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h-STABILITY FOR LINEAR IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. We study the h-stability for linear impulsive differential equations and their perturbations by using the impulsive integral inequalities.

1. Introduction

Recently, stability theory of impulsive differential equations has been popularly applied in variety fields of science and technology: theoretical physics, mechanics, population dynamics, pharmacokinetics, impulse technique, industrial robotics, chemical technology, biotechnology, economics, ets.

Impulsive differential equations and application ware introduced by some authors: A. M. Samoilenko and N. A. Perestyuk [11], V. Lakashmikantham, D. D. Bainov and P. S. Simeonov[7], Bainov and Simeonov[1, 3].

Pinto [9] introduced the notion of h-stability with the intention of obtaining results about stability for a weakly stable system (at least, weaker than thouse given exponential asymptotic stability) under some perturbations.

In this paper we examine the h-stability for linear impulsive differential equations at fixed moments and their perturbations by using impulsive inequality of Gronwall type.

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2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space and $|\cdot|$ denotes the norm on $\mathbb{R}^n$. Let $\mathbb{N}_0 = \{n_0, n_0+1, \cdots\}$ (n_0 is a fixed nonnegative integer).

Let $\nu = \{t_k\}_{k=1}^{\infty} \subset [t_0, \infty)$ be an unbounded and increasing sequence. Denoted by $PC([t_0,\infty), \mathbb{R}^n \times \mathbb{R}^n)$ the set of functions $\varphi : [t_0,\infty) \to \mathbb{R}^n \times \mathbb{R}^n$ which are continuous for $t \in [t_0,\infty) \setminus \nu$, are continuous from the left for $t \in [t_0,\infty)$, and have discontinuities of the first type at the points t_k for each $k \in \mathbb{N}$.

We consider the linear impulsive system

$$\begin{cases} x' = A(t)x, & t \neq t_k, \\ \Delta x = B_k x, & t = t_k, \\ x(t_0^+) = x_0, \end{cases} \tag{2.1}$$

where $A \in PC([t_0,\infty), \mathbb{R}^n \times \mathbb{R}^n)$, and its perturbed linear system with fixed moments of impulse

$$\begin{cases} y' = A(t)y + C(t)y, & t \neq t_k, \\ \Delta y = B_k y + R_k y, & t = t_k, \\ y(t_0^+) = y_0, \end{cases} \tag{2.2}$$

where $C \in PC([t_0,\infty), \mathbb{R}^n \times \mathbb{R}^n)$, and B_k, R_k are $n \times n$ matrices.

We assume that the solution $y(t)$ of system (2.2) is left continuous at the moments of impulsive effect t_k, i.e., $y(t_k^-) = y(t_k)$, and $\Delta y(t_k) = y(t_k^+) - y(t_k)$.

Lemma 2.1. [2, Theorem 1.5] *Let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Then the following statements hold:*

1. *There exists a unique solution of equation (2.1) with $x(t_0^+) = x_0$ (or $x(t_0) = x_0$) and this solution is defined for $t > t_0$ (or $t \geq t_0$).*
2. *If $det(E + B_k) \neq 0$ for each $k \in \mathbb{Z}$, then this solution is defined for all $t \in \mathbb{R}$.*

The next result follows from a simple calculation.

Lemma 2.2. [2] *Each solution $y(t)$ of (2.2) satisfies the integro-summary equation*

$$\begin{aligned} y(t) = W(t,s)y(s) &+ \int_s^t W(t,\tau)C(\tau)y(\tau)d\tau \\ &+ \sum_{s<\tau_k<t} W(t,\tau_k^+)R_k y(\tau_k),\ t \geq s, \end{aligned}$$

where $W(t,s)$ is a Cauchy matrix for equation (2.1).

LEMMA 2.3. [2, Lemma 1.4] *suppose that for* $t \geq t_0$ *the inequality*

$$u(t) \leq c + \int_{t_0}^{t} b(s)u(s)ds + \sum_{t_0 \leq \tau_k < t} \beta_k u(\tau_k) \tag{2.3}$$

holds, where $u \in PC(\mathbb{R}, \mathbb{R})$, $b \in PC(\mathbb{R}, \mathbb{R}^+)$ *and* $\beta_k \geq 0$, $k \in \mathbb{Z}$ *and* c *are constants. Then we have*

$$u(t) \leq c \prod_{t_0 \leq \tau_k < t} (1 + \beta_k) \exp(\int_{t_0}^{t} b(s)ds) \tag{2.4}$$

$$\leq c \exp(\int_{t_0}^{t} b(s)ds + \sum_{t_0 \leq \tau_k < t} \beta_k), \quad t \geq t_0. \tag{2.5}$$

We will prove that under a general "small" mean condition on the perturbations C and R_k, h-stability of system (2.1) is inherited by the perturbed system (2.2).

DEFINITION 2.4. [6, Definition 2.5] The zero solution $\upsilon = 0$ of (2.1) (or system (2.1)) is called h-stable if there exist a positive bounded left continuous function $h : \mathbb{R}^+ \to \mathbb{R}$, and a constant $c \geq 1$ such that

$$\|x(t, t_0, x_0)\| \leq c\|x_0\| h(t) h(t_0)^{-1}, \; t \geq t_0 \tag{2.6}$$

for x_0 small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

We need the following assumption for our lemmas.
(A_1): Let $\lambda : N_0 \to \mathbb{R}^+$ be a function that satisfies

$$\overline{\lim}_{n \to \infty} \Lambda_m(n) < \gamma < \infty, m \in N \text{ fixed}, \tag{2.7}$$

where

$$\Lambda_m(n) = \frac{1}{m+1} \sum_{s=n}^{n+m} \lambda(s). \tag{2.8}$$

LEMMA 2.5. [8, Lemma 1] *Suppose that* $\lambda : N_0 \to \mathbb{R}^+$ *is a function. Then*

$$\sum_{s=n}^{M} \lambda(s) \leq \sum_{s=n-m}^{M} \Lambda_m(s)$$

for all $M \geq n \geq m$, *with* $m \in N$ *fixed.*

In the following lemmas we will use the norms

$$\|\Lambda_m\|_\infty = \sup_{n \geq n_0} \{ \frac{1}{m+1} \sum_{s=n}^{n+m} |\lambda(s)| : m \in N \quad \text{fixed} \} \tag{2.9}$$

and

$$\|\Lambda\|_\infty = \sup_{t\geq t_0}\{\int_t^{t+1} |\lambda(s)|ds\}, \tag{2.10}$$

respectively.

LEMMA 2.6. [8, Lemma 2] *Assume that λ satisfies condition (A_1). Then there exists $M \geq m$ big enough such that for all $n \geq M \geq m$*

$$\sum_{s=n_0}^{n} \lambda(s) \leq \|\Lambda_m\|_\infty(m+1+M) + \gamma(n-n_0), \quad n \geq n_0. \tag{2.11}$$

We need the following result as the continuous version of Lemma 1 and Lemma 2 [9].

LEMMA 2.7. [8, Lemma 3] *Let $\lambda : [t_0,\infty) \to \mathbb{R}^+$ be a function satisfying*

$$\overline{\lim_{t\to\infty}} \Lambda(t) < \beta < \infty,$$

where $\Lambda(t) = \int_t^{t+1} \lambda(s)ds$. Then

1. *$\int_{t_0}^{t} \lambda(s)ds \leq \int_{t_0-1}^{t} \Lambda(s)ds$ for all $t \geq t_0 - 1$.*
2. *for each $\beta_1 > \beta$ there exists $T > t_0$ such that*

$$\int_{t_0}^{t} \lambda(s)ds \leq \|\Lambda\|_\infty(T+1) + \beta_1(t-t_0), \ \ t > t_0.$$

Also, we need the following assumption for our lemma.
(A_2):

$$\overline{\lim_{t\to\infty}} |C(s)|ds = \gamma_1 < \infty,$$

and

$$\overline{\lim_{n\to\infty}} \frac{1}{m+1}\sum_{s=n}^{n+m} |R_s| = \gamma_2 < \infty, \ m \in \mathbb{N}.$$

REMARK 2.8. *In order to prove Theorem 3.1, we will use the following fact:*
If $t_{n_0-1} < t_0 < t_{n_0} < \cdots < t_n < t \leq t_{n+1}$, then it follows from Lemma 2.6 that

$$\sum_{t_0<t_k<t} |R_k| \leq d_2 + \gamma_2(n-n_0).$$

If, in addition, we assume that
(A_3): If the moments $\{t_k\}$ satisfy $t_k - t_{k-1} \geq \theta > 0, k \in \mathbb{N}$,
then

$$\sum_{t_0<t_k<t} |R_k| \leq d_2 + \gamma_2(t - t_0), t > t_0.$$

3. Main results

Theorem 3.1. *Assume that conditions (A_1)-(A_3) hold, and the zero solution $x = 0$ of (2.1) is h-stable. If $\frac{\log h(t)}{t} \leq -c(\gamma_1+\gamma_2)$ for each $t > 0$, then the zero solution $y = 0$ of the (2.2) is h-stable.*

Proof. It follows from Lemma 2.2 that the solution $y(t)$ of (2.2) is given by

$$\begin{aligned} y(t) = W(t, t_0^+)y_0 + \int_{t_0}^{t} W(t, s)C(s)y(s)ds \\ + \sum_{t_0<t_k<t} W(t, t_k)R_k y(t_k),\ t \geq t_0. \end{aligned}$$

Then, we have

$$\begin{aligned} |y(t)| \leq ch(t)h(t_0)^{-1}|y_0| + \int_{t_0}^{t} ch(t)h(s)^{-1}|C(s)||y(s)|ds \\ + \sum_{t_0<t_k<t} ch(t)h(t_k)^{-1}|R_k||y(t_k)|,\ t \geq t_0. \end{aligned}$$

Letting $\nu(t) = \frac{|y(t)|}{h(t)}$, we obtain

$$\nu(t) \leq c\nu(t_0) + \int_{t_0}^{t} c|C(s)|\nu(s)ds + \sum_{t_0<t_k<t} c|R_k|\nu(t_k).$$

By the Gronwall impulsive inequalities of Lemma 2.3, we obtain

$$\nu(t) \leq c\nu(t_0)exp[\int_{t_0}^{t} c|C(s)|ds + \sum_{t_0<t_k<t} c|R_k|],\ t > t_0.$$

In view of Lemma 2.6 and Lemma 2.7, and condition (A_3), we obtain

$$|y(t)| \leq c|y_0|h(t)h(t_0)^{-1}exp[d_1 + c\gamma_1(t - t_0) + d_2 + c\gamma_2(t - t_0)],\ t \geq t_0,$$

where d_1 and d_2 are constants. Thus we have

$$|y(t)| \leq c_1|y_0|H(t)H(t_0)^{-1},\ t \geq t_0,$$

where $c_1 = c\exp(d_1 + d_2)$ and $H(t) = e^{c(\gamma_1+\gamma_2)t}h(t)$. Since $\frac{\log h(t)}{t} \le -c(\gamma_1 + \gamma_2)$ for each $t > 0$, it is easy to show that $H(t)$ is a positive bounded left continuous function. Hence the zero solution $y = 0$ of (2.2) is h-stable. The proof is complete. □

Corollary 3.2. *If* $(\int_{t_0}^{\infty} |C(s)| + \sum_{t_0<t_k<t} |R_k|) < \infty$, *then the zero solution* $y = 0$ *of (2.2) is* h*-stable.*

In order to obtain h-stability of solutions of nonlinear impulsive differential systems, we need the following assumption.
(A_4):

(i) $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $(t_{k-1}, t_k] \times \mathbb{R}^n$ and for every $y \in \mathbb{R}^n$, $n \in \mathbb{N}$

$$\lim_{(t,x)\to(t_k,y)} f(t,x) \text{ exists for } t > t_k.$$

In addition, there exists $\lambda \in PC(\mathbb{R}^+, \mathbb{R})$ such that

$$|f(t,y)| \le \lambda(t)|y|$$

for $(t, y) \in \mathbb{R}^+ \times \mathbb{R}^n$, where λ satisfies Lemma 2.7.

(ii) For every $k \in \mathbb{N}$, B_k is an $n \times n$ matrix, and $I_k : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and satisfies

$$|I_k(y)| \le \lambda_k|y|,\ y \in \mathbb{R}^n, \lambda_k > 0,$$

where the sequence λ_k satisfies condition (A_1).

We consider nonlinear impulsive differential system

$$\begin{cases} y' = A(t)y + f(t,y),\ t \ne t_k, \\ \Delta y = B_k y + I_k y, \quad t = t_k, \\ y(t_0^+) = y_0, \end{cases} \tag{3.1}$$

where $f(t, 0) = 0$.

We can obtain the various stability results from Theorem 3.1.

Corollary 3.3. *Suppose that the assumptions of Theorem 3.1 hold. Let* α *be a positive constant.*

1. *If* $h(t) = e^{-\alpha t}$ *for each* $t \in \mathbb{R}^+$ *in Theorem 3.1, then the zero solution* $y = 0$ *of (2.2) is* h*-stable* [8, Theorem 1].
2. *If we set h(t)=c for each* $t \in \mathbb{R}^+$, *then the zero solution* $y = 0$ *of (2.2) is uniformly stable.*
3. *If* $h(t) \to 0$ *as* $t \to \infty$, *then the zero solution* $y = 0$ *of (2.2) is asymptotically stable.*

Theorem 3.4. *Assume that conditions (A_4) holds, If the zero solution $x = 0$ of 2.1 is h-stable with condition $\frac{\log h(t)}{t} \leq -c(\gamma_1 + \gamma_2)$, for each $t > 0$, then the zero solution $y = 0$ of (3.1) is h-stable.*

The proof of Theorem 3.4 can be proved in a similar manner as that of Theorem 3.1. So we omit the proof.

Corollary 3.5. *Assume that the ordinary differential system:*

$$x'(t) = A(t)x(t)$$

is h-stable. Furthermore, suppose that f, $I_k(k \in \mathbb{N})$ and $h(t)$ satisfy the hypothesis of Theorme 3.4. Then the impulsive system

$$y'(t) = A(t)y(t) + f(t, y),\ t \neq t_k,$$
$$\Delta y(t_k) = I_k(y),\ t = t_k.$$

is h-stable.

To illustrate our results, we will give an example about h-stability of linear impulsive differential system.

Example 3.6. [8, Example] *Let $t_{k+1} - t_k = T$, $k \in \mathbb{N}$, and let a, b, ω be constants. We consider the linear impulsive differential system*

$$\begin{cases} x' = A(t)x = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x,\ t \neq t_k, \\ \Delta x = B_k x_k = \begin{bmatrix} 0 & 0 \\ \frac{a}{\omega} & b \end{bmatrix} x_k,\ t = t_k, \\ x(t_0^+) = x_0, \end{cases} \tag{3.2}$$

where

$$A(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix},\ B_k = \begin{bmatrix} 0 & 0 \\ \frac{a}{\omega} & b \end{bmatrix};$$

and its perturbed linear impulsive system

$$\begin{cases} y' = (A(t) + C(t))y = (\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} + \begin{bmatrix} 0 & e^{-t} \\ e^{-t^2} & 0 \end{bmatrix})y,\ t \neq t_k, \\ \Delta y = (B_k + R_k)y_k = (\begin{bmatrix} 0 & 0 \\ \frac{a}{\omega} & b \end{bmatrix} + \begin{bmatrix} 2^{-k} & d \\ \frac{1}{k} & 0 \end{bmatrix})y_k,\ t = t_k, \\ y(t_0^+) = y_0, \end{cases} \tag{3.3}$$

where

$$C(t) = \begin{bmatrix} 0 & e^{-t} \\ e^{-t^2} & 0 \end{bmatrix},\ R_k = \begin{bmatrix} 2^{-k} & d \\ \frac{1}{k} & 0 \end{bmatrix}.$$

If $-2 + |\frac{a}{\omega}\sin\omega T| < b < 0$*, then the multipliers of system (3.2) are the modules less than 1 (see* [1]*), and there exist a constant* $c \geq 1$ *and a positive bounded continuous function* $h : \mathbb{R}^+ \to \mathbb{R}$ *such that*

$$|W(t,s)| \leq ch(t)h(t_0)^{-1},\ t \geq t_0 \geq 0,$$

where $W(t,t_0)$ *is a Cauchy matrix of (3.2) and* $h(t) = e^{-\alpha t}$ *for soma* $\alpha > 0$*. Also, the solution* $x(t,t_0,x_0) = W(t,t_0)x_0$ *of equation (3.2) satisfies*

$$|x(t,t_0,x_0)| \leq c|x_0|h(t)h(t_0)^{-1},\ t > t_0.$$

Furthermore, for a suitable norm, we have

$$\gamma_1 = \overline{\lim_{n\to\infty}} \frac{1}{m+1}\sum_{k=n}^{n+m}|R_k| = |d|$$

and

$$\gamma_2 = \overline{\lim_{t\to\infty}} \int_t^{t+1} |C(s)|ds = 0.$$

If $\frac{\log h(t)}{t} = -\alpha \leq -c|d|$ *for each* $t > 0$*, then the zero solution* $y = 0$ *of (3.3) is* h*-stable by Theorem 3.1.*

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