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A NOTE ON THE PAPER TITLED SOME VARIANTS OF OSTROWSKI'S METHOD WITH SEVENTH-ORDER CONVERGENCE

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Abstract. Kou et al. presented a class of new variants of Ostrowski's method in their paper (J. of Comput. Appl. Math., 209(2007), pp.153-159) whose title is "Some variants of Ostrowski's method with seventh-order convergence". They proposed an incorrect error equation, although they showed a correct seventh-order of convergence. The main objective of this note is to establish the correct error equation of the method and confirm its validity via concrete numerical examples.

1. Introduction

Kou et al.[1] developed some variants of Ostrowski's method for finding a simple root of nonlinear equations. For these variants, although they showed a correct seventh-order of convergence, they obtained an incorrect error equation. The aim of this paper is not only to investigate the convergence of this method but also to derive the correct error equation. Some numerical experiments will play an important role to support our analysis regarding their faulty error equation.

2. Convergence analysis

Let $x_0, x_1, x_2, \cdots, x_n, \cdots$ be a sequence converging to α and $e_n = x_n - \alpha$ be the nth iterate error. If there exist real numbers $p \in \mathbb{R}$ and $c \in \mathbb{R} - \{0\}$ such that the following error equation holds

$$e_{n+1} = c\, {e_n}^p + O({e_n}^{p+1}),$$

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then the absolute value of c is called the asymptotic error constant[2] and p is called the order of convergence. Let $f : \mathbb{R} \to \mathbb{R}$ have a simple zero α and be sufficiently smooth in small neighborhood of α. Kou et al. proposed in their paper[1] the following scheme

$$x_{n+1} = z_n - \frac{[(1 + H_2(x_n, y_n))^2 + H_k(y_n, z_n)]f(z_n)}{f'(x_n)} \tag{2.1}$$

where

$$k \in \mathbb{R},\ y_n = x_n - \frac{f(x_n)}{f'(x_n)},\ H_2(x_n, y_n) = \frac{f(y_n)}{f(x_n) - 2f(y_n)},$$

$$z_n = x_n - [1 + H_2(x_n, y_n)]\frac{f(x_n)}{f'(x_n)},\ \text{and } H_k(y_n, z_n) = \frac{f(z_n)}{f(y_n) - kf(z_n)}.$$

They showed the error equation of (1) below

$$e_{n+1} = -2(c_3^2 - 2c_2^2c_3 + c_2c_4)({c_2}^2 - c_3)e_n^7 + O(e_n^8),$$

which is apparently incorrect. This incorrectness motivates our analysis to develop the following theorem which gives the seventh-order of convergence and yields the correct error equation of (2.1).

THEOREM 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ have a simple zero α and be sufficiently smooth in a small neighborhood of α. Then the methods developed by (2.1) are of order seven for any $k \in \mathbb{R}$ and induce the following error equation:*

$$e_{n+1} = 4c_2^2(c_2^2 - c_3)^2 e_n^7 + O(e_n^8), \tag{2.2}$$

where $e_n = x_n - \alpha$ and $c_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}, j = 2, 3, \cdots$.

Proof. Using Taylor's series expansion, we have

$$\begin{aligned} f(x_n) = \quad & f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 \\ & +c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + O(e_n^8)]. \end{aligned} \tag{2.3}$$

$$\begin{aligned} f'(x_n) = \quad & f'(\alpha)[1 + 2c_2e_n^1 + 3c_3e_n^2 \\ & +4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + O(e_n^7)]. \end{aligned} \tag{2.4}$$

Dividing (2.3) by (2.4), we get

$$\begin{aligned}\frac{f(x_n)}{f'(x_n)} &= e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4 \\ &+(8c_2^4 - 20c_2^2c_3 + 6c3^2 + 10c_2c_4 - 4c_5)e_n^5 \\ &+(-16c_2^5 + 52c_2^3c_3 - 33c_2c_3^2 - 28c_2^2c_4 + 17c_3c_4 + 13c_2c_5 - 5c_6)e_n^6 \\ &+2[16c_2^6 - 64c_2^4c_3 - 9c_3^3 + 36c_2^3c_4 + 6c_4^2 + 9c_2^2(7c_3^2 - 2c_5) + 11c_3c_5 \\ &+c_2(-46c_3c_4 + 8c_6) - 3c_7)]e_n^7 + O(e_n^8).\end{aligned} \tag{2.5}$$

Due to the above relation (2.5) and $y_n - \alpha = e_n - f'(x_n)^{-1}f(x_n)$, we obtain

$$\begin{aligned}y_n - \alpha &= c_2e_n^2 - 2(c_2^2 - c3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ &-(8c_2^4 - 20c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 \\ &+(16c_2^5 - 52c_2^3c_3 + 33c_2c_3^2 + 28c_2^2c_4 - 17c_3c_4 - 13c_2c_5 + 5c_6)e_n^6 \\ &-2[16c_2^6 - 64c_2^4c_3 - 9c_3^3 + 36c_2^3c_4 + 6c_4^2 + 9c_2^2(7c_3^2 - 2c_5) + 11c_3c_5 \\ &+c_2(-46c_3c_4 + 8c_6) - 3c_7)]e_n^7 + O(e_n^8).\end{aligned} \tag{2.6}$$

Expanding $f(y_n)$ about α leads us to relation:

$$\begin{aligned}f(y_n) &= f'(\alpha)\{c_2e_n^2 - 2(c_2^2 - c3)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ &-2(6c_2^4 - 12c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)e_n^5 \\ &+(28c_2^5 - 73c_2^3c_3 + 37c_2c_3^2 + 34c_2^2c_4 - 17c_3c_4 - 13c_2c_5 + 5c_6)e_n^6 \\ &-2[32c_2^6 - 103c_2^4c_3 - 9c_3^3 + 52c_2^3c_4 + 6c_4^2 + c_2^2(80c_3^2 - 22c_5) \\ &+11c_3c_5 + c_2(-52c_3c_4 + 8c_6) - 3c_7]e_n^7 + O(e_n^8)\}.\end{aligned} \tag{2.7}$$

Using (2.3) and (2.7), we have

$$\begin{aligned}z_n &= x_n - [1 + H_2(x_n, y_n)]\frac{f(x_n)}{f'(x_n)} \\ &= \alpha + (c_2^3 - c_2c_3)e_n^4 - 2(2c_2^4 - 4c_2^2c_3 + c_3^2 + c_2c_4)e_n^5 \\ &+(10c_2^5 - 30c_2^3c_3 + 18c_2c_3^2 + 12c_2^2c_4 - 7c_3c_4 - 3c_2c_5)e_n^6 \\ &-2[10c_2^6 - 40c_2^4c_3 - 6c_3^3 + 20c_2^3c_4 + 3c_4^2 + 8c_2^2(5c_3^2 - c_5) \\ &+5c_3c_5 + c_2(-26c_3c_4 + 2c_6)]e_n^7 + O(e_n^8).\end{aligned} \tag{2.8}$$

Taylor expansion about $f(z_n)$ about α gives the following relation:

$$\begin{aligned} f(z_n) &= f'(\alpha)\{(c_2^3 - c_2c_3)e_n^4 - 2(2c_2^4 - 4c_2^2c_3 + c_3^2 + c_2c_4)e_n^5 \\ &+ (10c_2^5 - 30c_2^3c_3 + 18c_2c_3^2 + 12c_2^2c_4 - 7c_3c_4 - 3c_2c_5)e_n^6 \\ &- 2[10c_2^6 - 40c_2^4c_3 - 6c_3^3 + 20c_2^3c_4 + 3c_4^2 + 8c_2^2(5c_3^2 - c_5) \\ &+ 5c_3c_5 + c_2(-26c_3c_4 + 2c_6)]e_n^7 + O(e_n^8)\}. \end{aligned} \tag{2.9}$$

By means of (2.7) and (2.9), we find

$$\begin{aligned} H_k(y_n, z_n) &= \frac{f(z_n)}{f(y_n) - kf(z_n)} \\ &= (c_2^2 - c_3)e_n^2 - 2(c_2^3 - 2c_2c_3 + c_4)e_n^3 \\ &+[5c_2c_4 - 3c_5 + c_2^4(1+k) - 2c_2^2c_3(3+k) + c_3^2(3+k)]e_n^4 \\ &-2[2c_6 + c_2^3c_3(2-6k) + 2c_2^5(-1+k) + 2c_2^2c_4(1+k) \\ &-c_3c_4(3+2k) + c_2(-3c_5 + c_3^2(2+4k))]e_n^5 + O(e_n^6). \end{aligned} \tag{2.10}$$

Substituting (2.10) into (2.1) together with $e_{n+1} = x_{n+1} - \alpha$ and simplifying by the aid of symbolic computation of Mathematica[3], we finally arrive at the relation below:

$$e_{n+1} = 4c_2^2(c_2^2 - c_3)^2 e_n^7 + O(e_n^8).$$

This states that the method defined by (2.1) has seventh-order convergence and relation (2.2) represents the correct error equation as desired, completing the proof. □

CLAIM 2.2. Remark: It is interesting to observe that the error equation is independent of k, although it explictly involves $H_k(y_n, z_n)$.

3. Numerical examples and conclusion

Kou et al. made some mistakes in the course of derivation of their error equation. Specifically, the flaw initiated by the second term "$\frac{2}{c_2}(c_2^2 - c_3)^2 e_n^3$"of equation (20) on page 156 of their paper[1] has badly affected their error equation. The corresponding term should have been corrected as "$-2(c_2^3 - 2c_2c_3 + c_4)e_n^3$". To convince our analysis, two concrete numerical examples are presented and displayed in Tables 1 and 2. As Tables 1 and 2 suggest, we find that the order of convergence p is clearly seven and the asymptotic error constant η defined by, with $e_n = x_n - \alpha,\ n = 0, 1, 2, \cdots$,

$$\eta = \lim_{n\to\infty} \left|\frac{e_{n+1}}{{e_n}^p}\right|$$

is apparently reached.

In these experiments, we assign 350 as the minimum number of digits of precision to achieve the specified sufficient accuracy in Mathematica Version7. We set the error bound ϵ to 10^{-300} for $\mid x_n - \alpha \mid < \epsilon$. Although computed values of x_n are truncated to be accurate up to 300 significant digits and inexact value of α is accurate enough about up to 400 significant digits, the limited space allows us to list them up to 15 significant digits.

As a first example, we select a test function

$$f(x) = \cos(\pi x/2) + \log(x^2 + 2x + 2)/(1 + x^2)$$

having a simple zero $\alpha = -1$. We choose $x_0 = -0.8$ as an initial guess. The order of convergence and the asymptotic error constant are clearly shown in Table 1 revealing a good agreement with the theory in Section 2.

Table 1. Convergence behavior with

$$f(x) = \cos(\pi x/2) + \log(x^2 + 2x + 2)/(1 + x^2)$$

n	x_n	$\lvert f(x_n)\rvert$	$\mid x_n - \alpha \mid$	$\lvert e_{n+1}/e_n{}^7\rvert$	η
0	-0.800000000000000	0.332932	0.200000		0.0152918
1	-0.999999900952812	1.55583×10^{-7}	9.90472×10^{-8}	0.0077380615	
2	-1.00000000000000	2.24633×10^{-51}	1.43006×10^{-51}	0.0152918	
3	-1.00000000000000	0.0×10^{-349}	0.0×10^{-350}		

We take another test function $f(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$ with a root $\alpha = -1.20764782713092$. We select $x_0 = -1.25$ as an initial value. In this example, we also find that the order of convergence is seven and the computed asymptotic error constant $|e_{n+1}/e_n{}^7|$ well approaches the theoretical value η.

Table 2. Convergence behavior with

$$f(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$$

n	x_n	$\lvert f(x_n)\rvert$	$\mid x_n - \alpha \mid$	$\lvert e_{n+1}/e_n{}^7\rvert$	η
0	-1.25000000000000	0.918021	0.0423522		0.58290597
1	-1.20764782730259	3.48612^{-9}	1.71667^{-10}	0.70235609	
2	-1.20764782713092	5.20072×10^{-68}	2.56100×10^{-69}	0.58290597	
3	-1.20764782713092	$0. \times 10^{-349}$	$0. \times 10^{-349}$		

This paper has not only confirmed seventh-order convergence but also proved the correct error equation for iterative method (2.1) proposed by Kou et al.[1].

References

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