''}(t) - \frac{4n}{n+1} k^2 u(t)^{1-\frac{4}{n+1}} + R(t,x) u(t) = 0. \tag{2.13}$$

If $R(t,x) = R(t)$ is the bounded function of only t-variable, our first main theorem is as follows :

Theorem 2.6. *Suppose that $R(g) = \frac{4n}{n+1} k^2$ for $n \geq 3$ and $R(t,x) = R(t) \in C^\infty([a,\infty))$. Assume that for $t > t_0$, there exists a positive solution*

$u(t)$ of equation (2.13) with $-M < R(t) < 0$ for some positive constant M. Then $u(t) \geq t^{\alpha}$ for large t and all $\alpha > 0$.

Proof. Suppose $u(t) > 0$ satisfies equation (2.13), i.e.,

$$\frac{4n}{n+1}u^{''}(t) = -R(t)u(t) + \frac{4n}{n+1}k^2u(t)^{1-\frac{4}{n+1}}. \tag{2.14}$$

Since $R(t) \leq 0$ and $\frac{4n}{n+1}k^2u(t)^{1-\frac{4}{n+1}} > 0$, integrating equation (2.14) from $\tau(\geq a)$ to t, we have

$$u^{'}(t) - u^{'}(\tau) > 0$$

for all $t(> \tau)$.

Here we have two following cases :

[Case 1] There exists $\tau(\geq a)$ such that $u^{'}(\tau) \geq 0$. Then $u^{'}(t) \geq 0$, so $u(t)$ is an increasing function. Thus $u(t) \geq u(\tau) > 0$. Therefore from equation (2.14) we have

$$u^{''}(t) \geq k^2u(t)^{1-\frac{4}{n+1}} \geq c_0$$

for large t and some positive constant c_0. Hence we have

$$u(t) \geq \frac{c_0}{2}t^2 + c_1t + c_2 \tag{2.15}$$

for some constants c_1, c_2. Again substituting equation (2.15) to equation (2.14), we have

$$u^{''}(t) \geq k^2\left(\frac{c_0}{2}t^2 + c_1t + c_2\right)^{1-\frac{4}{n+1}}$$

$$u(t) > c_3t^{2+2\left(1-\frac{4}{n+1}\right)}$$

for some positive constant c_3. Reiterating this method, we complete the theorem.

[Case 2] There does not exist $\tau(\geq a)$ such that $u^{'}(\tau) \geq 0$. In other words, we have $u^{'}(t) < 0$ for all $t(\geq a)$. Then $u(t)$ is a decreasing function. If $u(t) \geq c_0$ for some positive constant c_0, then by [Case 1] our theorem holds. Otherwise, since $u(t)$ is a decreasing function, $u(t) \to +0$ as $t \to \infty$. From equation (2.14) $R(t)$ is bounded and $1 - \frac{4}{n+1} > 0$, so $u^{''}(t) \to +0$ as $t \to \infty$ and $u^{'}(t)$ is an increasing function.

Put $u(t) = e^{-g(t)}$. Then $u^{'}(t) = -e^{-g(t)} g^{'}(t) < 0$, so $g^{'}(t) > 0$ and $g(t)$ is an increasing function. Thus we have

$$u^{''}(t) = e^{-g(t)}((g^{'}(t))^2 - g^{''}(t)) \to +0$$

as $t \to \infty$, so $g^{''}(t) > 0$ because $\dfrac{(g^{'}(t))^2}{e^{g(t)}} \approx \dfrac{2g^{'}(t)g^{''}(t)}{e^{g(t)}g^{'}(t)} = \dfrac{2g^{''}(t)}{e^{g(t)}}$ by L'Hospital's Theorem.

Therefore from equation (2.14) we have

$$(g^{'}(t))^2 - g^{''}(t) = -\frac{n+1}{4n} R(t) + k^2 e^{\frac{4}{n+1} g(t)}.$$

Since $g^{''}(t) > 0$ and $R(t) \leq 0$, we have

$$(g^{'}(t))^2 \geq k^2 e^{\frac{4}{n+1} g(t)}.$$

And since $g^{'}(t) > 0$, we have

$$g^{'}(t) \geq k e^{\frac{2}{n+1} g(t)}. \tag{2.16}$$

Since $g(t)$ is increasing, from equation (2.16) $g^{'}(t) \geq c_0$ for some positive constant c_0. Hence we have

$$g(t) \geq c_0 t + c_1 \tag{2.17}$$

for some constant c_1. Again, substituting equation (2.17) to equation (2.16), we have

$$g^{'}(t) \geq k e^{\frac{2}{n+1}(c_0 t + c_1)}$$

and, integrating the above equation, we have

$$g(t) \geq a_1 e^{\frac{2}{n+1} c_0 t} \tag{2.18}$$

for some positive constants a_1 and c_0. Again plugging equation (2.18) into equation (2.16), we have

$$g^{'}(t) \geq k e^{\frac{2}{n+1} a_1 e^{\frac{2}{n+1} c_0 t}}$$

and, integrating the above equation, we have

$$g(t) \geq a_2 e^{\frac{2}{n+1} a_1 e^{\frac{2}{n+1} c_0 t}}$$

for some positive constant a_2. And again, iterating this way, we have

$$g(t) \geq a_n e^{\frac{2}{n+1} a_{n-1} e^{\frac{2}{n+1} \cdot^{\cdot^{\cdot e^{a_1 e^{\frac{2}{n+1} c_0 t}}}}}},$$

which is impossible.

From [Case 1] and [Case 2], we complete the theorem. □

If $R(t,x)$ is also the function of only t-variable, our second main theorem is as follows :

Theorem 2.7. *Suppose that* $R(g) = \frac{4n}{n+1} k^2$. *Assume that* $R(t,x) = R(t) \in C^\infty([a,\infty))$ *is a function such that*

$$-\frac{n(n-1)}{t^2} \leq R(t) < 0 \quad \text{for} \quad t > t_0,$$

where $t_0 > a$ *and* $1 \leq C$ *is a constant. Then equation* (2.13) *has no positive solution on* $[a,\infty)$.

Proof. Assume that for $t > t_0$, there exists a solution $u(t)$ of equation (2.13) with $0 > R(t) \geq -M$. Then by Theorem 2.6 u(t) is an increasing function such that $u(t) \geq t^\alpha$ for large t and all $\alpha > 0$. From equation (2.13), we have

$$\frac{4n}{n+1} \frac{u''(t)}{u(t)} = -R(t) + \frac{4n}{n+1} k^2 u(t)^{-\frac{4}{n+1}} \leq \frac{C}{t^2}$$

for some constant $C \geq 1$ and large t. Hence the Lemma 2.4 implies that for all large t

$$u(t) \leq C_0 t^\epsilon$$

for some positive constant C_0 and $\epsilon > 1$, which is a contradiction to the fact that $u(t) \geq t^\alpha$ for large t and all $\alpha > 0$. Therefore equation (2.13) has no positive solution on $[a,\infty)$. □

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