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A SIMPLE PROOF OF QUOTIENTS OF THETA SERIES AS RATIONAL FUNCTIONS OF J

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Abstract. For two even unimodular positive definite integral quadratic forms $A[X]$, $B[X]$ in n-variables, J. K. Koo [1, Theorem 1] showed that $\theta_A(\tau)/\theta_B(\tau)$ is a rational function of J, satisfying a certain condition. Where $\theta_A(\tau)$ and $\theta_B(\tau)$ are theta series related to $A[X]$ and $B[X]$, respectively, and J is the classical modular invariant. In this paper we give a simple proof of Theorem 1 of [1].

1. Introduction

Let $Q(n,1)$ be the set of even unimodular positive definite integral quadratic forms in n-variables, where $n \equiv 0 \bmod 8$. For $A[X] \in Q(n,1)$, the theta series $\theta_A(\tau) := \sum_{n\in\mathbb{Z}^n} e^{\pi i\tau A[X]}$ is a modular form of weight $n/2$ for $SL_2(\mathbb{Z})$. If $n \geq 24$ and $A[X]$, $B[X]$ are two quadratic forms in $Q(n,1)$ then the quotient $\theta_A(\tau)/\theta_B(\tau)$ is a modular function for $SL_2(\mathbb{Z})$. J. K. Koo [1, Theorem 1] showed that $\theta_A(\tau)/\theta_B(\tau)$ is a rational function of J, satisfying a certain condition where J is the classical modular invariant and τ is contained in the complex upper half plane $\mathbb{H}$. In this paper we give a simple proof of Theorem 1 of [1].

Theorem 1.1. [1] *Let $n \geq 24$. For any two quadratic forms $A[X]$ and $B[X]$ in $Q(n,1)$*

$$\frac{\theta_A(\tau)}{\theta_B(\tau)} = \frac{f(J(\tau))}{g(J(\tau))} \qquad (\tau \in \mathbb{H}),$$

where $f(x)$ and $g(x)$ are polynomials over $\mathbb{Q}$ of degree $[n/24]$, and $[x]$ is the greatest integer less than or equal to x.

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2. A simple proof of Theorem 1.1

Proof. Let Δ_k be the unique normalized modular form of weight k for $SL_2(\mathbb{Z})$ with the zero of the maximum order at infinity. We denote the order of the zero of Δ_k at infinity by r_k. Then the vector space M_k of modular forms of weight k for $SL_2(\mathbb{Z})$ is spanned by $J^{r_k}\Delta_k, J^{r_k-1}\Delta_k, \ldots, J\Delta_k$ and Δ_k which implies that $r_k = \dim M_k - 1$. We know that M_k is generated by Eisenstein series $E_4(\tau)^i$, $E_6(\tau)^j$ with $4j + 6j = k$. So the Fourier expansion of Δ_k at infinity has rational coefficients. Since $\theta_A(\tau)$ and $\theta_B(\tau)$ are contained in $M_{n/2}$ and they have non-zero constant terms in the Fourier expansion at infinity, we have that $\theta_A(\tau) = \Delta_{n/2} f(J(\tau))$ and $\theta_B(\tau) = \Delta_{n/2} g(J(\tau))$ for some polynomials $f(x)$ and $g(x)$ over $\mathbb{Q}$ of degree $r_{n/2}$. We note that $\dim M_{n/2} = [n/24]+1$ for $n/2 \not\equiv 2 \bmod 12$. This prove the assertion. □

REMARK 2.1. By the same argument we can obtain that $\theta_A(\tau)/\theta_B(\tau)$ is a rational function of a Hauptmodul t_Γ for a congruence subgroup Γ of genus zero. More precisely

$$\frac{\theta_A(\tau)}{\theta_B(\tau)} = \frac{f(t_\Gamma(\tau))}{g(t_\Gamma(\tau))} \qquad (\tau \in \mathbb{H}),$$

where $f(x)$ and $g(x)$ are polynomials over $\mathbb{C}$ in x of degree $r = \dim M_k(\Gamma) - 1$.

References

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