JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **24**, No. 4, December 2011

GALOIS ACTIONS OF A CLASS INVARIANT OVER QUADRATIC NUMBER FIELDS WITH DISCRIMINANT $D \equiv 21 \pmod{36}$

Daeyeol Jeon*

Abstract. A class invariant is the value of a modular function that generates a ring class field of an imaginary quadratic number field such as the singular moduli of level 1. In this paper, using Shimura Reciprocity Law, we compute the Galois actions of a class invariant from a generalized Weber function $\mathfrak{g}_2$ over quadratic number fields with discriminant $D \equiv 21 \pmod{36}$.

1. Introduction

Let K be an imaginary quadratic number field with the discriminant D with ring of integer $\mathcal{O} = \mathbb{Z}[\theta]$ where

$$\theta := \begin{cases} \frac{\sqrt{D}}{2}, & \text{if } D \equiv 0 \pmod 4; \\ \frac{-1+\sqrt{D}}{2}, & \text{if } D \equiv 1 \pmod 4). \end{cases} \tag{1.1}$$

Then the theory of complex multiplication states that the modular invariant $j(\mathcal{O}) = j(\theta)$ generates the *ring class field* $H_{\mathcal{O}}$ over K with degree $[H_{\mathcal{O}} : K] = h(\mathcal{O})$, the class number of $\mathcal{O}$, and the conjugates of $j(\theta)$ under the action of $Gal(H_{\mathcal{O}}/K)$ are *singular moduli* $j(\tau)$, where $\tau := \tau_Q$ is the Heegner point determined by $Q(\tau_Q, 1) = 0$ for a positive definite integral primitive binary quadratic forms

$$Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$$

with discriminant $D = b^2 - 4ac$.

Received October 21, 2011; Accepted December 02, 2011.
2010 Mathematics Subject Classification: Primary 11F37; Secondary 11F30.
Key words and phrases: class invariants, Galois Actions, Weber Functions.
This work was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2010-0023942).

In his Lehrbuch der Algebra [5], H. Weber calls the value of a modular function $f(\theta)$ a *class invariant* if we have

$$K(f(\theta)) = K(j(\theta)).$$

Gee determined the class invariants from a generalized Weber function $\mathfrak{g}_2$ by using the Shimura Reciprocity Law as follows:

THEOREM 1.1. [3, p.73, Theorem 1] *Let K be an imaginary quadratic number field of discriminant $D \equiv 21 \pmod{36}$ with the ring of integer $\mathcal{O} = \mathbb{Z}[\theta]$. Suppose $\theta = \frac{-B+\sqrt{D}}{2}$ as defined in (1.1). Then $\frac{1}{3\sqrt{-3}}\mathfrak{g}_2^6(\theta)$ gives an integral generator for $H_{\mathcal{O}}$ over K.*

In this paper, we compute the Galois actions of the class invariant $\frac{1}{3\sqrt{-3}}\mathfrak{g}_2^6(\theta)$ under $Gal(H_{\mathcal{O}}/K)$.

2. Preliminary

Let $\mathcal{Q}_D^0$ be the set of primitive quadratic forms and $C(D) = \mathcal{Q}_D^0/\Gamma(1)$ denote the form class group of discriminant D. Since $Gal(H_{\mathcal{O}}/K)$ is isomorphic to $C(D)$, it suffices to compute the action of a primitive quadratic form $Q = [a, b, c]$ on the class invariant $\frac{1}{3\sqrt{-3}}\mathfrak{g}_2^6(\theta)$.

THEOREM 2.1. [1, 2] *Let $\mathbb{Z}[\theta]$ be the ring of integers of an imaginary quadratic number field K of discriminant D and let $Q = [a, b, c]$ be a primitive quadratic form of discriminant D. Let $\theta = \frac{-B+\sqrt{D}}{2}$ as defined in (1.1) and $\tau_Q = \frac{-b+\sqrt{-D}}{2a}$. Let $M = M_{[a,b,c]} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ be given as follows: For $D \equiv 0 \pmod 4$,*

$$(2.1) \qquad M \equiv \begin{cases} \begin{pmatrix} a & \frac{b}{2} \\ 0 & 1 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \nmid a; \\ \begin{pmatrix} -\frac{b}{2} & -c \\ 1 & 0 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \mid a \text{ and } p \nmid c; \\ \begin{pmatrix} -\frac{b}{2} - a & -\frac{b}{2} - c \\ 1 & -1 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \mid a \text{ and } p \mid c, \end{cases}$$

and for $D \equiv 1 \pmod 4$,

$$(2.2) \qquad M \equiv \begin{cases} \begin{pmatrix} a & \frac{b-1}{2} \\ 0 & 1 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \nmid a; \\ \begin{pmatrix} \frac{-b-1}{2} & -c \\ 1 & 0 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \mid a \text{ and } p \nmid c; \\ \begin{pmatrix} \frac{-b-1}{2} - a & -\frac{1-b}{2} - c \\ 1 & -1 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \mid a \text{ and } p \mid c. \end{cases}$$

where p runs over all prime factors of N and $p^{r_p}||N$. Then the Galois action of the class of $[a,-b,c]$ in $C(D)$ with respect to the Artin map is given by

$$f(\theta)^{[a,-b,c]} = f^M(\tau_Q)$$

for any modular function f of level N such that $f(\theta) \in H_{\mathcal{O}}$. Here f^M denote the image of f under the action of M.

The action of M depends only on M_m for all primes $p|N$ where $M_m \in \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$ is the reduction modulo m of M. Every M_m with determinant x decomposes as $M_m = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$. Since $\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$ is generated by $S_m \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T_m \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, it suffices to find the action of $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_{p^{r_p}}$, $S_{p^{r_p}}$ and $T_{p^{r_p}}$ on f for all $p|N$. Denote ζ_n by a primitive nth root of unity. For $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_{p^{r_p}}$, the action on f is given by lifting the automorphism of $\mathbb{Q}(\zeta_N)$ determined by

$$\zeta_{p^{r_p}} \mapsto \zeta_{p^{r_p}}^x \quad \text{and} \quad \zeta_{q^{r_q}} \mapsto \zeta_{q^{r_q}}$$

for all prime factors $q|N$ with $q \neq p$. In order that the actions of the matrices at different primes commute with each other, we lift $S_{p^{r_p}}$ and $T_{p^{r_p}}$ to matrices in $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ such that they reduce to the identity matrix in $\mathrm{SL}_2(\mathbb{Z}/q^{r_q}\mathbb{Z})$ for all $q \neq p$.

The Dedekind-eta function

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad \text{with} \quad q = e^{2\pi i z}$$

is holomorphic and non-zero for z in the complex upper half plane $\mathbb{H}$ and $\Delta(z) = \eta^{24}(z)$ is modular form of weight 12 with no poles or zeros on $\mathbb{H}$. Then we have generalized Weber functions as follows:

(2.3)

$$\mathfrak{g}_0(z) = \frac{\eta(\frac{z}{3})}{\eta(z)},\ \mathfrak{g}_1(z) = \zeta_{24}^{-1}\frac{\eta(\frac{z+1}{3})}{\eta(z)},\ \mathfrak{g}_2(z) = \frac{\eta(\frac{z+2}{3})}{\eta(z)},\ \mathfrak{g}_3(z) = \sqrt{3}\frac{\eta(3z)}{\eta(z)}.$$

Note that the functions in (2.3) are modular of level 72. For the generating matrices S, $T \in SL_2(\mathbb{Z})$ given by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T =$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the transformation rules $\eta \circ S(z) = \sqrt{-iz}\eta(z)$ and $\eta \circ T(z) = \zeta_{24}\eta(z)$ hold. Hence

$$(2.4) \qquad \begin{aligned} (\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ S &= (\mathfrak{g}_3, \zeta_{24}^{-2}\mathfrak{g}_2, \zeta_{24}^{2}\mathfrak{g}_1, \mathfrak{g}_0), \\ (\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ T &= (\mathfrak{g}_1, \zeta_{24}^{-2}\mathfrak{g}_2, \mathfrak{g}_0, \zeta_{24}^{2}\mathfrak{g}_3). \end{aligned}$$

3. Results

In this section, we compute the action of a primitive quadratic form $Q = [a, b, c]$ on the class invariant $\frac{1}{3\sqrt{-3}}\mathfrak{g}_2^6(\theta)$. For that we need to find the action of $M_m \in GL_2(\mathbb{Z}/m\mathbb{Z})$ with $m = 8, 9$. The author with C.H. Kim and S.-Y. Kang[4] obtain the following transformation rule:

Lemma 3.1. *The actions of* $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_m$, S_m *and* T_m $(m = 8, 9)$ *on* $\mathfrak{g}_i^2$ $(i = 0, 1, 2, 3)$ *are given by*

	$\mathfrak{g}_0^6$	$\mathfrak{g}_1^6$	$\mathfrak{g}_2^6$	$\mathfrak{g}_3^6$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_8$	$\mathfrak{g}_0^6$	$\mathfrak{g}_1^6$	$\mathfrak{g}_2^6$	$\mathfrak{g}_3^6$
S_8	$-\mathfrak{g}_0^6$	$-\mathfrak{g}_1^6$	$-\mathfrak{g}_2^6$	$-\mathfrak{g}_3^6$
T_8	$-\mathfrak{g}_0^6$	$-\mathfrak{g}_1^6$	$-\mathfrak{g}_2^6$	$-\mathfrak{g}_3^6$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_9$, $3 \mid (x-1)$	$\mathfrak{g}_0^6$	$\mathfrak{g}_1^6$	$\mathfrak{g}_2^6$	$\mathfrak{g}_3^6$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_9$, $3 \mid (x-2)$	$\mathfrak{g}_0^6$	$\mathfrak{g}_2^6$	$\mathfrak{g}_1^6$	$\mathfrak{g}_3^6$
S_9	$-\mathfrak{g}_3^6$	$\mathfrak{g}_2^6$	$\mathfrak{g}_1^6$	$-\mathfrak{g}_0^6$
T_9	$-\mathfrak{g}_1^6$	$\mathfrak{g}_2^6$	$-\mathfrak{g}_0^6$	$\mathfrak{g}_3^6$

Using this, together with Theorem 2.1, we have the following theorems.

Theorem 3.2. *Let* $D \equiv 21 \pmod{36}$ *be a discriminant of an order* $\mathcal{O} = [\theta, 1]$ *in an imaginary quadratic field. Let* $\theta = \frac{-1+\sqrt{D}}{2}$, $\tau_Q = \frac{-b+\sqrt{D}}{2a}$ *and* $u = (-1)^{\frac{b+1}{2}+ac+a+c}$. *If* $[a, b, c]$ *be a reduced primitive quadratic form of discriminant* D, *then the actions of* $[a, -b, c]$ *on* $\frac{1}{3\sqrt{-3}}\mathfrak{g}_2^6(\theta)$ *are as follows:*

(1) *The case $3 \nmid a$.*

$$\frac{1}{3\sqrt{-3}}\mathfrak{g}_2^6(\theta)^{[a,-b,c]} = \begin{cases} \left(\frac{a}{3}\right) \frac{u}{3\sqrt{-3}}\mathfrak{g}_0^6(\tau_Q) & \text{if } b \equiv 0 \pmod 3; \\ \left(\frac{a}{3}\right) \frac{u}{3\sqrt{-3}}\mathfrak{g}_1^6(\tau_Q) & \text{if } a+b \equiv 0 \pmod 3; \\ \left(\frac{a}{3}\right) \frac{u}{3\sqrt{-3}}\mathfrak{g}_2^6(\tau_Q) & \text{if } a-b \equiv 0 \pmod 3. \end{cases}$$

(2) *The cases $3|a$ and $3 \nmid c$.*

$$\frac{1}{3\sqrt{-3}}\mathfrak{g}_2^6(\theta)^{[a,-b,c]} = \begin{cases} \left(\frac{c}{3}\right) \frac{u}{3\sqrt{-3}}\mathfrak{g}_3^6(\tau_Q) & \text{if } b \equiv 0 \pmod 3; \\ \left(\frac{c}{3}\right) \frac{u}{3\sqrt{-3}}\mathfrak{g}_1^6(\tau_Q) & \text{if } a+b \equiv 0 \pmod 3; \\ \left(\frac{c}{3}\right) \frac{u}{3\sqrt{-3}}\mathfrak{g}_2^6(\tau_Q) & \text{if } a-b \equiv 0 \pmod 3. \end{cases}$$

(3) *The cases $3|a$ and $3|c$.*

$$\frac{1}{3\sqrt{-3}}\mathfrak{g}_2^6(\theta)^{[a,-b,c]} = \left(\frac{b}{3}\right) \frac{u}{3\sqrt{-3}}\mathfrak{g}_1^6(\tau_Q)$$

Here $\left(\,-\,\right)$ denotes the Legendre symbol.

References

[1] H. H. Chan, A. Gee and V. Tan, *Cubic singular moduli, Ramanujan's class invariants λ_n and the explicit Shimura reciprocity law*, Pacific J. Math. **208** (2003), 23–37.

[2] A. Gee, *Class invariants by Shimura's reciprocity law,* J. Théor. Nombre Bordeaux **11** (1999), 45–72.

[3] A. Gee, *Class fields by Shimura reciprocity,* Ph. D. Thesis, Universiteit van Amsterdam, 2001.

[4] D. Jeon, C. H. Kim and S.-Y. Kang, *Modularity of Galois traces of class invariants,* to appear in Math. Ann.

[5] H. Weber, *Lehrbuch der Algebra,* dritter Band, Friedrich Vieweg und Sohn, 1908.

*

Department of Mathematics Education
Kongju National University
Kongju 314-701, Republic of Korea
E-mail: `dyjeon@kongju.ac.kr`