

**SOME PROPERTIES INVOLVING THE HIGHER  
ORDER  $q$ -GENOCCHI NUMBERS AND POLYNOMIALS  
WITH WEIGHT  $(\alpha, \beta)$  VIA THE  $p$ -ADIC  $q$ -INTEGRAL  
ON  $\mathbb{Z}_p$**

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ABSTRACT. The main properties of this paper is to describe the higher order  $q$ -Genocchi polynomials with weight  $(\alpha, \beta)$ . However, we derive some interesting properties concerning this type of polynomials.

## 1. Introduction, definitions and notations

The main motivations of this paper are the papers [8], [14] by *Kim et al.* and *Hwang et al.*, in which they introduced and studied on higher order  $q$ -Euler numbers and polynomials with weight  $\alpha$  and higher order  $q$ -Bernoulli numbers and polynomials with weight  $\alpha$ , respectively.

By using  $q$ -Volkenborn integral, *Kim* introduced the  $q$ -Bernoulli numbers and polynomials with weight  $\alpha$  which are derived some interesting properties of  $q$ -Bernoulli numbers and polynomials with weight  $\alpha$ . However, several mathematicians have studied on the special numbers and polynomials with weight  $\alpha$  (see for details [3],[4], [8], [9], [10], [11], [14],[17], [22]).

Assume that  $p$  is a fixed odd prime number. Throughout this paper  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$ ,  $\mathbb{C}$  and  $\mathbb{C}_p$ , will, respectively, denote the ring of  $p$ -adic rational integers, the field of  $p$ -adic rational numbers, the complex number field and the completion of algebraic closure of  $\mathbb{Q}_p$ . Let  $\mathbb{N}$  be the set of natural numbers and  $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ . Let  $v_p$  be the normalized exponential

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valuation of  $\mathbb{C}_p$  with  $|p|_p = p^{-v_p(p)} = \frac{1}{p}$  (see [1-6,8-17]). If  $q \in \mathbb{C}$ , then we always assume that  $|q| < 1$ . If  $q \in \mathbb{C}_p$ , then we usually assume that  $|1 - q|_p < p^{-\frac{1}{p-1}}$ , which yields  $q^x = \exp(\log q)$  for  $|x|_p \leq 1$ .

The  $q$ -basic naturel numbers are defined by  $[x]_q = \frac{q^x - 1}{q - 1}$  ( $x \in \mathbb{N}$ ). Hence,  $\lim_{q \rightarrow 1} [x]_q = x$  for any  $x$  with  $|x|_p \leq 1$  in the present  $p$ -adic case. The  $q$ -factorial is defined as  $[n]_q! = [n]_q \cdot [n - 1]_q \cdot [n - 2]_q \cdots [2]_q \cdot [1]_q$ , and the Gaussian binomial coefficients is defined by  $\binom{n}{k}_q = \frac{[n]_q!}{[n - k]_q! [k]_q!}$ . Note that  $\lim_{q \rightarrow 1} \binom{n}{k}_q = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$  (see [5,8-16]).

In [5,10,11,16], the  $q$ -binomial formulas are known,

$$(1.1) \quad \begin{aligned} (b; q)_n &= (1 - b)(1 - qb) \cdots (1 - q^{n-1}b) \\ &= \sum_{k=0}^n \binom{n}{k}_q q^{\binom{k}{2}} (-1)^k b^k, \end{aligned}$$

$$(1.2) \quad \begin{aligned} (b; q)_n^{-1} &= \frac{1}{(1 - b)(1 - qb) \cdots (1 - q^{n-1}b)} \\ &= \sum_{k=0}^{\infty} \binom{n + k - 1}{k}_q b^k. \end{aligned}$$

We say that  $f$  (and write  $f \in UD(\mathbb{Z}_p)$ ) is a uniformly differentiable function at a point  $a \in \mathbb{Z}_p$ , if the quotients  $F_f(x, y) = \frac{f(x) - f(y)}{x - y}$  have a limit  $f'(a)$  as  $(x, y) \rightarrow (a, a)$ . For  $f \in UD(\mathbb{Z}_p)$ , *T. Kim defined the fermionic  $p$ -adic  $q$ -integral* on  $\mathbb{Z}_p$  ( or  $q$ -Volkenborn integral in the sense of fermionic) is defined as

$$(1.3) \quad \begin{aligned} I_{-q}(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) \\ &= \frac{[2]_q}{2} \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N - 1} (-1)^x f(x) q^x, \text{ (see [1-5,8-17]).} \end{aligned}$$

The Genocchi polynomials are defined by

$$(1.4) \quad \frac{2te^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi \text{ (see [6,7,10,11]).}$$

The  $q$ -extension of Genocchi numbers are defined by

$$(1.5) \quad G_{0,q} = 0, \quad q(qG_q + 1)^n + G_{n,q} = \begin{cases} [2]_q, & n = 1 \\ 0, & n \neq 1 \end{cases}$$

with usual the convention about replacing  $(G_q)^n$  by  $G_{n,q}$  (see [1]).  
 The  $(h, q)$ -extension of Genocchi numbers are defined by

$$(1.6) \quad G_{0,q}^{(h)} = 0, \quad q^{h-2} \left( qG_q^{(h)} + 1 \right) + G_{n,q}^{(h)} = \begin{cases} [2]_q, & n = 1 \\ 0, & n \neq 1 \end{cases}$$

with usual convention about replacing  $(G_q^{(h)})^n$  by  $G_{n,q}^{(h)}$  (see[2]).

Recently, For  $\alpha \in \mathbb{N}$ , the weighted  $q$ -Genocchi numbers are defined by

$$(1.7) \quad \tilde{G}_{0,q}^{(\alpha)} = 0, \quad q^{1-\alpha} \left( q^\alpha \tilde{G}_q^{(\alpha)} + 1 \right)^n + \tilde{G}_{n,q}^{(\alpha)} = \begin{cases} [2]_q, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0, \end{cases}$$

with usual convention about replacing  $(\tilde{G}_q^{(\alpha)})^n$  by  $\tilde{G}_{n,q}^{(\alpha)}$  (see[4]).

By same the motivation of [8], [14], we consider on the higher order  $q$ -Genocchi polynomials with weight  $(\alpha, \beta)$ . However, we derive some interesting properties concerning this type of polynomials.

**2. Higher Order  $q$ -Genocchi numbers and polyomials with weight  $(\alpha, \beta)$**

DEFINITION 1. Let be  $h \in \mathbb{Z}$ ,  $\alpha, k, \beta \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . Then we defined higher-order  $q$ -Genocchi polynomials with weight  $(\alpha, \beta)$  as follows:

$$(2.1) \quad \frac{\tilde{G}_{n+k,q}^{(\alpha,\beta)}(h, k | x)}{k! \binom{n+k}{k}} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{\sum_{j=1}^k x_j(h-j)} d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_k)$$

By (1.3) and (2.1), we arrive at the following theorem:

THEOREM 1. For  $h \in \mathbb{Z}$ ,  $\alpha, k, \beta \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . We obtain

$$(2.2) \quad \frac{\tilde{G}_{n+k,q}^{(\alpha,\beta)}(h, k | x)}{k! \binom{n+k}{k}} = \frac{[2]_{q^\beta}^k}{(1 - q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{(-q^{\alpha l + h - k + \beta}; q)_k}$$

We simply get,

$$\begin{aligned}
 & [x_1 + x_2 + \dots + x_k]_{q^\alpha}^n q^{x_1(h-1)+\dots+x_k(h-k)} \\
 = & [x_1 + x_2 + \dots + x_k]_{q^\alpha}^n q^{x_1(h-\alpha-1)+\dots+x_k(h-\alpha-k)} q^{\alpha(x_1+x_2+\dots+x_k)} \\
 = & [x_1 + x_2 + \dots + x_k]_{q^\alpha}^n q^{x_1(h-\alpha-1)+\dots+x_k(h-\alpha-k)} ((q^\alpha - 1) \frac{q^{\alpha(x_1+x_2+\dots+x_k)} - 1}{q^\alpha - 1} + 1) \\
 = & [x_1 + x_2 + \dots + x_k]_{q^\alpha}^n q^{x_1(h-\alpha-1)+\dots+x_k(h-\alpha-k)} ((q^\alpha - 1) [x_1 + x_2 + \dots + x_k]_{q^\alpha} + 1)
 \end{aligned}$$

Namely, we have Corollary 1 as follows:

COROLLARY 1. For  $\alpha, k \in \mathbb{N}$ . Then we get

$$\begin{aligned}
 (2.3) \quad & [x_1 + x_2 + \dots + x_k]_{q^\alpha}^n q^{x_1(h-1)+\dots+x_k(h-k)} \\
 = & \left( \begin{array}{c} (q^\alpha - 1) [x_1 + x_2 + \dots + x_k]_{q^\alpha}^{n+1} q^{x_1(h-\alpha-1)+\dots+x_k(h-\alpha-k)} \\ + [x_1 + x_2 + \dots + x_k]_{q^\alpha}^n q^{x_1(h-\alpha-1)+\dots+x_k(h-\alpha-k)} \end{array} \right)
 \end{aligned}$$

With regard to (2.3) and by using fermionic  $p$ -adic  $q$ -integral, we obtain the following theorem:

THEOREM 2. For  $h \in \mathbb{Z}$ ,  $\alpha, k, \beta \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . We obtain

$$\begin{aligned}
 (2.4) \quad & \tilde{G}_{n+k,q}^{(\alpha,\beta)}(h, k) \\
 = & (q - 1) [\alpha]_q \frac{\binom{n+k}{k}}{\binom{n+k+1}{k}} \tilde{G}_{n+k+1,q}^{(\alpha,\beta)}(h - \alpha, k) + \tilde{G}_{n+k}^{(\alpha,\beta)}(h - \alpha, k)
 \end{aligned}$$

By (2.1), we have,

$$\begin{aligned}
 & \frac{\tilde{G}_{k,q}^{(\alpha,\beta)}(m\alpha, k + 1)}{k!} \\
 = & \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^{k+1} (m\alpha-j)x_j} d\mu_{-q^\beta}(x_1) \dots d\mu_{-q^\beta}(x_{k+1}) \\
 = & \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \left( \frac{q^{\alpha(x_1+\dots+x_{k+1})} - 1}{q^\alpha - 1} (q^\alpha - 1) + 1 \right)^m q^{-\sum_{j=1}^{k+1} jx_j} \\
 & d\mu_{-q^\beta}(x_1) \dots d\mu_{-q^\beta}(x_{k+1}) \\
 = & \sum_{l=0}^m \binom{m}{l} (q^\alpha - 1)^l \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} [x_1 + \dots + x_{k+1}]_{q^\alpha}^l q^{-\sum_{j=1}^{k+1} jx_j} \\
 & d\mu_{-q^\beta}(x_1) \dots d\mu_{-q^\beta}(x_{k+1}) \\
 = & \sum_{l=0}^m \binom{m}{l} (q^\alpha - 1)^l \frac{\tilde{G}_{l+k}^{(\alpha,\beta)}(0, k + 1)}{k! \binom{l+k}{k}}
 \end{aligned}$$

Therefore, we obtain the following theorem:

**THEOREM 3.** For  $h \in \mathbb{Z}$ ,  $\alpha, k, \beta \in \mathbb{N}$  and  $m \in \mathbb{Z}_+$ . We obtain

$$(2.5) \quad \tilde{G}_{k,q}^{(\alpha,\beta)}(m\alpha, k+1) = \sum_{l=0}^m \frac{\binom{m}{l}}{\binom{l+k}{k}} (q^\alpha - 1)^l \tilde{G}_{l+k}^{(\alpha,\beta)}(0, k+1)$$

Moreover, we readily see,

$$(2.6) \quad \tilde{G}_{k,q}^{(\alpha,\beta)}(m\alpha, k+1) = \frac{k! [2]_{q^\beta}^{k+1}}{(-q^{m\alpha-k+\beta}; q)_k}$$

By (2.5) and (2.6), we get following theorem:

**THEOREM 4.** For  $h \in \mathbb{Z}$ ,  $\alpha, k, \beta \in \mathbb{N}$  and  $m \in \mathbb{Z}_+$ . We obtain

$$(2.7) \quad \sum_{l=0}^m \frac{\binom{m}{l}}{\binom{l+k}{k}} (q^\alpha - 1)^l \tilde{G}_{l+k}^{(\alpha,\beta)}(0, k+1) = \frac{k! [2]_{q^\beta}^{k+1}}{(-q^{m\alpha-k+\beta}; q)_k}$$

From (2.1), we can derive the following equation:

$$\begin{aligned} & \sum_{j=0}^i \binom{i}{j} (q^\alpha - 1)^j \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^{n-i+j} q^{\sum_{l=1}^k (h-\alpha-l)x_l} \\ & \qquad \qquad \qquad d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_k) \\ = & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^{n-i} q^{\sum_{l=1}^k (h-l)x_l} q^{\alpha(x_1+\cdots+x_k)(i-1)} \\ & \qquad \qquad \qquad d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_k) \\ = & \sum_{j=0}^{i-1} \binom{i-1}{j} (q^\alpha - 1)^j \frac{\tilde{G}_{n-i+j+k}^{(\alpha,\beta)}(h, k)}{k! \binom{n-i+j+k}{k}}. \end{aligned}$$

Thus, we obtain the following theorem:

**THEOREM 5.** For  $h \in \mathbb{Z}$ ,  $\alpha, k, \beta \in \mathbb{N}$  and  $n, i \in \mathbb{Z}_+$ . We obtain

$$(2.8) \quad \begin{aligned} & \sum_{j=0}^i \frac{\binom{i}{j}}{\binom{n-i+j+k}{k}} (q^\alpha - 1)^j \tilde{G}_{n-i+j+k}^{(\alpha,\beta)}(h - \alpha, k) \\ = & \sum_{j=0}^{i-1} \frac{\binom{i-1}{j}}{\binom{n-i+j+k}{k}} (q^\alpha - 1)^j \tilde{G}_{n-i+j+k}^{(\alpha,\beta)}(h, k). \end{aligned}$$

**3. Polynomials**  $\tilde{G}_{n,q}^{(\alpha,\beta)}(0, k | x)$

In this section, we discuss the polynomials  $\tilde{G}_{n,q}^{(\alpha,\beta)}(0, k | x)$  by

$$\begin{aligned}
 & \frac{\tilde{G}_{n+k,q}^{(\alpha,\beta)}(0, k | x)}{k! \binom{n+k}{k}} \\
 (3.1) \quad &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{-\sum_{j=1}^k jx_j} \\
 & \qquad \qquad \qquad d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_k).
 \end{aligned}$$

From (3.1), we get

$$(3.2) \quad \frac{\tilde{G}_{n+k,q}^{(\alpha,\beta)}(0, k | x)}{k! \binom{n+k}{k}} = \frac{[2]_{q^\beta}^k}{(1 - q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{lx}}{(-q^{\alpha l - k + \beta}; q)_k}$$

By (3.1), we can easily derive the equation.

$$\begin{aligned}
 (3.3) \quad & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^k (\alpha n - j)x_j + \alpha n x} d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_k) \\
 &= \sum_{l=0}^n (q^\alpha - 1)^l \frac{\binom{n}{l}}{k! \binom{l+k}{k}} \tilde{G}_{l+k}^{(\alpha,\beta)}(0, k | x)
 \end{aligned}$$

and

$$\begin{aligned}
 (3.4) \quad & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^k (\alpha n - j)x_j + \alpha n x} d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_k) \\
 &= \frac{[2]_{q^\beta}^k q^{\alpha n x}}{(-q^{\alpha n - k + \beta}; q)_k}
 \end{aligned}$$

Therefore, we get the following theorem.

**THEOREM 6.** For  $\alpha, k, \beta \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . We get

$$\frac{\tilde{G}_{n+k,q}^{(\alpha,\beta)}(0, k | x)}{k! \binom{n+k}{k}} = \frac{[2]_{q^\beta}^k}{(1 - q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{(-q^{\alpha l - k + \beta}; q)_k}$$

and

$$\sum_{l=0}^n (q^\alpha - 1)^l \frac{\binom{n}{l}}{k! \binom{l+k}{k}} \tilde{G}_{l+k,q}^{(\alpha,\beta)}(0, k | x) = \frac{[2]_{q^\beta}^k q^{\alpha n x}}{(-q^{\alpha n - k + \beta}; q)_k}.$$

Assume that  $d \in \mathbb{N}$  with  $d \equiv 1 \pmod{2}$ . We readily see, namely,

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[ x + \sum_{j=1}^k x_j \right]_{q^\alpha}^n q^{-\sum_{j=1}^k jx_j} d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_k) \\ &= \frac{[d]_{q^\alpha}^n}{[d]_{-q^\beta}^n} \sum_{a_1, \dots, a_k=0}^{d-1} q^{-\sum_{j=2}^k (j-1)a_j} (-1)^{\sum_{j=1}^k a_j} \times \\ & \quad \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[ \frac{x + \sum_{j=1}^k a_j}{d} + \sum_{j=1}^k x_j \right]_{q^{\alpha d}}^n q^{-d \sum_{j=1}^k jx_j} \\ & \quad d\mu_{(-q^\beta)^d}(x_1) \cdots d\mu_{(-q^\beta)^d}(x_k) \end{aligned}$$

As well as, we obtain the following theorem.

**THEOREM 7.** For  $\alpha, k, \beta \in \mathbb{N}$ ,  $n \in \mathbb{Z}_+$  and  $d \equiv 1 \pmod{2}$ . We get

$$\begin{aligned} & \tilde{G}_{n+k,q}^{(\alpha,\beta)}(0, k | x) \\ &= \frac{[d]_{q^\alpha}^n}{[d]_{-q^\beta}^n} \sum_{a_1, \dots, a_k=0}^{d-1} q^{-\sum_{j=2}^k (j-1)a_j} (-1)^{\sum_{j=1}^k a_j} \\ & \quad \times \tilde{G}_{n+k,q^d}^{(\alpha,\beta)}\left(0, k \mid \frac{x + a_1 + a_2 + \cdots + a_k}{d}\right) \end{aligned}$$

From (3.1), we have

$$\tilde{G}_{n+k,q}^{(\alpha,\beta)}(0, k | x) = \sum_{l=0}^n \binom{n+k}{l+k} q^{\alpha l x} [x]_{q^\alpha}^{n-l} \tilde{G}_{l+k,q}^{(\alpha,\beta)}(0, k).$$

Thus, we note that

$$\tilde{G}_{n+k,q}^{(\alpha,\beta)}(0, k | x + y) = \sum_{l=0}^n \binom{n+k}{l+k} q^{\alpha l y} [y]_{q^\alpha}^{n-l} \tilde{G}_{l+k,q}^{(\alpha,\beta)}(0, k | x).$$

#### 4. Polynomials $\tilde{G}_{n,q}^{(\alpha,\beta)}(h, 1 | x)$

We now consider polynomials  $\tilde{G}_{n,q}^{(h)}(h, 1 | x)$  as follows:

$$(4.1) \quad \frac{\tilde{G}_{n+1,q}^{(\alpha,\beta)}(h, 1 | x)}{n+1} = \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{(h-1)x_1} d\mu_{-q^\beta}(x_1).$$

By (4.1), we get

$$(4.2) \quad \frac{\tilde{G}_{n+1,q}^{(\alpha,\beta)}(h, 1 | x)}{n + 1} = \frac{[2]_{q^\beta}}{(1 - q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{1 + q^{\alpha l + h - 1 + \beta}}$$

By applying *fermionic p-adic q-integral* on  $\mathbb{Z}_p$ , we see that

$$\begin{aligned} & q^{\alpha x} \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n d\mu_{-q^\beta}(x_1) \\ &= (q - 1) [\alpha]_q \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^{n+1} q^{x_1(h-\alpha-1)} d\mu_{-q^\beta}(x_1) \\ & \quad + \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{x_1(h-\alpha-1)} d\mu_{-q^\beta}(x_1) \end{aligned}$$

Therefore, we obtain the following theorem:

**THEOREM 8.** *For  $h \in \mathbb{Z}$ ,  $\alpha, \beta \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . We get*

$$(4.3) \quad \begin{aligned} & q^{\alpha x} \tilde{G}_{n+1,q}^{(\alpha,\beta)}(h, 1 | x) \\ &= \frac{n + 1}{n + 2} (q - 1) [\alpha]_q \tilde{G}_{n+2,q}^{(\alpha,\beta)}(h - \alpha, 1 | x) + \tilde{G}_{n+1,q}^{(\alpha,\beta)}(h - \alpha, 1 | x) \end{aligned}$$

It is simple to indicate

$$(4.4) \quad \begin{aligned} \tilde{G}_{n+1,q}^{(\alpha,\beta)}(h, 1 | x) &= (n + 1) \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n d\mu_{-q^\beta}(x_1) \\ &= q^{-\alpha x} \sum_{l=0}^n \binom{n + 1}{l + 1} [x]_{q^\alpha}^{n+1-(l+1)} q^{\alpha(l+1)x} \tilde{G}_{l+1,q}^{(\alpha,\beta)}(h, 1) \\ &= q^{-\alpha x} \left( q^{\alpha x} \tilde{G}_q^{(\alpha,\beta)}(h, 1) + [x]_{q^\alpha} \right)^{n+1}, \text{ for } n \geq 0, \end{aligned}$$

with usual the convention about replacing  $\left(\tilde{G}_q^{(\alpha,\beta)}(h, 1)\right)^n$  by  $\tilde{G}_{n,q}^{(\alpha,\beta)}(h, 1)$ .

By  $qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0)$  where  $f_1(x) = f(x + 1)$ . We get,

$$(4.5) \quad \begin{aligned} & q^{\beta+h-1} \int_{\mathbb{Z}_p} [x + x_1 + 1]_{q^\alpha}^n q^{(h-1)x_1} d\mu_{-q^\beta}(x_1) \\ & \quad + \int_{\mathbb{Z}_p} [x + x_1]_{q^\alpha}^n q^{(h-1)x_1} d\mu_{-q^\beta}(x_1) \\ &= [2]_{q^\beta} [x]_{q^\alpha}^n. \end{aligned}$$

From (4.1) and (4.5), we simply see,

$$(4.6) \quad \frac{q^{\beta+h-1}}{n + 1} \tilde{G}_{n+1}^{(\alpha,\beta)}(h, 1 | x) + \frac{1}{n + 1} \tilde{G}_{n+1}^{(\alpha,\beta)}(h, 1 | x) = [2]_{q^\beta} [x]_{q^\alpha}^n.$$

For  $x = 0$  into (4.6), we obtain

$$\frac{q^{\beta+h-1}}{n+1} \tilde{G}_{n+1}^{(\alpha, \beta)}(h, 1 | x) + \frac{1}{n+1} \tilde{G}_{n+1}^{(\alpha, \beta)}(h, 1 | x) = \begin{cases} [2]_{q^\beta}, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0. \end{cases}$$

From the *fermionic  $p$ -adic  $q$ -integral* on  $\mathbb{Z}_p$ , we simply get,

$$\tilde{G}_{1,q}^{(\alpha, \beta)}(h, 1 | x) = \int_{\mathbb{Z}_p} q^{x_1(h-1)} d\mu_{-q^\beta}(x_1) = \frac{[2]_{q^\beta}}{[2]_{q^{h+\beta-1}}}.$$

By (4.1), we see that,

$$\begin{aligned} & \frac{\tilde{G}_{n+1,q^{-1}}^{(\alpha, \beta)}(h, 1 | 1-x)}{n+1} = \int_{\mathbb{Z}_p} [1-x+x_1]_{q^{-\alpha}}^n q^{-(h-1)x_1} d\mu_{-q^{-\beta}}(x_1) \\ = & (-1)^n q^{n\alpha+h-1} \left( \frac{[2]_{q^\beta}}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{1+q^{\alpha l+h-1+\beta}} \right) \\ = & (-1)^n q^{n\alpha+h-1} \frac{\tilde{G}_{n+1,q}^{(\alpha, \beta)}(h, 1 | x)}{n+1} \end{aligned}$$

Therefore, we obtain the following theorem,

**THEOREM 9.** For  $h \in \mathbb{Z}$ ,  $\alpha, \beta \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . We get

$$\tilde{G}_{n+1,q^{-1}}^{(\alpha, \beta)}(h, 1 | 1-x) = (-1)^n q^{n\alpha+h-1} \tilde{G}_{n+1,q}^{(\alpha, \beta)}(h, 1 | x)$$

Assume that  $d \in \mathbb{N}$  with  $d \equiv 1 \pmod{2}$ . Then we have

$$\begin{aligned} (4.7) \quad & \int_{\mathbb{Z}_p} [dx+x_1]_{q^\alpha}^n q^{x_1(h-1)} d\mu_{-q^\beta}(x_1) \\ = & \frac{[d]_{q^\alpha}^n}{[d]_{-q^\beta}} \sum_{a=0}^{d-1} q^{ha} (-1)^a \int_{\mathbb{Z}_p} \left[x + \frac{a}{d} + x_1\right]_{q^{\alpha d}}^n q^{d(h-1)x_1} d\mu_{(-q^\beta)^d}(x_1) \end{aligned}$$

Thus, from (4.7) we obtain the following theorem,

**THEOREM 10.** For  $h \in \mathbb{Z}$ ,  $d \equiv 1 \pmod{2}$ ,  $\alpha, \beta \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . We get

$$\tilde{G}_{n+1,q}^{(\alpha, \beta)}(h, 1 | dx) = \frac{[d]_{q^\alpha}^n}{[d]_{-q^\beta}} \sum_{a=0}^{d-1} q^{ha} (-1)^a \tilde{G}_{n+1,q^d}^{(\alpha, \beta)}\left(h, 1 \mid x + \frac{a}{d}\right).$$

**5. Polynomials**  $\tilde{G}_{n,q}^{(\alpha,\beta)}(k, k | x)$

In (2.1), we know that

$$\begin{aligned}
 (5.1) \quad & \frac{\tilde{G}_{n+k,q}^{(\alpha,\beta)}(h, k | x)}{k! \binom{n+k}{k}} \\
 &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{\sum_{j=1}^k (h-j)x_j} \\
 & \qquad \qquad \qquad d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_k)
 \end{aligned}$$

As well as we get,

$$\begin{aligned}
 (5.2) \quad & \frac{\tilde{G}_{n+k,q}^{(\alpha,\beta)}(h, k | x)}{k! \binom{n+k}{k}} \\
 &= [2]_{q^\beta}^k \left( \frac{1}{1 - q^\alpha} \right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{lx}}{(-q^{\alpha l + h - k + \beta}; q)_k}
 \end{aligned}$$

and

$$\begin{aligned}
 (5.3) \quad & q^{h+\beta-1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + 1 + x_1 + \cdots + x_k]_{q^\alpha}^n \times \\
 & q^{\sum_{j=1}^k (h-j)x_j} d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_k) \\
 &= \left( \begin{array}{l} - \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{\sum_{j=1}^k (h-j)x_j} d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_k) \\ + \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{\sum_{j=2}^k (h-j)x_j} d\mu_{-q^\beta}(x_2) \cdots d\mu_{-q^\beta}(x_k) \end{array} \right).
 \end{aligned}$$

From (5.1) and (5.3), we obtain the following theorem.

**THEOREM 11.** For  $h \in \mathbb{Z}$ ,  $\alpha, k, \beta \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . We get

$$\begin{aligned}
 & q^{h+\beta-1} \tilde{G}_{n+k}^{(\alpha,\beta)}(h, k | x + 1) + \tilde{G}_{n+k}^{(\alpha,\beta)}(h, k | x) \\
 &= (n + k) \tilde{G}_{n+k-1}^{(\alpha,\beta)}(h - 1, k - 1 | x)
 \end{aligned}$$

Assume that  $d \in \mathbb{N}$  with  $d \equiv 1 \pmod{2}$ . Then we obtain

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[ x + \sum_{j=1}^k x_j \right]_{q^{\alpha d}}^n q^{\sum_{j=1}^k (h-j)x_j} d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_k) \\
 &= \frac{[d]_{q^\alpha}^n}{[d]_{-q^\beta}^n} \sum_{a_1, \dots, a_k=0}^{d-1} q^{h \sum_{j=1}^k a_j - \sum_{j=2}^k (j-1)a_j} (-1)^{\sum_{j=1}^k a_j} \times
 \end{aligned}$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[ \frac{x + \sum_{j=1}^k a_j}{d} + \sum_{j=1}^k x_j \right]_{q^{\alpha d}}^n q^{d \sum_{j=1}^k (h-j)x_j} d\mu_{(-q^\beta)^d}(x_1) \cdots d\mu_{(-q^\beta)^d}(x_k)$$

Thus, we obtain the following theorem.

**THEOREM 12.** For  $h \in \mathbb{Z}$ ,  $\alpha, k, \beta \in \mathbb{N}$ ,  $n \in \mathbb{Z}_+$  with  $d \equiv 1 \pmod{2}$ . We get

$$\begin{aligned} & \tilde{G}_{n+k,q}^{(\alpha,\beta)}(h, k | x) \\ &= \frac{[d]_{q^\alpha}^n}{[d]_{-q^\beta}^n} \sum_{a_1, \dots, a_k=0}^{d-1} q^{h \sum_{j=1}^k a_j - \sum_{j=2}^k (j-1)a_j} (-1)^{\sum_{j=1}^k a_j} \\ & \quad \times \tilde{G}_{n+k,q^d}^{(\alpha,\beta)}\left(h, k \mid \frac{x + \sum_{j=1}^k a_j}{d}\right) \end{aligned}$$

Assume that  $\tilde{G}_{n,q}^{(\alpha,\beta)}(k, k | x) = \tilde{G}_{n,q}^{(\alpha,\beta)}(k | x)$ . Then we obtain

$$(5.4) \quad \frac{\tilde{G}_{n+k,q}^{(\alpha,\beta)}(k | x)}{k! \binom{n+k}{k}} = \frac{[2]_{q^\beta}^k}{(1 - q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{lx}}{(-q^{\alpha l + \beta}; q)_k}$$

and

$$\begin{aligned} (5.5) \quad & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [k - x + x_1 + x_2 + \cdots + x_k]_{q^{-\alpha}}^n \times \\ & q^{-((k-1)x_1 + \cdots + (k-k)x_k)} d\mu_{-q^{-\beta}}(x_1) \cdots d\mu_{-q^{-\beta}}(x_k) \\ &= (-1)^n q^{n\alpha + \binom{k}{2}} \frac{[2]_{q^\beta}^k}{(1 - q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{(-q^{\alpha l + \beta}; q)_k} \\ &= (-1)^n q^{n\alpha + \binom{k}{2}} \frac{\tilde{G}_{n+k,q}^{(\alpha,\beta)}(k | x)}{k! \binom{n+k}{k}} \end{aligned}$$

By (5.5), we get the following theorem.

**THEOREM 13.** For  $h \in \mathbb{Z}$ ,  $\alpha, k, \beta \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . We get

$$\tilde{G}_{n+k,q^{-1}}^{(\alpha,\beta)}(k - x | x) = (-1)^n q^{n\alpha + \binom{k}{2}} \tilde{G}_{n+k,q}^{(\alpha,\beta)}(k | x)$$

By simple calculation, we get

$$\begin{aligned} & \sum_{l=0}^n (q-1)^l [\alpha]_q^l \binom{n}{l} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^n q^{\sum_{i=1}^k (k-l)x_i} \\ & \qquad \qquad \qquad d\mu_{-q^{-\beta}}(x_1) \cdots d\mu_{-q^{-\beta}}(x_k) \\ &= \frac{[2]_{q^\beta}^k}{(-q^{\alpha l + \beta}; q)_k} \end{aligned}$$

and

$$\begin{aligned} & \frac{\tilde{G}_{n+k,q}^{(\alpha,\beta)}(k|x)}{k! \binom{n+k}{k}} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{(k-1)x_1 + \cdots + (k-k)x_k} \\ & \qquad \qquad \qquad d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_k) \\ &= \sum_{l=0}^n \binom{n}{l} q^{\alpha l x} [x]_{q^\alpha}^{n-l} \frac{\tilde{G}_{l+k,q}^{(\alpha,\beta)}(k|0)}{k! \binom{l+k}{k}}. \end{aligned}$$

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