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GENERALIZED ULAM-HYERS STABILITY OF $C^\star$-TERNARY ALGEBRA 3-HOMOMORPHISMS FOR A FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate the Ulam-Hyers stability of $C^\star$-ternary algebra 3-homomorphisms for the functional equation

$$f(x_1+x_2, y_1+y_2, z_1+z_2) = \sum_{1\le i,j,k\le 2} f(x_i, y_j, z_k)$$

in $C^\star$-ternary algebras.

1. Introduction and preliminaries

Ternary algebraic operations were considered in the 19th century by several mathematicians, such as Cayley [8] who introduced the notion of cubic matrix, which, in turn, was generalized by Kapranov [13] et al. The simplest example of such nontrivial ternary operation is given by the following composition rule:

$$\{a,b,c\}_{ijk} = \sum_{1\le l,m,n\le N} a_{nil} b_{ljm} c_{mkn} \quad (i,j,k = 1,2,\cdots,N).$$

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their applications in physics. Some significant physical applications are as follows (see [14, 15]):

(1) The algebra of *'nonions'* generated by two matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \& \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix} \quad \left(\omega = e^{\frac{2\pi i}{3}}\right)$$

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was introduced by Sylvester as a ternary analog of Hamilton's quaternions [1].

(2) The quark model inspired a particular brand of ternary algebraic systems. The so-called *'Nambu mechnics'* is based on such structures [9].

There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics, supersymmetric theory, and Yang–Baxter equation [1, 15, 24].

A $C^\star$-ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \le \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x, x, x]\| = \|x\|^3$ (see [2, 25]). Every left Hilbert $C^\star$-module is a $C^\star$-ternary algebra via the ternary product $[x, y, z] := \langle x, y\rangle z$.

If a $C^\star$-ternary algebra $(A, [\cdot, \cdot, \cdot])$ has an identity, i.e., an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [x, e, y]$ and $x^* := [e, x, e]$, is a unital $C^\star$-algebra. Conversely, if $(A, \circ)$ is a unital $C^\star$-algebra, then $[x, y, z] := x \circ y^* \circ z$ makes A into a $C^\star$-ternary algebra.

Let A and B are $C^\star$-ternary algebras. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *$C^\star$-ternary algebra homomorphism* if

$$H([x, y, z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in A$.

Definition 1.1. Let A and B are $C^\star$-ternary algebras. A 3-linear mapping $H : A \times A \times A \to B$ over $\mathbb{C}$ is called a *$C^\star$-ternary algebra 3-homomorphism* if it satisfies

$$\begin{aligned} &H([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) \\ &\qquad = [H(x_1, x_2, x_3), H(y_1, y_2, y_3), H(z_1, z_2, z_3)] \end{aligned}$$

for all $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3 \in A$.

In 1940, S. M. Ulam [23] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \varepsilon$ for all $x \in G$?

In 1941, Hyers [12] gave the first partial solution to Ulam's question for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Then, Aoki [3] and Bourgin [7] considered the stability problem with unbounded Cauchy differences. In 1978, Rassias [21] generalized the theorem of Hyers [12] by considering the stability problem with unbounded Cauchy differences. In 1991, Gajda [10] following the same approach as by Rassias [21] gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [10] as well as by Rassias and Šemrl [22], that one cannot prove a Rassias-type theorem when $p = 1$. Găvruta [11] obtained the generalized result of Rassias's theorem which allows the Cauchy difference to be controlled by a general unbounded function. During the past two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings, for example, Cauchy-Jensen mappings, k-additive mappings, invariant means, multiplicative mappings, bounded nth differences, convex functions, generalized orthogonality mappings, Euler-Lagrange functional equations, generalized Jensen's functional equations [4], n-dimensional quadratic functional equations [5], bi-quadratic functional equations [18], differential equations, and Navier-Stokes equations. On the other hand, Park [17] and the authors [6] have contributed works to the stability problem of ternary homomorphisms and ternary derivations.

Let X and Y be real or complex vector spaces. For a mapping $f : X \times X \times X \to Y$, consider the functional equation:

$$f(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \sum_{1 \le i,j,k \le 2} f(x_i, y_j, z_k). \tag{1.1}$$

In 2006, the authors [19] showed that a mapping $f : X \times X \times X \to Y$ satisfies the equation (1.1) if and only if the mapping f is 3-additive. We investigate the generalized Ulam stability in $C^\star$-ternary algebras for the 3-additive mappings satisfying (1.1).

2. Ulam-Hyers stability of $C^\star$-ternary algebra 3-homomorphisms

LEMMA 2.1. *Let X and Y be complex vector spaces and let $f : X \times X \times X \to Y$ be a 3-additive mapping such that $f(\lambda x, \mu y, \nu z) = \lambda\mu\nu f(x, y, z)$ for all $\lambda, \mu, \nu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, z \in X$, then f is 3-linear over $\mathbb{C}$.*

Proof. Since f is 3-additive, we get $f(\frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}z) = \frac{1}{8}f(x,y,z)$ for all $x,y,z \in X$. Now let $\rho, \sigma, \tau \in \mathbb{C}$ and M an integer greater than $2(|\rho|+|\sigma|+|\tau|)$. Since $|\frac{\rho}{M}| < \frac{1}{2}, |\frac{\sigma}{M}| < \frac{1}{2}$ and $|\frac{\tau}{M}| < \frac{1}{2}$, there is $r,s,t \in (\frac{\pi}{3}, \frac{\pi}{2}]$ such that $|\frac{\rho}{M}| = \cos r = \frac{e^{ir}+e^{-ir}}{2}$, $|\frac{\sigma}{M}| = \cos s = \frac{e^{is}+e^{-is}}{2}$ and $|\frac{\tau}{M}| = \cos t = \frac{e^{it}+e^{-it}}{2}$. Now $\frac{\rho}{M} = |\frac{\rho}{M}|\lambda$, $\frac{\sigma}{M} = |\frac{\sigma}{M}|\mu$ and $\frac{\tau}{M} = |\frac{\tau}{M}|\nu$ for some $\lambda, \mu, \nu \in \mathbb{T}^1$. Thus we have

$$\begin{aligned}
& f(\rho x, \sigma y, \tau z) = f\Big(M\frac{\rho}{M}x, M\frac{\sigma}{M}y, M\frac{\tau}{M}z\Big) \\
&= M^3 f\Big(\frac{\rho}{M}x, \frac{\sigma}{M}y, \frac{\tau}{M}z\Big) = M^3 f\Big(\Big|\frac{\rho}{M}\Big|\lambda x, \Big|\frac{\sigma}{M}\Big|\mu y, \Big|\frac{\tau}{M}\Big|\nu z\Big) \\
&= M^3 f\bigg(\frac{e^{ir}+e^{-ir}}{2}\lambda x, \frac{e^{is}+e^{-is}}{2}\mu y, \frac{e^{it}+e^{-it}}{2}\nu z\bigg) \\
&= \frac{1}{8}M^3 f(e^{ir}\lambda x + e^{-ir}\lambda x, e^{is}\mu y + e^{-is}\mu y, e^{it}\nu z + e^{-it}\nu z) \\
&= \frac{1}{8}M^3\,[\,e^{ir}e^{is}e^{it}\lambda\mu\nu f(x,y,z) + e^{ir}e^{is}e^{-it}\lambda\mu\nu f(x,y,z) \\
&\quad + e^{ir}e^{-is}e^{it}\lambda\mu\nu f(x,y,z) + e^{ir}e^{-is}e^{-it}\lambda\mu\nu f(x,y,z) \\
&\quad + e^{-ir}e^{is}e^{it}\lambda\mu\nu f(x,y,z) + e^{-ir}e^{is}e^{-it}\lambda\mu\nu f(x,y,z) \\
&\quad + e^{-ir}e^{-is}e^{it}\lambda\mu\nu f(x,y,z) + e^{-ir}e^{-is}e^{-it}\lambda\mu\nu f(x,y,z)\,] \\
&= \rho\sigma\tau f(x,y,z)
\end{aligned}$$

for all $x,y,z \in X$. So the mapping $f : X \times X \times X \to Y$ is 3-linear over $\mathbb{C}$. □

Using the above lemma, one can obtain the following result.

LEMMA 2.2. *Let X and Y be complex vector spaces and let $f : X \times X \times X \to Y$ be a mapping such that*

$$f(\lambda x_1 + \lambda x_2, \mu y_1 + \mu y_2, \nu z_1 + \nu z_2) = \lambda\mu\nu \sum_{1\le i,j,k\le 2} f(x_i, y_j, z_k) \tag{2.1}$$

for all $\lambda, \mu, \nu \in \mathbb{T}^1$ and all $x_1, x_2, y_1, y_2, z_1, z_2 \in X$. Then f is 3-linear over $\mathbb{C}$.

Proof. Letting $\lambda = \mu = \nu = 1$, by Theorem 3.1 in [19], f is 3-additive. Putting $x_2 = y_2 = z_2 = 0$ in (2.1), we get $f(\lambda x_1, \mu y_1, \nu z_1) = \lambda\mu\nu f(x_1, y_1, z_1)$ for all $\lambda, \mu, \nu \in \mathbb{T}^1$ and all $x_1, y_1, z_1 \in X$. So by Lemma 2.1, the mapping f is 3-linear over $\mathbb{C}$. □

From now on, assume that A is a $C^\star$-ternary algebra with norm $\|\cdot\|_A$ and that B is a $C^\star$-ternary algebra with norm $\|\cdot\|_B$.

For a given mapping $f : A \times A \times A \to B$, we define

$$D_{\lambda,\mu,\nu} f(x_1, x_2, y_1, y_2, z_1, z_2) := f(\lambda x_1 + \lambda x_2, \mu y_1 + \mu y_2, \nu z_1 + \nu z_2) - \lambda\mu\nu \sum_{1\le i,j,k\le 2} f(x_i, y_j, z_k)$$

for all $\lambda, \mu, \nu \in \mathbb{T}^1$ and all $x_1, x_2, y_1, y_2, z_1, z_2 \in A$.

We prove the generalized stability of homomorphisms in $C^\star$-ternary algebras for the functional equation $D_{\lambda,\mu,\nu} f(x_1, x_2, y_1, y_2, z_1, z_2) = 0$.

Theorem 2.3. *Let $p, q, r \in (0, \infty)$ with $p+q+r < 3$ and $\theta \in (0, \infty)$, and let $f : A \times A \times A \to B$ be a mapping such that*

$$\begin{aligned} &\|D_{\lambda,\mu,\nu} f(x_1, x_2, y_1, y_2, z_1, z_2)\|_B \\ &\le \theta \max\{\|x_1\|_A, \|x_2\|_A\}^p \cdot \max\{\|y_1\|_A, \|y_2\|_A\}^q \\ &\qquad \cdot \max\{\|z_1\|_A, \|z_2\|_A\}^r, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\|f([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) \\ &\qquad - [f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)]\|_B \\ &\le \theta \sum_{i=1}^{3} \|x_i\|_A^p \cdot \|y_i\|_A^q \cdot \|z_i\|_A^r \end{aligned} \tag{2.3}$$

for all $\lambda, \mu, \nu \in \mathbb{T}^1$ and all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique $C^\star$-ternary algebra 3-homomorphism $H : A \times A \times A \to B$ such that

$$\|f(x, y, z) - H(x, y, z)\|_B \le \frac{\theta}{8 - 2^{p+q+r}} \|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r \tag{2.4}$$

for all $x, y, z \in A$.

Proof. Letting $\lambda = \mu = \nu = 1$, $x_1 = x_2 = x$, $y_1 = y_2 = y$ and $z_1 = z_2 = z$ in (2.2), we gain

$$\|f(2x, 2y, 2z) - 8f(x, y, z)\|_B \le \theta \|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r \tag{2.5}$$

for all $x, y, z \in A$. Thus we have

$$\begin{aligned} &\left\| \frac{1}{8^{j+1}} f(2^{j+1}x, 2^{j+1}y, 2^{j+1}z) - \frac{1}{8^j} f(2^j x, 2^j y, 2^j z) \right\|_B \\ &\qquad \le 2^{(p+q+r-3)j-3} \theta \|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r \end{aligned} \tag{2.6}$$

for all $x, y, z \in A$ and all $j \in \mathbb{N}$. For given integer l, m $(0 \le l < m)$, we obtain that

$$
\begin{aligned}
&\left\| \frac{1}{8^m} f(2^m x, 2^m y, 2^m z) - \frac{1}{8^l} f(2^l x, 2^l y, 2^l z) \right\|_B \\
&\le \sum_{j=l}^{m-1} 2^{(p+q+r-3)j-3} \theta \|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r
\end{aligned} \tag{2.7}
$$

for all $x, y, z \in A$. Since $p+q+r < 3$, the sequence $\left\{ \frac{1}{8^j} f(2^j x, 2^j y, 2^j z) \right\}$ is a Cauchy sequence for all $x, y, z \in A$. Since B is complete, the sequence $\left\{ \frac{1}{8^j} f(2^j x, 2^j y, 2^j z) \right\}$ converges for all $x, y, z \in A$. Define $H : A \times A \times A \to B$ by

$$
H(x, y, z) := \lim_{j \to \infty} \frac{1}{8^j} f(2^j x, 2^j y, 2^j z)
$$

for all $x, y, z \in A$. Putting $l = 0$ and taking $m \to \infty$ in (2.7), one can obtain the inequality (2.4). By (2.2), we see that

$$
\begin{aligned}
&\Bigg\| \frac{1}{8^s} f\big(2^s(x_1 + x_2), 2^s(y_1 + y_2), 2^s(z_1 + z_2)\big) \\
&\quad - \sum_{1 \le i,j,k \le 2} \frac{1}{8^s} f(2^s x_i, 2^s y_j, 2^s z_k) \Bigg\|_B \\
&\le 2^{(p+q+r-3)s} \, \theta \, \max\{\|x_1\|_A, \|x_2\|_A\}^p \cdot \max\{\|y_1\|_A, \|y_2\|_A\}^q \\
&\quad \cdot \max\{\|z_1\|_A, \|z_2\|_A\}^r
\end{aligned}
$$

for all $x_1, x_2, y_1, y_2, z_1, z_2 \in A$ and all s. Since $p + q + r < 3$, letting $s \to \infty$ in the above inequality, H satisfies (1.1). By Theorem 3.1 in [19], H is 3-additive.

Letting $x_1 = x_2 = x$, $y_1 = y_2 = y$ and $z_1 = z_2 = z$ in (2.2), we gain

$$
\|f(2\lambda x, 2\mu y, 2\nu z) - 8\lambda\mu\nu f(x, y, z)\|_B \le \theta \|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r
$$

for all $\lambda, \mu, \nu \in \mathbb{T}^1$ and all $x, y, z \in A$. Thus we have

$$
\begin{aligned}
&\|f(2^n \lambda x, 2^n \mu y, 2^n \nu z) - 8\lambda\mu\nu f(2^{n-1} x, 2^{n-1} y, 2^{n-1} z)\|_B \\
&\le 2^{(p+q+r)(n-1)} \theta \|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r
\end{aligned}
$$

for all $\lambda, \mu, \nu \in \mathbb{T}^1$, all $x, y, z \in A$ and all $n \in \mathbb{N}$. So we get

$$
\begin{aligned}
&\|f(2^n x, 2^n y, 2^n z) - 8 f(2^{n-1} x, 2^{n-1} y, 2^{n-1} z)\|_B \\
&\le 2^{(p+q+r)(n-1)} \theta \|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r
\end{aligned}
$$

for all $x, y, z \in A$ and all $n \in \mathbb{N}$. And one can show that

$$\begin{aligned}
&\|\lambda\mu\nu f(2^n x, 2^n y, 2^n z) - 8\lambda\mu\nu f(2^{n-1}x, 2^{n-1}y, 2^{n-1}z)\|_B \\
&\quad = |\lambda\mu\nu| \cdot \left\|f(2^n x, 2^n y, 2^n z) - 8f(2^{n-1}x, 2^{n-1}y, 2^{n-1}z)\right\|_B \\
&\quad \le 2^{(p+q+r)(n-1)}\theta\|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r
\end{aligned}$$

for all $\lambda, \mu, \nu \in \mathbb{T}^1$, all $x, y, z \in A$ and all $n \in \mathbb{N}$. So

$$\begin{aligned}
&\|f(2^n\lambda x, 2^n\mu y, 2^n\nu z) - \lambda\mu\nu f(2^n x, 2^n y, 2^n z)\|_B \\
&\quad \le \|f(2^n\lambda x, 2^n\mu y, 2^n\nu z) - 8\lambda\mu\nu f(2^{n-1}x, 2^{n-1}y, 2^{n-1}z)\|_B \\
&\qquad + \|8\lambda\mu\nu f(2^{n-1}x, 2^{n-1}y, 2^{n-1}z) - \lambda\mu\nu f(2^n x, 2^n y, 2^n z)\|_B \\
&\quad \le 2^{(p+q+r)(n-1)+1}\theta\|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r
\end{aligned}$$

for all $\lambda, \mu, \nu \in \mathbb{T}^1$, all $x, y, z \in A$ and all $n \in \mathbb{N}$. Since $p + q + r < 3$, we have

$$\frac{1}{8^n}\|f(2^n\lambda x, 2^n\mu y, 2^n\nu z) - \lambda\mu\nu f(2^n x, 2^n y, 2^n z)\|_B \to 0$$

as $n \to \infty$ for all $\lambda, \mu, \nu \in \mathbb{T}^1$ and all $x, y, z \in A$. Hence

$$\begin{aligned}
H(\lambda x, \mu y, \nu z) &= \lim_{n\to\infty} \frac{f(2^n\lambda x, 2^n\mu y, 2^n\nu z)}{8^n} \\
&= \lim_{n\to\infty} \lambda\mu\nu \frac{f(2^n x, 2^n y, 2^n z)}{8^n} = \lambda\mu\nu H(x, y, z)
\end{aligned}$$

for all $\lambda, \mu, \nu \in \mathbb{T}^1$ and all $x, y, z \in A$. By Lemma 2.1, the mapping $H : A \times A \times A \to B$ is 3-linear over $\mathbb{C}$.

It follows from (2.3) that

$$\begin{aligned}
&\|H([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) \\
&\qquad - [H(x_1, x_2, x_3), H(y_1, y_2, y_3), H(z_1, z_2, z_3)]\|_B \\
&= \lim_{n\to\infty} \frac{1}{8^n}\|f([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) \\
&\qquad - [f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)]\|_B \\
&\le \lim_{n\to\infty} \frac{\theta}{8^n} \sum_{i=1}^{3} \|x_i\|_A^p \cdot \|y_i\|_A^q \cdot \|z_i\|_A^r = 0
\end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. So

$$\begin{aligned}
H([x_1, y_1, z_1],&[x_2, y_2, z_2], [x_3, y_3, z_3]) \\
&= [H(x_1, x_2, x_3), H(y_1, y_2, y_3), H(z_1, z_2, z_3)]
\end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$.

Now, let $T : A\times A\times A \to B$ be another 3-additive mapping satisfying (2.4). Then we have

$$\begin{aligned}
&\|H(x,y,z) - T(x,y,z)\|_B \\
&\qquad = \frac{1}{8^n}\|H(2^n x, 2^n y, 2^n z) - T(2^n x, 2^n y, 2^n z)\|_B \\
&\qquad \le \frac{1}{8^n}\|H(2^n x, 2^n y, 2^n z) - f(2^n x, 2^n y, 2^n z)\|_B \\
&\qquad\quad + \frac{1}{8^n}\|f(2^n x, 2^n y, 2^n z) - T(2^n x, 2^n y, 2^n z)\|_B \\
&\qquad \le \frac{2^{(p+q+r-3)n+1}\theta}{8 - 2^{p+q+r}}\|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x, y, z \in A$. So we can conclude that $H(x,y,z) = T(x,y,z)$ for all $x, y, z \in A$. This proves the uniqueness of H.

Thus the mapping $H : A \to B$ is a unique $C^\star$-ternary algebra 3-homomorphism satisfying (2.4). □

Putting $p = q = r = 0$ and $\theta = \varepsilon$ in Theorem 2.3, we obtain the Ulam stability for the 3-additive functional equation (1.1).

Corollary 2.4. *Let $\varepsilon \in (0,\infty)$ and let $f : A\times A\times A \to B$ be a mapping satisfying*

$$\|D_{\lambda,\mu,\nu} f(x,y,z,v,w)\|_B \le \varepsilon$$

and

$$\begin{aligned}
&\|f([x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]) \\
&\qquad - [f(x_1,x_2,x_3), f(y_1,y_2,y_3), f(z_1,z_2,z_3)]\|_B \le 3\varepsilon
\end{aligned}$$

for all $\lambda, \mu, \nu \in \mathbb{T}^1$ and all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique $C^\star$-ternary algebra 3-homomorphism $H : A\times A\times A \to B$ such that

$$\|f(x,y,z) - H(x,y,z)\|_B \le \frac{\varepsilon}{7}$$

for all $x, y, z \in A$.

THEOREM 2.5. *Let $p \in (0,3)$ and $\theta \in (0,\infty)$, and let $f : A\times A\times A \to B$ be a mapping such that*

$$\begin{aligned}
&\|D_{\lambda,\mu,\nu} f(x_1,x_2,y_1,y_2,z_1,z_2)\|_B \\
&\qquad \le \theta \sum_{i=1}^{2} (\|x_i\|_A^p + \|y_i\|_A^p + \|z_i\|_A^p), \qquad (2.8)\\
&\|f([x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]) \\
&\qquad\qquad - [f(x_1,x_2,x_3), f(y_1,y_2,y_3), f(z_1,z_2,z_3)]\|_B \\
&\qquad \le \theta \sum_{i=1}^{3} (\|x_i\|_A^p + \|y_i\|_A^p + \|z_i\|_A^p) \qquad (2.9)
\end{aligned}$$

for all $\lambda,\mu,\nu \in \mathbb{T}^1$ and all $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3 \in A$. Then there exists a unique $C^\star$-ternary algebra 3-homomorphism $H : A\times A\times A \to B$ such that

$$\|f(x,y,z) - H(x,y,z)\|_B \le \frac{2\theta}{8-2^p}(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)$$

for all $x,y,z \in A$.

Proof. The proof is similar to the proof of Theorem 2.3. □

THEOREM 2.6. *Let $p,q,r \in (0,\infty)$ with $p+q+r<3$, $s \in (0,3)$ and $\theta,\eta \in (0,\infty)$, and let $f : A\times A\times A \to B$ be a mapping such that*

$$\begin{aligned}
&\|D_{\lambda,\mu,\nu} f(x_1,x_2,y_1,y_2,z_1,z_2)\|_B \\
&\qquad \le \theta \max\{\|x_1\|_A, \|x_2\|_A\}^p \cdot \max\{\|y_1\|_A, \|y_2\|_A\}^q \\
&\qquad\qquad \cdot \max\{\|z_1\|_A, \|z_2\|_A\}^r \\
&\qquad\quad + \eta \sum_{i=1}^{2} (\|x_i\|_A^s + \|y_i\|_A^s + \|z_i\|_A^s), \qquad (2.10)\\
&\|f([x_1,y_1,z_1],[x_2,y_2,z_2],[x_3,y_3,z_3]) \\
&\qquad - [f(x_1,x_2,x_3), f(y_1,y_2,y_3), f(z_1,z_2,z_3)]\|_B \\
&\qquad \le \theta \sum_{i=1}^{3} \|x_i\|_A^p \cdot \|y_i\|_A^q \cdot \|z_i\|_A^r \\
&\qquad\quad + \eta \sum_{i=1}^{3} (\|x_i\|_A^s + \|y_i\|_A^s + \|z_i\|_A^s) \qquad (2.11)
\end{aligned}$$

for all $\lambda,\mu,\nu \in \mathbb{T}^1$ and all $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3 \in A$. Then there exists a unique $C^\star$-ternary algebra 3-homomorphism $H : A\times A\times A \to B$

such that

$$\|f(x,y,z)-H(x,y,z)\|_B \le \frac{\theta}{8-2^{p+q+r}}\|x\|_A^p\cdot\|y\|_A^q\cdot\|z\|_A^r + \frac{2\eta}{8-2^s}(\|x\|_A^s+\|y\|_A^s+\|z\|_A^s) \tag{2.12}$$

for all $x,y,z\in A$.

Proof. The proof is similar to the proof of Theorem 2.3. □

THEOREM 2.7. *Let $p,q,r\in(0,\infty)$ with $p+q+r>3$ and $\theta\in(0,\infty)$, and let $f:A\times A\times A\to B$ be a mapping satisfying* (2.2), (2.3) *and $f(0,0,0)=0$. Then there exists a unique $C^\star$-ternary algebra 3-homomorphism $H:A\times A\times A\to B$ such that*

$$\|f(x,y,z)-H(x,y,z)\|_B \le \frac{\theta}{2^{p+q+r}-8}\|x\|_A^p\cdot\|y\|_A^q\cdot\|z\|_A^r \tag{2.13}$$

for all $x,y,z\in A$.

Proof. It follows from (2.5) that

$$\left\|f(x,y,z)-8f\Big(\frac{x}{2},\frac{y}{2},\frac{z}{2}\Big)\right\|_B \le \frac{\theta}{2^{p+q+r}}\|x\|_A^p\cdot\|y\|_A^q\cdot\|z\|_A^r$$

for all $x,y,z\in A$. So

$$\begin{aligned}&\left\|8^l f\Big(\frac{x}{2^l},\frac{y}{2^l},\frac{z}{2^l}\Big)-8^m f\Big(\frac{x}{2^m},\frac{y}{2^m},\frac{z}{2^m}\Big)\right\|_B\\ &\le \sum_{j=l}^{m-1}\left\|8^j f\Big(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^j}\Big)-8^{j+1} f\Big(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},\frac{z}{2^{j+1}}\Big)\right\|_B\\ &\le \frac{\theta}{2^{p+q+r}}\sum_{j=l}^{m-1}\frac{8^j}{2^{(p+q+r)j}}\|x\|_A^p\cdot\|y\|_A^q\cdot\|z\|_A^r \end{aligned}\tag{2.14}$$

for all nonnegative integers m and l with $m>l$ and all $x,y,z\in A$. It follows from (2.14) that the sequence $\{8^n f(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n})\}$ is a Cauchy sequence for all $x,y,z\in A$. Since B is complete, the sequence

$$\Big\{8^n f\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big)\Big\}$$

converges for all $x,y,z\in A$. So one can define the mapping $H:A\times A\times A\to B$ by

$$H(x,y,z):=\lim_{n\to\infty}8^n f\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big)$$

for all $x, y, z \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.14), we get (2.13).

The rest of the proof is similar to the proof of Theorem 2.3. $\square$

EXAMPLE 2.8. *We present the following counterexample modified by the well-known counterexample of Z. Gajda* [10] *for the functional equation (1.1). Fix* $\theta > 0$ *and put* $\mu := \frac{\theta}{144}$.

Define a function $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(x, y, z) := \sum_{n=0}^{\infty} \frac{1}{2^n} \phi_\mu(2^n x, y, z)$$

for all $x, y, z \in \mathbb{R}$, where $\phi_\mu : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the function given by

$$\phi_\mu(x, y, z) := \begin{cases} \mu & \text{if} \quad xyz \geq 1 \\ \mu xyz & \text{if} \quad -1 < xyz < 1 \\ -\mu & \text{if} \quad xyz \leq -1 \end{cases}$$

for all $x, y, z \in \mathbb{R}$. Define another function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) := f(x, 1, 1) = \sum_{n=0}^{\infty} \frac{1}{2^n} \phi_\mu(2^n x, 1, 1)$$

for all $x \in \mathbb{R}$.

It was proved in [10] that

$$|g(x + y) - g(x) - g(y)| \leq \frac{\theta}{24}(|x| + |y|)$$

for all $x, y \in \mathbb{R}$. By the above inequality, we can obtain that

$$\begin{aligned} &|g(x + y + z + w) - g(x) - g(y) - g(z) - g(w)| \\ &\quad \leq |g(x + y + z + w) - g(x + y) - g(z + w)| \\ &\quad\qquad + |g(x + y) + g(z + w) - g(x) - g(y) - g(z) - g(w)| \\ &\quad \leq \frac{\theta}{24}(|x + y| + |z + w|) + \frac{\theta}{24}(|x| + |y|) + \frac{\theta}{24}(|z| + |w|) \\ &\quad \leq \frac{\theta}{12}(|x| + |y| + |z| + |w|) \end{aligned}$$

and

$$
\begin{aligned}
&\left|g\left(\sum_{i=1}^{8} x_i\right) - \sum_{i=1}^{8} g(x_i)\right| \\
&\quad \le \left|g\left(\sum_{i=1}^{8} x_i\right) - g\left(\sum_{i=1}^{4} x_i\right) - g\left(\sum_{i=5}^{8} x_i\right)\right| \\
&\qquad + \left|g\left(\sum_{i=1}^{4} x_i\right) + g\left(\sum_{i=5}^{8} x_i\right) - \sum_{i=1}^{8} g(x_i)\right| \\
&\quad \le \frac{\theta}{24}\left(\left|\sum_{i=1}^{4} x_i\right| + \left|\sum_{i=5}^{8} x_i\right|\right) + \frac{\theta}{12}\left(\sum_{i=1}^{4} |x_i| + \sum_{i=5}^{8} |x_i|\right) \\
&\quad \le \frac{1}{8}\,\theta \sum_{i=1}^{8} |x_i| \qquad (2.15)
\end{aligned}
$$

for all $x, y, z, w, x_1, \cdots, x_8 \in \mathbb{R}$. Note that

$$
\begin{aligned}
f(x,y,z) &= \begin{cases} \mu & \text{if} \quad 2^n xyz \ge 1 \\ \mu 2^n xyz & \text{if} \quad -1 < 2^n xyz < 1 \\ -\mu & \text{if} \quad 2^n xyz \le -1 \end{cases} \\
&= f(xyz, 1, 1) = g(xyz) \qquad (2.16)
\end{aligned}
$$

for all $x, y, z \in \mathbb{R}$. By the inequality (2.15) and the above equality (2.16), we see that

$$
\begin{aligned}
&\left| f(x_1 + x_2, y_1 + y_2, z_1 + z_2) - \sum_{1 \le i,j,k \le 2} f(x_i, y_j, z_k) \right| \\
&\quad = \left| g\big((x_1 + x_2)(y_1 + y_2)(z_1 + z_2)\big) - \sum_{1 \le i,j,k \le 2} g(x_i y_j z_k) \right| \\
&\quad = \left| g\Bigg(\sum_{1 \le i,j,k \le 2} x_i y_j z_k \Bigg) - \sum_{1 \le i,j,k \le 2} g(x_i y_j z_k) \right| \\
&\quad \le \frac{1}{8}\,\theta \sum_{1 \le i,j,k \le 2} |x_i y_j z_k| \le \frac{1}{8}\,\theta(|x_1| + |x_2|)(|y_1| + |y_2|)(|z_1| + |z_2|) \\
&\quad = \theta \max\{|x_1|, |x_2|\} \cdot \max\{|y_1|, |y_2|\} \cdot \max\{|z_1|, |z_2|\}
\end{aligned}
$$

for all $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$. But we observe from [10] that

$$\frac{g(x^3)}{x^3} \to \infty \quad \text{as} \quad x \to \infty.$$

And so

$$\frac{f(x,x,x)}{x^3} \to \infty \quad \text{as} \quad x \to \infty.$$

Thus

$$\frac{|f(x,x,x) - h(x,x,x)|}{x^3} \ (x \neq 0) \text{ is unbounded,}$$

where $h : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the function given by

$$h(x,y,z) := \lim_{n\to\infty} \frac{1}{8^n} f(2^n x, 2^n y, 2^n z)$$

for all $x, y, z \in \mathbb{R}$. Hence the function f is a counterexample for the common singular case $p + q + r = 3$ of Theorems 2.3 and 2.7.

THEOREM 2.9. *Let $p \in (3, \infty)$ and $\theta \in (0, \infty)$, and let $f : A \times A \times A \to B$ be a mapping satisfying* (2.8), (2.9) *and* $f(0,0,0) = 0$. *Then there exists a unique $C^\star$-ternary algebra* 3*-homomorphism $H : A \times A \times A \to B$ such that*

$$\|f(x,y,z) - H(x,y,z)\|_B \leq \frac{2\theta}{2^p - 8}(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p)$$

for all $x, y, z \in A$.

Proof. The proof is similar to the proof of Theorem 2.7. □

EXAMPLE 2.10. *We present the following counterexample modified by the well-known counterexample of Z. Gajda* [10] *for the functional equation (1.1). Fix $\theta > 0$ and put $\mu := \frac{\theta}{24}$.*

Let $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the same as in Example A. By the same argument as in Example 2.8, one can obtain that g satisfies the inequality

$$\left| g\left(\sum_{i=1}^{8} x_i\right) - \sum_{i=1}^{8} g(x_i) \right| \leq \frac{3}{4}\theta \sum_{i=1}^{8} |x_i|$$

for all $x_1, \cdots, x_8 \in \mathbb{R}$. By the equality (2.16) and the above inequality, we see that

$$\begin{aligned}
&\left| f(x_1+x_2, y_1+y_2, z_1+z_2) - \sum_{1\le i,j,k\le 2} f(x_i, y_j, z_k) \right| \\
&\quad = \left| g\left(\sum_{1\le i,j,k\le 2} x_i y_j z_k \right) - \sum_{1\le i,j,k\le 2} g(x_i y_j z_k) \right| \\
&\quad \le \frac{3}{4}\theta \sum_{1\le i,j,k\le 2} |x_i y_j z_k| \le \frac{\theta}{4} \sum_{1\le i,j,k\le 2} \left(|x_i|^3 + |y_i|^3 + |z_i|^3 \right) \\
&\quad = \theta \sum_{i=1}^{2} \left(|x_i|^3 + |y_i|^3 + |z_i|^3 \right)
\end{aligned}$$

for all $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$. By the same reason as Example 2.8, the function f is a counterexample for the common singular case $p = 3$ of Theorems 2.5 and 2.9.

THEOREM 2.11. *Let $p, q, r \in (0, \infty)$ with $p + q + r > 3$, $s \in (3, \infty)$ and $\theta, \eta \in (0, \infty)$, and let $f : A \times A \times A \to B$ be a mapping such that (2.10), (2.11) and $f(0, 0, 0) = 0$. Then there exists a unique $C^\star$-ternary algebra 3-homomorphism $H : A \times A \times A \to B$ such that*

$$\begin{aligned}
&\|f(x,y,z) - H(x,y,z)\|_B \\
&\quad \le \frac{\theta}{2^{p+q+r} - 8} \|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r + \frac{2\eta}{2^s - 8} (\|x\|_A^s + \|y\|_A^s + \|z\|_A^s)
\end{aligned}$$

for all $x, y, z \in A$.

Proof. The proof is similar to the proof of Theorem 2.7. □

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