JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **24**, No. 4, December 2011

AN EINSTEIN'S CONNECTION WITH ZERO TORSION VECTOR IN EVEN-DIMENSIONAL UFT X_n

Jong Woo Lee*

Abstract. The main goal in the present paper is to obtain a necessary and sufficient condition for a new connection with zero torsion vector to be an Einstein's connection and derive some useful representation of the vector defining the Einstein's connection in even-dimensional UFT X_n.

1. Introduction

Einstein([1]) proposed a new unified field theory that would include both gravitation and electromagnetism. Hlavatý([8]) gave the mathematical foundation of the Einstein's unified field theory in a 4-dimensional generalized Riemannian space X_4 (i.e., space-time) for the first time. And the n-dimensional generalization of this theory in a generalized Riemannian manifold X_n, the so-called *Einstein's n-dimensional unified field theory*(UFT X_n), had been obtained by Mishra([7]). Since then many consequences of this theory has been obtained by a number of mathematicians. However, it has been unable yet to represent a general n-dimensional Einstein's connection in a surveyable tensorial form, probably due to the complexity of the higher dimensions. The purpose of the present paper is to introduce a new connection with zero torsion vector in UFT X_n. In the next, we obtain a necessary and sufficient condition for the connection to be an Einstein's connection and derive some useful representation of the vector defining the Einstein's connection in UFT X_n. The obtained results and discussions in the present paper will be useful for the even-dimensional considerations of the unified field theory.

Received October 12, 2011; Accepted November 18, 2011.
2010 Mathematics Subject Classification: Primary 53A45, 53B50, 53C25.
Key words and phrases: Einstein's connection, Einstein's equation, Einstein's vector, torsion tensor, torsion vector, unified field theory, UFT.

2. Preliminary

Let X_n be an n-dimensional generalized Riemannian manifold covered by a system of real coordinate neighborhoods $\{U; x^\nu\}$, where, here and in the sequel, Greek indices run over the range $\{1, 2, \cdots, n\}$ and follow the summation convention. The algebraic structure on X_n is imposed by a basic real non-symmetric tensor $g_{\lambda\mu}$, which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{2.1}$$

where we assume that

$$\begin{aligned} &(a)\ G = det(g_{\lambda\mu}) \neq 0, \\ &(b)\ H = det(h_{\lambda\mu}) \neq 0, \\ &(c)\ T = det(k_{\lambda\mu}) \neq 0. \end{aligned} \tag{2.2}$$

Since $det(h_{\lambda\mu}) \neq 0$, we may define a unique tensor $h^{\lambda\nu}(= h^{\nu\lambda})$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.3}$$

We use the tensors $h^{\lambda\nu}$ and $h_{\lambda\mu}$ as tensors for raising and/or lowering indices for all tensors defined on X_n in the usual manner. Then we may define new tensors by

$$(a)\ k^\alpha{}_\mu = k_{\lambda\mu} h^{\lambda\alpha}, \quad (b)\ k_\lambda{}^\alpha = k_{\lambda\mu} h^{\mu\alpha}, \quad (c)\ k^{\alpha\beta} = k_{\lambda\mu} h^{\lambda\alpha} h^{\mu\beta}. \tag{2.4}$$

Since $k_{\lambda\mu}$ is skew-symmetric, and $T \neq 0$, the dimension of X_n is even. That is, n is *even*. Hence *all our further considerations in the present paper are dealt in even-dimensional UFT X_n*. The manifold X_n is assumed to be connected by a general real connection $\Gamma^\nu_{\lambda\mu}$ which may also be split into its symmetric part $\Lambda^\nu_{\lambda\mu}$ and skew-symmetric part $S_{\lambda\mu}{}^\nu$, called the *torsion tensor* of $\Gamma^\nu_{\lambda\mu}$:

$$\begin{aligned} &(a)\ \Lambda^\nu_{\lambda\mu} = \Gamma^\nu_{(\lambda\mu)} = \frac{1}{2}(\Gamma^\nu_{\lambda\mu} + \Gamma^\nu_{\mu\lambda}), \\ &(b)\ S_{\lambda\mu}{}^\nu = \Gamma^\nu_{[\lambda\mu]} = \frac{1}{2}(\Gamma^\nu_{\lambda\mu} - \Gamma^\nu_{\mu\lambda}). \end{aligned} \tag{2.5}$$

The *Einstein's n-dimensional unified field theory on X_n*(UFT X_n) is governed by the following set of equations :

$$\partial_\omega g_{\lambda\mu} - g_{\alpha\mu}\Gamma^\alpha_{\lambda\omega} - g_{\lambda\alpha}\Gamma^\alpha_{\omega\mu} = 0 \qquad (\partial_\nu = \frac{\partial}{\partial x^\nu}), \tag{2.6}$$

and

$$(a)\ S_\lambda = S_{\lambda\alpha}{}^\alpha = 0, \quad (b)\ R_{[\lambda\mu]} = \partial_{[\lambda} P_{\mu]}, \quad (c)\ R_{(\lambda\mu)} = 0, \tag{2.7}$$

where P_μ is an arbitrary vector, called the *Einstein's vector*, and $R_{\lambda\mu}$ is the contracted curvature tensor $R^{\alpha}_{\lambda\mu\alpha}$ of the curvature tensor $R^{\omega}_{\lambda\mu\nu}$:

$$R^{\omega}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\omega}_{\lambda\nu} - \partial_\nu \Gamma^{\omega}_{\lambda\mu} + \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\omega}_{\alpha\mu} - \Gamma^{\alpha}_{\lambda\mu}\Gamma^{\omega}_{\alpha\nu}. \tag{2.8}$$

The equation (2.6) is called the *Einstein's equation*, and a solution $\Gamma^{\nu}_{\lambda\mu}$ of the Einstein's equation is called the *Einstein's connection*. And the vector S_λ, defined by (2.7)(a), is the called the *torsion vector*.

In UFT X_n, the following quantities are frequently used, where $p = 1, 2, 3, \dots$:

$$\begin{aligned} &(a)\quad g = \frac{G}{H}, \quad k = \frac{T}{H}, \\ &(b)\quad K_0 = 1, \qquad K_p = k_{[\alpha_1}{}^{\alpha_1} k_{\alpha_2}{}^{\alpha_2} \dots k_{\alpha_p]}{}^{\alpha_p}, \\ &(c)\quad {}^{(0)}k_\lambda{}^\nu = \delta^\nu_\lambda, \quad {}^{(p)}k_\lambda{}^\nu = k_\lambda{}^{\alpha}\,{}^{(p-1)}k_\alpha{}^\nu = {}^{(p-1)}k_\lambda{}^\alpha\, k_\alpha{}^\nu, \\ &(d)\quad \phi = {}^{(2)}k_\alpha{}^\alpha. \end{aligned} \tag{2.9}$$

It should be remarked that the tensor ${}^{(p)}k_{\lambda\nu}$ is symmetric if p is even, and skew-symmetric if p is odd. It has been shown by Chung([5]) that the following relations hold in UFT X_n.

$$\begin{aligned} &(a)\quad K_n = k, \quad K_p = 0 \ (p \ is \ odd), \\ &(b)\quad g = \sum_{s=0}^{n} K_s, \\ &(c)\quad \sum_{s=0}^{n} K_s\,{}^{(n-s)}k_\lambda{}^\nu = 0. \end{aligned} \tag{2.10}$$

Here and in what follows, the index s is assumed to take the values 0, 2, 4, ... , n in the specified range. The following quantities are also used in our further considerations.

$$\Omega_0 \;=\; 0, \qquad \Omega_s \;=\; \phi\Omega_{s-2} + K_{s-2}, \tag{2.11}$$

where ϕ is given in (2.9)(d). A direct calculation shows that

$$\begin{aligned} \Omega_{n+2} &= \phi^{\frac{n}{2}} K_0 + \phi^{\frac{n-2}{2}} K_2 + \phi^{\frac{n-4}{2}} K_4 + \\ &\quad \dots + \phi K_{n-2} + K_n \\ &= \sum_{s=0}^{n} \{\sqrt{\phi}\}^{n-s} K_s \end{aligned} \tag{2.12}$$

Furthermore the characteristic polynomial corresponding to $k_{\lambda\mu}$, that is, the basic polynomial(Chung([6])) may be given by

$$\begin{aligned} det(Mh_{\lambda\mu}+k_{\lambda\mu}) =& H(M^n+M^{n-2}K_2+...+M^2K_{n-2}+k) \\ =& H\sum_{s=0}^{n} M^{n-s}K_s, \end{aligned} \tag{2.13}$$

for an arbitrary scalar M.

REMARK 2.1. In virtue of (2.2)(c), since $T\neq 0$, there exists a unique skew-symmetric tensor $\overline{k}^{\lambda\mu}$ in UFT X_n satisfying

$$k_{\lambda\mu}\,\overline{k}^{\lambda\nu}=\delta_{\mu}^{\nu}. \tag{2.14}$$

It has been shown by Lee([3]) that in UFT X_n, the representation of the tensor $\overline{k}^{\lambda\mu}$ may be given by

$$\overline{k}^{\lambda\mu}=\frac{1}{k}\sum_{s=0}^{n-2}K_s\ {}^{(n-s-1)}k^{\lambda\mu}. \tag{2.15}$$

As useful results of the relations (2.10)(b) and (2.15) for the lower-dimensional cases $n=2,4$, we obtain the following Table 1, in virtue of (2.9)(b) and (d),

TABLE 1. For $n=2,4$, the representations of g and $\overline{k}^{\lambda\mu}$.

n	g	$\overline{k}^{\lambda\mu}$
2	$g=1+k$	$\overline{k}^{\lambda\mu}=\frac{1}{k}k^{\lambda\mu}$
4	$g=1-\frac{1}{2}\phi+k$	$\overline{k}^{\lambda\mu}=\frac{1}{k}({}^{(3)}k^{\lambda\mu}+(g-k-1)k^{\lambda\mu})$

3. An Einstein's connection with zero torsion vector in UFT X_n

The following theorem was proved by Hlavatý([8]).

THEOREM 3.1. *If the Einstein's equation (2.6) admits a solution $\Gamma^{\nu}_{\lambda\mu}$ in UFT X_n, then this solution must be of the form*

$$\Gamma^{\nu}_{\lambda\mu} = \{{}_{\lambda}{}^{\nu}{}_{\mu}\} + 2h^{\nu\alpha}S_{\alpha(\lambda}{}^{\beta}k_{\mu)\beta} + S_{\lambda\mu}{}^{\nu}, \tag{3.1}$$

where $\{{}_{\lambda}{}^{\nu}{}_{\mu}\}$ are the Christoffel symbols defined by $h_{\lambda\mu}$.

REMARK 3.2. In virtue of Theorem 3.1, the equation (3.1) reduces the investigation $\Gamma^{\nu}_{\lambda\mu}$ to the study of its torsion tensor $S_{\lambda\mu}{}^{\nu}$. Hence in order to know an Einstein's connection $\Gamma^{\nu}_{\lambda\mu}$, it is necessary and sufficient to know its torsion tensor $S_{\lambda\mu}{}^{\nu}$. For this, we introduces a torsion tensor $S_{\lambda\mu}{}^{\nu}$(Lee([2])) given by

$$S_{\lambda\mu}{}^{\nu} = 2\delta^{\nu}_{[\lambda}X_{\mu]} + k_{\lambda\mu}Y^{\nu}, \tag{3.2}$$

for some nonzero vectors X_λ and Y_λ. This torsion tensor (3.2) satisfies the condition (2.7)(a), that is, its torsion vector is zero if and only if the vectors X_λ and Y_λ defining (3.2) are related by

$$X_\lambda = \frac{1}{n-1}k_{\lambda\alpha}Y^{\alpha}. \tag{3.3}$$

Substituting (3.3) into (3.2), we obtain a new torsion tensor $S_{\lambda\mu}{}^{\nu}$ given by, for some nonzero vector Y_λ,

$$S_{\lambda\mu}{}^{\nu} = \frac{2}{n-1}\delta^{\nu}_{[\lambda}k_{\mu]\alpha}Y^{\alpha} + k_{\lambda\mu}Y^{\nu}, \tag{3.4}$$

which is a torsion tensor with zero torsion vector.

THEOREM 3.3. *In UFT X_n, if the connection (3.1) is a connection such that its torsion tensor is of the form (3.4) for some nonzero vector Y_λ, then the connection is given by*

$$\Gamma^{\nu}_{\lambda\mu} = \{{}_{\lambda}{}^{\nu}{}_{\mu}\} + \frac{2(2-n)}{n-1}k_{(\lambda}{}^{\nu}\,k_{\mu)\alpha}Y^{\alpha} + \frac{2}{n-1}\delta^{\nu}_{[\lambda}k_{\mu]\alpha}Y^{\alpha} + k_{\lambda\mu}Y^{\nu}. \tag{3.5}$$

Proof. Since the torsion tensor of the connection (3.1) is of the form (3.4), we obtain

$$2h^{\nu\alpha}S_{\alpha(\lambda}{}^{\beta}k_{\mu)\beta} = \frac{2(2-n)}{n-1}k_{(\lambda}{}^{\nu}\,k_{\mu)\alpha}Y^{\alpha} \tag{3.6}$$

by a straightforward computation. Substituting (3.4) and (3.6) into (3.1), we obtain (3.5). □

THEOREM 3.4. *In UFT X_n, the connection (3.5) is a Einstein's connection if and only if the vector Y_λ defining (3.5) satisfies the following condition*

$$\nabla_\nu k_{\lambda\mu} = \frac{2}{n-1} h_{\nu[\lambda} k_{\mu]\alpha} Y^\alpha - 2k_{\nu[\lambda} Y_{\mu]} + \frac{2(n-2)}{n-1} {}^{(2)}k_{\nu[\lambda} k_{\mu]\alpha} Y^\alpha, \tag{3.7}$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to $\{{}_\lambda{}^\nu{}_\mu\}$.

Proof. The connection (3.5) is an Einstein's connection if and only if the connection (3.5) satisfies the Einstein's equation (2.6). Substituting (2.1) and (3.5) into (2.6), and making use of $\nabla_\nu h_{\lambda\mu} = 0$, we obtain

$$\nabla_\nu k_{\lambda\mu} - \frac{2}{n-1} h_{\nu[\lambda} k_{\mu]\alpha} Y^\alpha + 2k_{\nu[\lambda} Y_{\mu]} - \frac{2(n-2)}{n-1} {}^{(2)}k_{\nu[\lambda} k_{\mu]\alpha} Y^\alpha = 0 \tag{3.8}$$

by a straightforward computation. Hence the connection (3.5) is an Einstein's connection if and only if the vector Y_λ defining (3.5) satisfies the condition (3.7). □

REMARK 3.5. In virtue of Remark 3.2, Theorem 3.3, and Theorem 3.4, if a vector Y_λ defining (3.5) satisfies the condition (3.7), then the connection (3.5) defined the vector Y_λ is an Einstein's connection with zero torsion vector. Since this Einstein's connection satisfies one of the field equations, that is, (2.7)(a), it will play an important role in the study of UFT X_n.

4. The representation of the vector Y^ν defining (3.5) and satisfying (3.7)

In virtue of Theorem 3.3 and Theorem 3.4, in order to know the Einstein's connection (3.5) it is necessary and sufficient to know the vector Y_λ defining (3.5) and satisfying (3.7), which is the main goal of this section.

REMARK 4.1. Multiplying $h^{\mu\alpha}$ on both sides of (3.7) and contracting for ν and α, we obtain

$$\nabla_\beta k_\lambda{}^\beta = -\frac{n-2}{n-1}\{\phi k_{\lambda\beta} - {}^{(3)}k_{\lambda\beta}\} Y^\beta = -\frac{n-2}{n-1} P_{\lambda\beta} Y^\beta, \tag{4.1}$$

where ϕ is given by (2.9)(d), and

$$P_{\lambda\mu} = \phi k_{\lambda\mu} - {}^{(3)}k_{\lambda\mu}, \tag{4.2}$$

Our investigation is based on the skew-symmetric tensor (4.2).

LEMMA 4.2. *In UFT X_n, if $\phi = 1$, then the determinant of the tensor $P_{\lambda\mu}$, given by (4.2), never vanishes, i.e.,*

$$det(k_{\lambda\mu} - {}^{(3)}k_{\lambda\mu}) \neq 0. \tag{4.3}$$

Proof. When $\phi = 1$, the tensor $P_{\lambda\mu}$ can be rewritten as

$$\begin{aligned} P_{\lambda\mu} =& k_{\lambda\mu} - {}^{(3)}k_{\lambda\mu} \\ =& (h_{\alpha\lambda} + k_{\alpha\lambda})k^{\alpha\beta}(h_{\beta\mu} + k_{\beta\mu}) = g_{\alpha\lambda}k^{\alpha\beta}g_{\beta\mu}. \end{aligned} \tag{4.4}$$

For any n-square matrices $A = (a^{\lambda\mu})$, $B = (b_{\lambda\mu})$ and $C = (c_{\lambda\mu})$, the determinant of their matrix product ${}^tBAC = (d_{\lambda\mu})$, where ${}^tB$ is the transpose of B, is given by

$$det({}^tBAC) = det(b_{\alpha\lambda}a^{\alpha\beta}c_{\beta\mu}) = detB\ detA\ detC, \tag{4.5}$$

in virtue of $det\,{}^tB = detB$. Hence in virtue of (2.2)(b) and (c), (2.3), (2.4)(c), and (4.5), we obtain

$$det(k^{\lambda\mu}) = det(h^{\alpha\lambda}k_{\alpha\beta}h^{\beta\mu}) = T(\frac{1}{H})^2, \tag{4.6}$$

since $detA^{-1} = 1/detA$. Hence in virtue of (2.2)(a), (2.9)(a), and (4.6), (4.4) implies

$$det(P_{\lambda\mu}) = G\{T(\frac{1}{H})^2\}G = T(\frac{G}{H})^2 = g^2T \neq 0. \tag{4.7}$$

□

LEMMA 4.3. *In UFT X_n, if $\phi = 1$, then Ω_{n+2}, given by (2.12), never vanishes, i.e.,*

$$\Omega_{n+2} \neq 0. \tag{4.8}$$

Proof. Since $\phi = 1$, in virtue of (2.2)(a), (2.9)(a), (2.10)(b), and (2.12), we obtain

$$\Omega_{n+2} = \sum_{s=0}^{n} K_s = g \neq 0. \tag{4.9}$$

□

LEMMA 4.4. *In UFT X_n, $det(P_{\lambda\mu}) \neq 0$ if and only if $\Omega_{n+2} \neq 0$.*

Proof. Applying the same method used to prove Lemma 4.2, we obtain

$$det(P_{\lambda\mu}) = \frac{T}{H^2}\{det(\sqrt{\phi}h_{\lambda\mu} + k_{\lambda\mu})\}^2. \tag{4.10}$$

On the other hand, in virtue of (2.12) and (2.13), we obtain

$$det(\sqrt{\phi}h_{\lambda\mu}+k_{\lambda\mu})=H\sum_{p=0}^{n}\{\sqrt{\phi}\}^{n-p}K_p=H\Omega_{n+2}. \tag{4.11}$$

Therefore, in virtue of (4.10) and (4.11), we obtain

$$det(P_{\lambda\mu})=\frac{T}{H^2}(H\Omega_{n+2})^2=T(\Omega_{n+2})^2, \tag{4.12}$$

which implies that $det(P_{\lambda\mu})\neq 0$ if and only if $\Omega_{n+2}\neq 0$. □

REMARK 4.5. In our further considerations in the present paper, we assume that

$$det(P_{\lambda\mu})=det(\phi k_{\lambda\mu}-{}^{(3)}k_{\lambda\mu})\neq 0, \tag{4.13}$$

and hence, in virtue of Lemma 4.4,

$$\Omega_{n+2}\neq 0. \tag{4.14}$$

In virtue of (4.13), there exists a unique skew-symmetric tensor $Q^{\lambda\nu}$ satisfying

$$P_{\lambda\mu}Q^{\lambda\nu}=\delta^{\nu}_{\mu}. \tag{4.15}$$

REMARK 4.6. According to Lemma 4.3, the assumption (4.14) is automatically satisfied for the case $\phi=1$. On the other hand, for the lower-dimensional cases $n=2,4$, we obtain the following Table 2, in virtue of (2.9)(b) and (d), (2.10)(a), (2.11), and Table 1, According to this Table 2, the assumption (4.14) is also automatically satisfied for the case $n=2$.

TABLE 2. For $n=2,4$, the representations of Ω_{n+2}.

n	Ω_4	Ω_6
2	$\Omega_4=-k\neq 0$	
4	$\Omega_4=1+k-g$	$\Omega_6=2(1+k-g)^2+k$

In our further considerations in the present paper, we use the following useful abbreviations for any tensor $Z_{\lambda\nu}$, for $p,q=1,2,3,...$

$${}^{(p)}Z_{\lambda\mu}={}^{(p-1)}k_{\lambda}{}^{\nu}\,Z_{\nu\mu}. \tag{4.16}$$

We then have

$$ {}^{(1)}Z_{\lambda\mu} = Z_{\lambda\mu}, \qquad {}^{(p)}k_\lambda{}^\nu\, {}^{(q)}Z_{\nu\mu} = {}^{(p+q)}Z_{\lambda\mu}. \tag{4.17} $$

Lemma 4.7. *The following recurrence relations in UFT X_n holds:*

$$ {}^{(p)}Q^{\omega\nu} = \phi\, {}^{(p-2)}Q^{\omega\nu} + {}^{(p-4)}k^{\omega\nu} \quad (p = 3, 4, 5, ...), \tag{4.18} $$

where

$$ {}^{(-1)}k^{\lambda\mu} = -\overline{k}^{\lambda\mu}, \tag{4.19} $$

where the tensor $\overline{k}^{\lambda\mu}$ is given by (2.15).

Proof. Substituting (4.2) into (4.15), we obtain, in virtue of (4.16),

$$ {}^{(4)}Q^{\mu\nu} = \phi\, {}^{(2)}Q^{\mu\nu} + h^{\mu\nu}. \tag{4.20} $$

Multiplying ${}^{(p-4)}k^\omega{}_\mu$ to both sides of (4.20), we obtain the relation (4.18) in virtue of (4.19). □

Theorem 4.8. *The representation of the tensor $Q^{\lambda\mu}$, given by (4.15), in UFT X_n may be given by*

$$ Q^{\lambda\mu} = -\frac{1}{\Omega_{n+2}} \sum_{s=0}^{n-2} \Omega_{s+2}\ {}^{(n-s-3)}k^{\lambda\mu}. \tag{4.21} $$

Proof. Multiplying $Q^{\nu\mu}$ to both sides of (2.10)(c), and using (4.16), we obtain

$$ \begin{aligned} &\sum_{s=0}^{n} K_s\ {}^{(n-s+1)}Q^{\lambda\mu} \\ &= K_0\ {}^{(n+1)}Q^{\lambda\mu} + K_2\ {}^{(n-1)}Q^{\lambda\mu} + \sum_{s=4}^{n} K_s\ {}^{(n-s+1)}Q^{\lambda\nu} = 0. \end{aligned} \tag{4.22} $$

Substituting ${}^{(n+1)}Q^{\lambda\mu}$ from (4.18) into the first term of (4.22), and using (2.9)(b) and (2.11), we obtain

$$ \begin{aligned} &{}^{(n-3)}k^{\lambda\mu} + \{\phi + K_2\}\, {}^{(n-1)}Q^{\lambda\mu} + \sum_{s=4}^{n} K_s\ {}^{(n-s+1)}Q^{\lambda\nu} \\ &= {}^{(n-3)}k^{\lambda\mu} + \Omega_4\, {}^{(n-1)}Q^{\lambda\mu} + K_4\, {}^{(n-3)}Q^{\lambda\mu} + \sum_{s=6}^{n} K_s\ {}^{(n-s+1)}Q^{\lambda\nu} \\ &= 0. \end{aligned} \tag{4.23} $$

Substituting again ${}^{(n-1)}Q_{\lambda\mu}$ from (4.18) into (4.23), and using (2.11), we obtain

$$\begin{aligned}
&{}^{(n-3)}k^{\lambda\mu} + \Omega_4\, {}^{(n-5)}k^{\lambda\mu} + \{\phi\Omega_4 + K_4\}\, {}^{(n-3)}Q^{\lambda\mu} \\
&\quad + \sum_{s=6}^{n} K_s\, {}^{(n-s+1)}Q^{\lambda\mu} \\
&= {}^{(n-3)}k^{\lambda\mu} + \Omega_4\, {}^{(n-5)}k^{\lambda\mu} + \Omega_6\, {}^{(n-3)}Q^{\lambda\mu} + K_6\, {}^{(n-5)}Q^{\lambda\mu} \\
&\quad + \sum_{s=8}^{n} K_s\, {}^{(n-s+1)}Q^{\lambda\mu} \\
&= 0.
\end{aligned} \tag{4.24}$$

After $(n-2)/2$ steps of successive repeat substituting for ${}^{(p)}Q^{\lambda\mu}$ from (4.18), we obtain

$$\sum_{s=0}^{n-4} \Omega_{s+2}\, {}^{(n-s-3)}k^{\lambda\mu} + \Omega_n\, {}^{(3)}Q^{\lambda\mu} + K_n\, Q^{\lambda\mu} = 0, \tag{4.25}$$

in virtue of (2.12). On the other hand, if $p = 3$ in (4.18), then we obtain

$${}^{(3)}Q^{\omega\nu} = \phi\, Q^{\omega\nu} + {}^{(-1)}k^{\omega\nu} = \phi\, Q^{\omega\nu} - \overline{k}^{\omega\nu} \tag{4.26}$$

Substituting (4.26) into (4.25), and using (2.11) and (4.19), we obtain

$$\begin{aligned}
&\sum_{s=0}^{n-4} \Omega_{s+2}\, {}^{(n-s-3)}k^{\lambda\mu} - \Omega_n\, \overline{k}^{\lambda\mu} + \{\phi\Omega_n + K_n\}\, Q^{\lambda\mu} \\
&= \sum_{s=0}^{n-2} \Omega_{s+2}\, {}^{(n-s-3)}k^{\lambda\mu} + \Omega_{n+2}\, Q^{\lambda\mu} = 0,
\end{aligned} \tag{4.27}$$

which is condensed to (4.21). □

THEOREM 4.9. *If the vector Y_λ defining (3.5) satisfies the condition (3.7) in UFT X_n, then*

$$Y^\alpha = -\frac{n-1}{n-2} Q^{\lambda\alpha}\, \nabla_\beta k_\lambda{}^\beta, \qquad (n \neq 2), \tag{4.28}$$

where $Q^{\lambda\mu}$ is given by (4.21).

Proof. In virtue of Remark 4.1, since the vector Y_λ defining (3.5) satisfies the condition (3.7), we obtain

$$\nabla_\beta k_\lambda{}^\beta = -\frac{n-2}{n-1} P_{\lambda\beta} Y^\beta. \tag{4.29}$$

Multiplying $Q^{\lambda\alpha}$ on both sides of (4.29) and making use of (4.15), we obtain (4.28). □

REMARK 4.10. When $n = 2$, that is, in UFT X_2, the connection (3.5) is given by

$$\Gamma^{\nu}_{\lambda\mu} = \{{}_{\lambda}{}^{\nu}{}_{\mu}\} + 2\delta^{\nu}_{[\lambda} k_{\mu]\alpha} Y^{\alpha} + k_{\lambda\mu} Y^{\nu}, \tag{4.30}$$

and the condition (3.7) is given by

$$\nabla_{\nu}\, k_{\lambda\mu} = 2h_{\nu[\lambda} k_{\mu]\alpha} Y^{\alpha} - 2k_{\nu[\lambda}\, Y_{\mu]}. \tag{4.31}$$

In particular, in virtue of (4.1),

$$\nabla_{\beta} k_{\lambda}{}^{\beta} = 0. \tag{4.32}$$

But since $n = 2$, in (4.30)

$$S_{\lambda\mu}{}^{\nu} = 0, \tag{4.33}$$

and in (4.31)

$$\nabla_{\nu}\, k_{\lambda\mu} = 0. \tag{4.34}$$

We can easily check the above results (4.33) and (4.34) in UFT X_2 : for instance,

$$S_{12}{}^{1} = \delta^{1}_{1} k_{2\alpha} Y^{\alpha} - \delta^{1}_{2} k_{1\alpha} Y^{\alpha} + k_{12} Y^{1} = 0, \tag{4.35}$$

and

$$\begin{aligned} \nabla_1\, k_{12} =& 2h_{1[1} k_{2]\alpha} Y^{\alpha} - 2k_{1[1}\, Y_{2]} \\ =& (h_{11} Y^{1} + h_{12} Y^{2}) k_{21} + k_{12} Y_{1} \\ =& h_{1\alpha} Y^{\alpha} k_{21} + k_{12} Y_{1} = 0. \end{aligned} \tag{4.36}$$

Therefore (4.30) reduces to

$$\Gamma^{\nu}_{\lambda\mu} = \{{}_{\lambda}{}^{\nu}{}_{\mu}\} \tag{4.37}$$

Hence the Einstein's equation (2.6) reduces

$$\partial_{\omega} g_{\lambda\mu} - g_{\alpha\mu}\{{}_{\lambda}{}^{\alpha}{}_{\omega}\} - g_{\lambda\alpha}\{{}_{\omega}{}^{\alpha}{}_{\mu}\} = 0, \tag{4.38}$$

or equivalently, making use of $\nabla_{\nu}\, h_{\lambda\mu} = 0$,

$$\nabla_{\omega}\, g_{\lambda\mu} = \nabla_{\omega}\, k_{\lambda\mu} = 0. \tag{4.39}$$

On the other hand,

$$R_{\lambda\mu} = H_{\lambda\mu}, \tag{4.40}$$

where $H_{\lambda\mu}$ is the contracted curvature tensor $H^{\alpha}_{\lambda\mu\alpha}$ of the curvature tensor $H^{\omega}_{\lambda\mu\nu}$ defined by $\{{}_{\lambda}{}^{\nu}{}_{\mu}\}$, so that the Einstein's vector (2.7)(b) is automatically satisfied by $P_\lambda = \partial_\lambda P$, and (2.7)(c) reduce to

$$H_{\lambda\mu} = 0. \tag{4.41}$$

Since, in virtue of (4.32) and (4.34)

$$\partial_{[\omega} k_{\lambda\mu]} = 0,\ \nabla_\alpha k^{\alpha\nu} = 0, \tag{4.42}$$

the tensor $k_{\lambda\mu}$ may be identified with the tensor of the electromagnetic field in UFT X_2.

Remark 4.11. In virtue of Theorem 3.3, Theorem 3.4, and Theorem 4.9, when $n \neq 2$, the connection (3.5) defined by the vector (4.28) is an Einstein's connection with zero torsion vector if and only if the vector (4.28) satisfies the condition (3.7). Hence the vector (4.28) will play an important role in the study of UFT X_n. For the lower-dimensional cases $n = 2, 4$, we obtain the following Table 3, in virtue of Table 1 and 2, Theorem 4.8, Theorem 4.9, and Remark 4.10.

Table 3. For $n = 2, 4$, the representations of $Q^{\lambda\mu}$ and Y_λ.

n	$Q^{\lambda\mu}$	Y_λ
2	$Q^{\lambda\mu} = -\frac{1}{k^2} k^{\lambda\mu}$	nonzero vector
4	$Q^{\lambda\mu} = \frac{\Omega_4{}^{(3)} k^{\lambda\mu} - (\Omega_4 + k) k^{\lambda\mu}}{k\{2(\Omega_4)^2 + k\}}$ $(\Omega_4 = 1 + k - g)$	$Y^\alpha = -\frac{3}{2} Q^{\lambda\alpha}\, \nabla_\beta k_\lambda{}^\beta$

References

[1] A. Einstein, *The meaning of relativity*, Princeton University Press, Princeton, New Jersey, 1950.
[2] J. W. Lee, *Field equations of SE(k)-manifold X_n*, International Journal of Theoretical Physics **30** (1991), 1343–1353.
[3] J. W. Lee, *Some reciprocal relations between the g-unified and *g-unified field tensor*, Comm. Korean Math. Soc. **23** (2008), no. 2, 229–239.
[4] J. W. Lee and K. T. Chung, *A solution of Einstein's unified field equations*, Comm. Korean Math. Soc. **11** (1996), no. 4, 1047–1053.

[5] K. T. Chung and D. H. Cheoi, *Relations of two n-dimensional unified field theories*, Acta Math. Hung. **45** (1985), 141–149.
[6] K. T. Chung and H. W. Lee, *n-dimensional considerations of indicators*, Yonsei Nonchong. Yonsei Univ. **12** (1975), 1–5.
[7] R. S. Mishra, *n-dimensional considerations of the unified field theory of relativity*, Tensor, N. S. **8** (1958), 95–122.
[8] V. Hlavatý, *Geometry of Einstein's unified field theory*, P. Noordhoff Ltd. New York, 1957.

*
Department of Mathematics
Yonsei University
Wonju, Gangwondo 220-710, Republic of Korea
E-mail: jwlee806@yonsei.ac.kr