JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **24**, No. 2, June 2011

A REMARK ON MULTI-VALUED GENERALIZED SYSTEM

Sangho Kum*

Abstract. Recently, Kazmi and Khan [7] introduced a kind of equilibrium problem called *generalized system* (GS) with a single-valued bi-operator F. In this note, we aim at an extension of (GS) due to Kazmi and Khan [7] into a multi-valued circumstance. We consider a fairly general problem called *the multi-valued quasi-generalized system* (in short, MQGS). Based on the existence of 1-person game by Ding, Kim and Tan [5], we give a generalization of (GS) in the name of (MQGS) within the framework of Hausdorff topological vector spaces. As an application, we derive an existence result of the generalized vector quasi-variational inequality problem. This result leads to a multi-valued vector quasi-variational inequality extension of the strong vector variational inequality (SVVI) due to Fang and Huang [6] in a general Hausdorff topological vector space.

1. Introduction

The equilibrium problem (EP) has been intensively studied, beginning with Blume and Oettli [2] where they proposed it as a generalization of optimization and variational inequality problem. It turns out that this problem includes, as special cases, other problems such as the fixed point and coincidence point problem, the complementarity problem, the Nash equilibrium problem, etc. Its numerous extensions and applications can be found in the literature. See, e.g., [1, 3, 4, 8] and the references therein. Recently, Kazmi and Khan [7] introduced a kind of EP called *generalized system* (GS) with a single-valued bi-operator

Received January 10, 2011; Accepted May 13, 2011.
2010 Mathematics Subject Classification: Primary 49J40, 47H10, 54H25.
Key words and phrases: equilibrium problem, vector variational inequality, generalized system.
This work was supported by the research grant of the Chungbuk National University in 2010.

F. Their result extends the strong vector variational inequality (SVVI) studied in Fang and Huang [6] in real Banach spaces.

In this note, we aim at an extension of (GS) due to Kazmi and Khan [7] into a multi-valued circumstance. We consider a fairly general problem called *the multi-valued quasi-generalized system* (in short, MQGS). Based on the existence of 1-person game by Ding, Kim and Tan [5], we give a generalization of (GS) due to Kazmi and Khan [7, Theorem 2.1] in the name of (MQGS) within the framework of Hausdorff topological vector spaces. As an application, we derive an existence result of the generalized vector quasi-variational inequality problem. This result leads to a multi-valued quasi-vector variational inequality extension of the strong vector variational inequality (SVVI) due to Fang and Huang [6] in a general Hausdorff topological vector space. From both theoretical and practical points of view, it is natural and useful to extend a single-valued case to a corresponding multi-valued one. This is a motivation for the current work.

2. Preliminaries

We begin with taking a brief look at several standard definitions and terminologies concerned with multi-valued functions. Let X, Y be nonempty topological spaces and $T : X \rightrightarrows Y$ be a multifunction. Then $T : X \rightrightarrows Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$. For each $y \in Y$, the set $T^{-1}(y) = \{x \in X \mid y \in T(x)\}$ is called *the lower section* of T at y. We denote by $\operatorname{cl} T$ the multi-valued function $\operatorname{cl} T(x) = \overline{T(x)}$ for all $x \in X$.

Let E be a Hausdorff topological vector space. A nonempty subset C of E is called a *convex cone* if $\lambda C \subseteq C$, for all $\lambda > 0$ and $C + C = C$. From now on, unless otherwise specified, we work under the following settings: Let X, Y be Hausdorff topological vector spaces and let K be a nonempty convex subset of X. Let C be a pointed closed convex cone in Y (not necessarily, $\operatorname{int} C \neq \emptyset$). Let $F : K \times K \rightrightarrows Y$ be a nonempty multi-valued bi-operator such that $F(x, x) \supseteq \{0\}$ for each $x \in K$. Then *a multi-valued generalized sytem* (MGS) is defined to be the problem of finding $x \in K$ such that

$$F(x, y) \nsubseteq -C \setminus \{0\} \quad \text{for all } y \in K. \qquad (MGS)$$

Let $A : K \rightrightarrows K$ be a multifunction. As a further generalized version, we define the *the multi-valued quasi-generalized system* (MQGS) to be the problem of seeking $x \in K$ such that $x \in \mathrm{cl}\, A(x)$ and,

$$F(x,y) \nsubseteq -C \setminus \{0\}, \quad \text{for all } \ y \in A(x). \qquad (MQGS)$$

When $A(x) \equiv K$ for each $x \in K$, (MQGS) reduces to the multi-valued generalized system(MGS), and (MGS) becomes the generalized system(GS) due to Kazmi and Khan [7] where $F : K \times K \to Y$ is a single-valued mapping.
A multi-valued mapping $G : K \rightrightarrows Y$ is said to be *C-convex* if $\forall x, y \in K,\ \forall \lambda \in [0,1]$,

$$G(\lambda x + (1-\lambda)y) \subseteq \lambda G(x) + (1-\lambda)G(y) - C.$$

As mentioned in the introduction, the following theorem is a basic tool to obtain the main result of this paper.

LEMMA 2.1. ([5, Theorem 2]) *Let $\Gamma = (X, A, Q)$ be an 1-person game such that*
(1) X is a nonempty convex subset of a locally convex Hausdorff topological vector space and D be a nonempty compact subset of X;
(2) $A : X \rightrightarrows D$ is a multifunction such that for each $x \in X$, $A(x)$ is nonempty convex and for each $y \in D$, $A^{-1}(y)$ is open in X;
(3) the multifunction $\mathrm{cl}\, A : X \rightrightarrows X$ is upper semicontinuous;
(4) the multifunction $Q : X \rightrightarrows D$ is such that $Q^{-1}(y)$ is open in X for each $y \in D$;
(5) for each $x \in X$, $x \notin \mathrm{co}\, Q(x)$ where $\mathrm{co}\, Q(x)$ stands for the convex hull of $Q(x)$.
Then Γ has an equilibrium choice $\hat{x} \in D$, i.e., $\hat{x} \in \mathrm{cl}\, A(\hat{x})$ and $A(\hat{x}) \cap Q(\hat{x}) = \emptyset$.

3. Main result

To establish the main result, we need the following lemma.

LEMMA 3.1. *A multifunction $G : K \rightrightarrows F$ is C-convex if and only if for every $n \geq 2$, whenever $x_1, \ldots, x_n \in K$ are given and for any $\lambda_i \in [0,1], i = 1, \ldots, n$, with $\sum_{i=1}^{n} \lambda_i = 1$, we have*

$$G\left(\sum_{i=1}^{n} \lambda_i\, x_i\right) \subseteq \lambda_1 G(x_1) + \cdots + \lambda_n G(x_n) - C. \qquad (1)$$

Proof. The sufficiency is clear. For the necessity, we shall use the induction argument on n. When $n = 2$, the condition (1) is exactly the same as the definition of C-convexity. Assume that the condition (1) holds for all $k \leq n-1 \ \ (n \geq 3)$. Let $\{x_1, \dots, x_n\} \subset K$ be given, and $\lambda_i \in [0,1], i = 1, \dots, n,$ with $\sum_{i=1}^{n} \lambda_i = 1$ be arbitrarily given. Without loss of generality, we may assume $\sum_{i=1}^{n-1} \lambda_i > 0$ by reindexing i. Then, for a given set $\{x_1, \dots, x_{n-1}\}$, the induction assumption assures that

$$G\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{j=1}^{n-1} \lambda_j}\, x_i\right) \subseteq \frac{\lambda_1}{\sum_{j=1}^{n-1} \lambda_j} G(x_1) + \cdots + \frac{\lambda_{n-1}}{\sum_{j=1}^{n-1} \lambda_j}\, G(x_{n-1}) - C. \tag{2}$$

Again applying C-convexity to $\left\{\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{j=1}^{n-1} \lambda_j}\, x_i,\ x_n\right\}$ in K, we see that

$$\begin{aligned} G\left(\sum_{i=1}^{n} \lambda_i\, x_i\right) &= G\left((\sum_{j=1}^{n-1} \lambda_j)\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{j=1}^{n-1} \lambda_j}\, x_i\right) + \lambda_n x_n\right) \\ &\subseteq (\sum_{j=1}^{n-1} \lambda_j) G\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{j=1}^{n-1} \lambda_j}\, x_i\right) + \lambda_n G(x_n) - C. \end{aligned}$$

Using the inclusion (2), we have

$$\begin{aligned} G\left(\sum_{i=1}^{n} \lambda_i\, x_i\right) &\subseteq \left[\lambda_1\, G(x_1) + \cdots + \lambda_{n-1}\, G(x_{n-1}) - C\right] + \lambda_n G(x_n) - C \\ &= \lambda_1 G(x_1) + \cdots + \lambda_{n-1} G(x_{n-1}) + \lambda_n G(x_n) - C \end{aligned}$$

because C is a convex cone. Therefore, by induction, for every $n \geq 2$, we can obtain the desired conclusion. □

THEOREM 3.2. *Let K be a nonempty compact convex subset of X. Let $F : K \times K \rightrightarrows Y$ be C-convex in the second variable. Assume that for each $y \in K$, the set $\{x \in K \mid F(x,y) \subseteq -C \setminus \{0\}\}$ is open. Let $A : K \rightrightarrows K$ be a multifunction such that $\mathrm{cl}\, A$ is upper semicontinuous, $A(x) \neq \emptyset$ for all $x \in K$, and A has an open lower section at any $y \in Y$. Then (MQGS) is solvable.*

Proof. First define a multifunction $Q : K \rightrightarrows K$ to be

$$Q(x) := \{y \in K \mid F(x,y) \subseteq -C \setminus \{0\}\}, \text{ for each } x \in K.$$

(i) For each $x \in K$, $x \notin \mathrm{co}\, Q(x)$. Indeed, suppose the contrary, i.e., there exists $x \in K$ such that $x \in \mathrm{co}\, Q(x)$. Then there exist $\{x_1, \cdots, x_n\} \subseteq$

$Q(x)$ and $0 < \lambda_1, \ldots, \lambda_n < 1$ such that $\sum_{i=1}^n \lambda_i = 1$ and $x = \sum_{i=1}^n \lambda_i x_i$. Hence, for each $i = 1, \ldots n$,

$$F(x, x_i) \subseteq -C \setminus \{0\}. \tag{3}$$

Since $F(x, \cdot)$ is C-convex, by Lemma 2 and (3), we have

$$\begin{aligned}\{0\} \subseteq F(x,x) &= F\left(x,\ \sum_{i=1}^n \lambda_i x_i\right)\\ &\subseteq \lambda_1 F(x,x_1) + \cdots + \lambda_n F(x,x_n) - C\\ &\subseteq -\big[\lambda_1(C \setminus \{0\}) + \cdots + \lambda_n(C \setminus \{0\}) + C\big]\\ &\subseteq (-C \setminus \{0\}) - C = -C \setminus \{0\}.\end{aligned}$$

This is a contradiction. Thus, we see that $x \notin \operatorname{co} Q(x)$ for all $x \in K$.

(ii) The multifunction Q has open lower sections, i.e., for each $y \in K$, $Q^{-1}(y)$ is open in K by the given hypothesis. Hence all the conditions of Lemma 1 are satisfied. So there is $x_0 \in K$ such that $x_0 \in \operatorname{cl} A(x_0)$, $A(x_0) \cap Q(x_0) = \emptyset$, that is,

$$F(x_0, x) \nsubseteq -C \setminus \{0\} \quad \text{for all} \quad x \in A(x_0).$$

This completes the proof. □

As an immediate consequence, we get directly the following multi-valued extension of Kazmi and Khan [7, Theorem 2.1].

Corollary 3.3. *Let K be a nonempty compact convex subset of X. Let $F : K \times K \rightrightarrows Y$ be C-convex in the second variable. Assume that for each $y \in K$, the set $\{x \in K \mid F(x,y) \subseteq -C \setminus \{0\}\}$ is open. Then (MGS) is solvable.*

Proof. Putting $A(x) \equiv K$ in Theorem 1 yields the desired result. □

As another application of Theorem 1, we derive an existence result of the generalized vector quasi-variational inequality as follows.

Corollary 3.4. *Let A be the same as in Theorem 1. Let $T : K \rightrightarrows L(X,Y)$ be a multifunction such that for each $y \in K$, the set $\{x \in K \mid \langle T(x), y-x\rangle \subseteq -C \setminus \{0\}\}$ is open where $L(X,Y)$ denotes the space of all continuous linear operators from X to Y. Then there exists $x_0 \in cl\, A(x_0)$ such that*

$$\langle T(x_0), y - x_0\rangle \nsubseteq -C \setminus \{0\} \quad \textit{for all} \quad x \in A(x_0).$$

Proof. Define the bi-operator $F : K \times K \rightrightarrows Y$ by $F(x, y) = \langle T(x), y - x\rangle$. To apply Theorem 1, it suffices to check that F is C-convex in the second variable. In fact, fix $x \in K$. For $\forall y_1, y_2 \in K,\ \forall\lambda \in [0, 1]$, we have

$$\begin{aligned} F(x, \lambda y_1 + (1-\lambda)y_2) &= \langle T(x),\, \lambda(y_1 - x) + (1-\lambda)(y_2 - x)\rangle \\ &\subseteq \lambda\langle T(x),\, y_1 - x\rangle + (1-\lambda)\langle T(x),\, y_2 - x\rangle - \{0\} \\ &\subseteq \lambda\langle T(x),\, y_1 - x\rangle + (1-\lambda)\langle T(x),\, y_2 - x\rangle - C \\ &= \lambda F(x, y_1) + (1-\lambda)F(x, y_2) - C. \end{aligned}$$

By Theorem 1, there exists $x_0 \in K$ such that $x_0 \in \mathrm{cl}\, A(x_0)$,

$$F(x_0, x) \nsubseteq -C \setminus \{0\} \quad \text{for all} \quad x \in A(x_0),$$

which implies that

$$\langle T(x_0), y - x_0\rangle \nsubseteq -C \setminus \{0\} \quad \text{for all} \quad x \in A(x_0).$$

□

REMARK 3.5. Corollary 2 is a multi-valued vector quasi-variational inequality extension of the strong vector variational inequality (SVVI) studied in a real Banach space by Fang and Huang [6, Theorem 2.1] in the context of a Hausdorff topological vector space.

References

[1] Q. H. Ansari, L. J. Lin and L. B. Su, Systems of simultaneous generalized vector quasiequilibrium problems and their applications, *J. Optim. Theory Appl.* **127** (2005), 27–44.

[2] E. Blume and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student* **63** (1994), 17–145.

[3] Y. Chiang, O. Chadli and J. C. Yao, Generalized vector equilibrium problems with trifunctions, *J. Global Optim.* **30** (2004), 135–154.

[4] Y. Chiang and J. C. Yao, Vector variational inequalities and the $(S)_+$ condition, *J. Optim. Theory Appl.* **17** (2004), 271–290.

[5] X.-P. Ding, W. K. Kim, and K.-K. Tan, Equilibria of non-compact generalized games with L^*-majorized preferences, *J. Math. Anal. Appl.* **164** (1992), 508–517.

[6] Y.-P. Fang and N.-J. Huang, Strong vector variational inequalities in Banach spaces, *Appl. Math. Lett.* **19** (2006), 362–368.

[7] K. R. Kazmi and S. A. Khan, Existence of solutions to a generalized system, *J. Optim. Theory Appl.* **142** (2009), 355–361.

[8] K. Kimura and J. C. Yao, Sensitivity Analysis of Vector Equilibrium Problems, *Taiwanese J. Math.* **12** (2008), 649–669.

*
Department of Mathematics Education
Chungbuk National University
Cheongju 361-763, Republic of Korea
E-mail: shkum@cbnu.ac.kr