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UNIFORM RECURRENCE IN DYNAMICAL SYSTEMS

Kishik Koo*

Abstract. The purpose of this paper is to study the orbit behaviours near almost periodic points. We introduce notions of uniformly recurrent and u-recurrent points and investigate some relationships between these properties.

1. Introduction and preliminaries

As an attempt to approach some recurrence problems of stability theory in dynamical systems, some recurrent points of homeomorphisms are studied together with the related concepts of various recurrent properties such as periodicity, recurrence and nonwanderingness.

Our purpose here is to study the orbit behaviours near almost periodic points. In [3], author studied the properties of uniformly almost recurrent motions with the related concept of prolongational limit sets.

In this work, we introduce notions of uniformly recurrent and u-recurrent points and investigate some relationships between these properties.

Throughout this paper, let X be a compact metric space with a metric function d anf f be a homeomorphism of X.

For x in X, $\alpha(x)$ and $\omega(x)$ denote the negative and positive limit set of x, and $J^+(x)$ denote the positive prolongational limit set of x. A point x in X is called almost periodic if for any positive number ε there is a positive integer N such that for any nonnegative integer n, there is an integer k with $n \leq k \leq n+N$ such that $d(x, f^k(x)) < \varepsilon$. A point x in X is calles *stable* if for any positive number ε, there is a positive number δ such that $d(x,y) < \delta$ implies $d(f^n(x), f^n(y)) < \varepsilon$ for every integer n.

For integers m, n with $m \geq n$, $[m, n]$ denote the set of integers k with $m \leq k \leq n$ and let $f^{[m,n]}(x)$ denote $\cup_{i=m}^{i=n} f^i(x)$.

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A subset M of X is called *minimal* if the orbit of every point in M is dense in M. Let $B(x, \varepsilon)$ denote $\{y \in X : d(x, y) < \varepsilon\}$ and $\overline{M}$ denote the closure of $M \subset X$.

Here, we introduce uniformly recurrent and u-recurrent concepts of points.

DEFINITION 1.1. A point x in X is called *uniformly recurrent* if, for any open neighborhood U_x of x, there is a positive integer T satisfying that $J^+(x) \subset f^{[0,T]}(U_x)$. A point x in X is called *u-recurrent* if, for any positive number ε, there exist a positive number δ and a positive integer N such that, for any point y in $B(x, \delta)$ and positive integer n,

$$f^{[n,n+N]}(y) \cap B(x, \varepsilon) \neq \emptyset$$

holds.

2. Main results

The following result is trivial.

THEOREM 2.1. *If a point x in X is u- recurrent, then x is almost periodic.*

THEOREM 2.2. *If a point x in X is stable and almost periodict, then x is u-recurrent.*

Proof. Let assume that $x \in X$ is stable and almost periodic. Then, for any positive number ε, there exist a positive number δ and a positive integer N such that, for any point y in $B(x, \delta)$ and positive integer n,

$$f^{[0,T]}(f^n(x)) \cap B(x, \frac{\varepsilon}{2}) \neq \emptyset$$

holds. In particular, since for any nonnegative integer i, $f^{[iT,\ (i+1)T]}(x) \cap B(x, \varepsilon/2)$ is nonempty, we can choose a set of integers $\{t_i : t_i \geq 0\}$ such that

$$t_i \in [(i-1)T,\ iT] \quad \text{and} \quad f^{t_i}(x) \in B(x, \frac{\varepsilon}{2}). \tag{1}$$

Also, since x is stable, there is a positive number δ such that

$$d(x, y) < \delta \quad \text{implies} \quad d(f^n(x), f^n(y)) < \frac{\varepsilon}{2}. \tag{2}$$

for every nonnegative integer n. By (1) and (2), for any point y in $B(x, \delta)$ and integer t_i in $\{t_i\}$, we get

$$d(x, f^{t_i}(y)) \leq d(x, f^{t_i}(x)) + d(f^{t_i}(x), f^{t_i}(y)) < \varepsilon.$$

Here, let $L = 2T$. Then, for any $y \in B(x,\delta)$ and an integer k, we can choose an integer t_{n_k} in $[k, k+L] \cap \{t_i\}$ satisfying that

$$f^{t_{n_k}}(y) \in f^{[0,\ L]}(f^k(y)) \cap B(x,\varepsilon).$$

Hence, the point x is u-recurrent and this completes the proof. □

THEOREM 2.3. *If a point x in X is u-recurrent, then, for any open neighborhood U_x of x, there exists a positive integer T satisfying that*

$$J^+(x) \subset \bigcup_{i=0}^{T} \overline{f^i(U_x)}.$$

Proof. Let U_x be an open neighborhood of x in X. Choose a positive number ε such that $\overline{B(x,\varepsilon)} \subset U_x$. Since x is u-recurrent, there exist a positive number δ and a positive integer T such that, for any $y \in B(x,\delta)$ and any nonnegative integer n,

$$f^{[n,n+T]} \cap B(x,\varepsilon) \neq \emptyset$$

holds. Assume that $n \geq T$. Then, for any $y \in B(x,\delta)$ and positive integer n, we have

$$f^{[0,T]}(f^{n-T}(y)) \cap B(x,\varepsilon) \neq \emptyset.$$

This implies that

$$f^T(f^{n-T}(y)) \in f^T(f^{[-T,0]}(B(x,\varepsilon)))$$

and therefore, we get $f^n(y) \in f^{[0,T]}(B(x,\varepsilon))$.

Let $z \in J^+(x)$. By the definition of prolongational limit set, there are a sequence of points $\{z_n\}$ and a sequence of integers $\{t_n\}$ satisfying that $z_n \to z$, $t_n \to \infty$, $t_n \geq T$ and $f^{t_n}(z_n)$ converges to z. Thus, we have

$$f^{t_n}(z_n) \in f^{[0,T]}(B(x,\varepsilon))$$

and this shows that

$$z \in \overline{f^{[0.T]}(B(x,\varepsilon))} \subset \bigcup_{i=0}^{T} \overline{f^i(U_x)}.$$

This completes the proof. □

THEOREM 2.4. *If a point x in X is uniformly recurrent, then x is u-recurrent.*

Proof. Suppose x is not u-recurrent. Then there exists a positive number ε_0 satisfying that, for every nonnegative integer n, there are a point x_n in X and nonnegative integers $t_n,\ r_n$ such that

$$x_n \in B(x, \frac{1}{n}),\ r_n - t_n \geq n \ \text{ and } \ d(x, f^{[t_n,r_n]}(x_n)) \geq \varepsilon_0. \qquad (1)$$

Here. let $s_n = [(t_n + r_n)/2]$, where $[k]$ denote the Gauss integer of real number k. First, we claim that $\{s_n\}$ is unbounded. To see this, assume that $\{s_n\}$ is bounded. Then, without loss of generality, we can let $s_n = s_0$ for all n. Let $f^{s_0}(x_n) = z_n$. Then, cleary, $z_n \to f^{s_0}(x)$. Since f is a homeomorphism, there is an open neigborhood W of $f^{s_0}(x)$ such that

$$f^{-s_0}(W) \subset B(x, \varepsilon_0/2).$$

Also, since $x_n \to x$, we can choose a positive integer m satisfying that

$$f^{s_0}(x_m) \in W \cap f^{[t_m,r_m]}(x_m) \ \text{ and } \ r_m - t_m > 3s_0.$$

Then, we have

$$d(f^{-s_0}(f^{s_m}(x_m)), x) = d(f^{s_m - s_0}(x_m), x) < \frac{\varepsilon_0}{2} \ \text{ and } \ t_n < s_m - s_0 < r_n.$$

But this contradicts to (1) and thus we get $\{s_n\}$ is unbounded.

Without loss of generality, assume that s_n goes to infinity. Let $f^{s_n}(x_n) \to y$. Then $y \in J^+(x)$. Choose a positive number ε with $\varepsilon < \varepsilon_0$. By assumption, there exists a positive integer T satisfying that

$$y \in J^+(x) \subset f^{[0,T]}(B(x, \varepsilon)).$$

Therefore, for some nonpositive integer t_0 with $-T \leq t_0 \leq 0$, $f^{t_0}(y) \in B(x, \varepsilon)$ holds.

Let $d(x, f^{t_0}(y)) = \xi$ and choose a positive number η such that $\xi + \eta < \varepsilon$. By the continuity of f^{t_0} we can choose a positive number ρ such that

$$d(z, y) < \rho \ \text{ implies } \ d(f^{t_0}(z), f^{t_0}(y)) < \eta.$$

Also, we can choose a sufficiently large integer l satisfying that

$$d(y, f^{s_l}(x_l)) < \rho \ \text{ and } \ s_l - t_l > t_0.$$

Then, we get

$$f^{t_0}(f^{s_l}(x_l)) \in f^{[t_l,s_l]}(x_l)$$

and thus

$$d(x, f^{t_0}(f^{s_l}(x_l))) \geq \varepsilon_0 > \varepsilon \qquad (2)$$

holds. On the otherhand, $d(y, f^{s_l}(x_l)) < \rho$ implies that

$$d(x, f^{t_0}(f^{s_l}(x_l))) \leq d(x, f^{t_0}(y)) + d(f^{t_0}(y), f^{t_0}(f^{s_l}(x_l))) < \eta + \xi < \varepsilon.$$

But this contradicts to (2) and we conclude that x is u-recurrent. Hence, the proof of this theorem is complete. □

PROPOSITION 2.5. *If X has only one minimal set, then f has only one uniformly recurrent closure.*

Proof. Let M be the unique minimal subset of X and $x \in M$. It is sufficient to show that x is uniformly recurrent. Since X is compact, for any y in X, $\omega(y)$ and $\alpha(y)$ is nonempty set and $x \in M \subset \omega(y) \cap \alpha(y)$. Cleary, $J^+(x) = X$ holds.

Let W be an open neighborhood of x and select an open neighborhood U of x with $\overline{U} \subset W$. For any point y in X define the integer t_y by

$$t_y = \min\{n \geq 0 : f^n(y) \in U\}.$$

Since $x \in \omega(y)$, the integer t_y is welldefined. Also define

$$T = \sup\{t_y \mid y \in \overline{U}\}.$$

Here, we claim that $T < \infty$. By the definition of t_y, for any point y in $\overline{U}$ $f^{t_y}(y) \in U$. So, we can choose an open neighborhood V_y of y such that $f^{t_y}(V_y) \subset U$. Note that, for any $z \in V_y$, $t_x \leq t_y$ holds. Then $\{V_y \mid y \in \overline{U}\}$ forms an open covering of $\overline{U}$. Therefore, there exists a finite open covering $\{V_{y_1}, V_{y_2}, \cdots, V_{y_n}\}$ of $\overline{U}$. Then for any z in $\overline{U}$,

$$t_z \leq \max\{t_{y_1}, t_{y_2}, \cdots, t_{y_n}\}$$

holds. Thus we get $T < \infty$.

Also, define the integer $r_y \leq 0$ by

$$r_y = \max\{n \leq 0 \mid f^n(y) \in U\}.$$

The integer r_y is well defined since $x \in \alpha(y)$. Therefore, we get

$$y \in f^{[0,t_y - r_y]}(f^{r_y}(y)) \in f^{[0,t_y - r_y]}(U) \quad \text{and} \quad t_y - r_y \leq T.$$

Thus, we get

$$y \in f^{[0.t_y - r_y]}(U) \subset f^{[0,T]}(\overline{U}) \subset f^{[0,T]}(W).$$

This shows that $J^+(x) \subset f^{[0,\ T]}(W)$ and so, we conclude that x is an uniformly recurrent point. This completes the proof of this proposition. □

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Department of Computer and Information Security
Daejon University
Daejon 300-716, Republic of Korea
E-mail: kskoo@dju.kr