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NORMALIZED DUALITY MAPPING AND GENERALIZED BEST APPROXIMATIONS

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Abstract. In this paper, we introduce certain concepts which provide us with a perspective and insight into the generalization of orthogonality with the normalized duality mapping. The material of this paper will be mainly, but not only, used in developing algorithms for the best approximation problem in a Banach space.

1. Introduction

Let E be a real Banach space with the norm $||\cdot||$ and let E^* be the dual space of E. Denote by $<\cdot,\cdot>$ the duality product. The normalized duality mapping J from E to E^* is defined by

$$Jx = \{x^* \in E^* :< x, x^* >= ||x||^2 = ||x^*||^2\}$$

for all $x \in E$. Hahn-Banach theorem guarantees that $Jx \neq \emptyset$ for every $x \in E$.

A Banach space E is said to be strictly convex if $||\frac{x+y}{2}|| < 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$. A Banach space E is said to be uniformly convex if $\lim_{n\to\infty} ||x_n - y_n|| = 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in E such that $||x_n|| = ||y_n|| = 1$ and $\lim_{n\to\infty} ||\frac{x_n+y_n}{2}|| = 1$. Let $S(E) = \{x \in E : ||x|| = 1\}$ be the unit sphere of E. The Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{||x + ty|| - ||x||}{t}$$

exists for each $x, y \in S(E)$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in S(E)$. It is well known that if E is smooth, then the duality mapping is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous

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on each bounded subset of E. Some properties of the normalized duality mapping have been given in [5, 6].

Let E be a smooth Banach space and let E^* be the dual of E. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(y, x) = ||y||^2 - 2 < y, Jx > +||x||^2$$

for all $x, y \in E$, where J is the normalized duality mapping from E to E^*. It is obvious from the definition of the function ϕ that

$$(||y|| - ||x||)^2 \leq \phi(y, x) \leq (||y|| + ||x||)^2 \tag{1}$$

for all $x, y \in E$.

In what follows we recall from [1] some examples for the mapping J in the uniformly convex and uniformly smooth Banach spaces ℓ^p and L^p, $p \in (1, \infty)$.

$\diamond$ For $\ell^p : Jx = ||x||_{\ell^p}^{2-p} y \in \ell^q$, $x = \{x_1, x_2, \cdots\}$,
$y = \{x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \cdots\}$,
$p^{-1} + q^{-1} = 1$.

$\diamond$ For $L^p : Jx = ||x||_{L^p}^{2-p} |x|^{p-2} x \in L^q$, $p^{-1} + q^{-1} = 1$.

In section 2, we define a new orthogonality concept, that is called a J-orthogonality in a smooth Banach space, by using the normalized duality mapping which is equivalent to the Birhkoff orthogonality in a Banach space, and give some basic properties of J-orthogonality in a smooth Banach space.

In [6], Matsushita and Takahashi gave a characterization of the generalized best approximation from a closed convex subset of a smooth Banach space E. In section 3, we find a best approximation to an element of a smooth Banach space E from a closed subspace of E and characterizations of the generalized best approximation.

2. J-orthogonality

In this section, we will study a kind of orthogonality by using the normalized duality mapping. First we will give some results about normalized duality mapping.

Proposition 2.1. [2] *(a) Jx is convex and $\sigma(E^*, E)$-closed. $J(\alpha x) = \alpha Jx$ for all $\alpha \in \mathbb{R}$.*

(b) For each $x \in S(E)$, Jx is a weak compact convex extremal subset of $S(E^*) = \{x^* \in E^* : ||x^*|| = 1\}$. In particular, Jx has extreme points, each extreme point of Jx is an extreme of $S(E^*)$, and Jx is the weak* closed convex hull of its set of extreme points.*

(c) J is norm-weak upper semi-continuous. That is, if $x_0 \in E$ and W is a weak* open sets with $Jx_0 \subset W$, then there exists an open neighborhood U of x_0 such that $Jx \subset W$ for all $x \in U$.*

(d) J is a map if and only if E^* is strictly convex. In particular, $J = I$ if E is Hilbert.

It is natural to ask under what conditions J is linear. It turns out that this completely characterize a Hilbert space.

Definition 2.2. A selection for the normalized duality mapping J is a function $s : E \to E^*$ such that $s(x) \in Jx$ for every $x \in E$. That is, $||s(x)|| = ||x||$ and $< x, s(x) >= ||x||^2$.

Theorem 2.3. [2] *The following statements are equivalent for a Banach space E.*

(1) E is a Hilbert space.

(2) Every selection for J is linear.

(3) There exists a selection for J which is linear.

(4) J is "additive", i.e., $J(x + y) = Jx + Jy$.

(5) J is "sub-additive", i.e., $Jx + Jy \subset J(x + y)$.

Proposition 2.4. [6] *If E is a strictly convex and smooth Banach space, then for any $x, y \in E$, $\phi(y, x) = 0$ if and only if $x = y$.*

Proof. It suffices to show that if $\phi(y, x) = 0$, then $x = y$. By (1), we have $||x|| = ||y||$. Then

$$< y, Jx >= ||y||^2 = ||x||^2 = ||Jx||^2.$$

By the definition of J, we have $Jx = Jy$. Since J is one-to-one, we have $x = y$. □

Now we define a new orthogonality concept in a Banach space.

Definition 2.5. Let E be a smooth Banach space and $x, y \in E$. If $< y, Jx >= 0$ or $\phi(y, x) = ||x||^2 + ||y||^2$, then x is J-orthogonal to y and denotes $x \perp^J y$.

Definition 2.6. Let E be a smooth Banach space and let $x_1, \cdots, x_n \in E \backslash \{0\}$.

(1) $\{x_1, \cdots, x_n\}$ is J-orthogonal if for any $i, j \in \{1, \cdots, n\}$ with $i \neq j$, $x_i \perp^J x_j$.

(2) If $\{x_1, \cdots, x_n\}$ is J-orthogonal and for each $i \in \{1, \cdots, n\}$, $||x_i|| = 1$, we say that $\{x_1, \cdots, x_n\}$ is J_1-orthogonal.

Lemma 2.7. *Let M be a closed subspace of a Banach space E and let $x \in E$. Then $0 \in P_M(x)$ if and only if there exists $f \in Jx$ such that $< m, f >= 0$ for all $m \in M$.*

Proof. By the characterization of a best approximation from a subspace, $0 \in P_M(x)$ if and only if there exists $f \in E^*$ such that $||f|| = 1$, $< m, f >= 0$ for all $m \in M$, and $< x, f >= ||x||$ if and only if there exists $f \in Jx$ such that $< m, f >= 0$ for all $m \in M$. □

With the above definition, we get the following properties.

Proposition 2.8. *Let E be a smooth Banach space and let $x_1, \cdots, x_n \in E \backslash \{0\}$.*

(1) If $\{x_1, \cdots, x_n\}$ is J-orthogonal, then $\{x_1, \cdots, x_n\}$ is linearly independent.

(2) $x \perp^J y$ if and only if $x \perp y$ in the Birkhoff sense, i.e., $||x + \alpha y||^2 \geq ||x||^2$ for all $\alpha \in \mathbb{R}$.

Proof. (1) Let $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$. Then for each $i \in \{1, \cdots, n\}$,

$$\begin{aligned} < \alpha_1 x_1 + \cdots \alpha_n x_n, Jx_i > &= \alpha_1 < x_1, Jx_i > + \cdots + \alpha_n < x_n, Jx_i > \\ &= \alpha_i ||x_i||^2 = 0, \end{aligned}$$

so $\alpha_i = 0$. Thus $\{x_1, \cdots, x_n\}$ is linearly independent.

(2) Suppose $x \perp^J y$. Then $< y, Jx >= 0$ and

$$\begin{aligned} \phi(x + \alpha y, x) &= ||x + \alpha y||^2 - 2 < x + \alpha y, Jx > + ||x||^2 \\ &= ||x + \alpha y||^2 - ||x||^2 - 2\alpha < y, Jx > \\ &= ||x + \alpha y||^2 - ||x||^2 \geq 0 \end{aligned}$$

for all $\alpha \in \mathbb{R}$. Thus $||x + \alpha y||^2 \geq ||x||^2$ for all $\alpha \in \mathbb{R}$. Hence $x \perp y$ in the Birkhoff sense.

Suppose that $x \perp y$ in the Birkhoff sense, i.e., $||x + \alpha y||^2 \geq ||x||^2$ for all $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} \phi(x + \alpha y, x) &= ||x + \alpha y||^2 - 2 < x + \alpha y, Jx > + ||x||^2 \\ &= ||x + \alpha y||^2 - ||x||^2 - 2\alpha < y, Jx > \\ &\geq 0 \end{aligned}$$

for all $\alpha \in \mathbb{R}$. If $< y, Jx > \neq 0$ then renotes by $\alpha' = \frac{||x+\alpha y||^2 - ||x||^2}{<y,Jx>}$,

$$\phi(x + \alpha' y, x) < 0.$$

This is a contradiction for $\phi(x, y) \geq 0$. □

As usual, we have the following properties.

PROPOSITION 2.9. *If $\{x_1, \cdots, x_n\}$ is a J_1-orthogonal set in a smooth Banach space E whose the dual space E^* is strictly convex, then $\{Jx_1, \cdots, Jx_n\}$ is linearly independent in the dual space E^*.*

Proof. Let $\alpha_1 Jx_1 + \cdots + \alpha_n Jx_n = 0$. Then for each $i \in \{1, \cdots, n\}$,

$$< x_i, \alpha_1 Jx_1 + \cdots \alpha_n Jx_n >= \alpha_i = 0.$$

Thus $\{Jx_1, \cdots, Jx_n\}$ is linearly independent in the dual space E^*. □

Let E be a Banach space and let E^* be the dual space of E. The normalized duality mapping J^* from E^* to E^{**} is defined by

$$J^*x^* = \{x^{**} \in E^{**} :< x^*, x^{**} >= ||x^*||^2 = ||x^{**}||^2\}$$

for all $x^* \in E^*$. If E is reflexive, then

$$J^*x^* = \{x \in E :< x, x^* >= ||x||^2 = ||x^*||^2\}$$

for all $x^* \in E^*$.

PROPOSITION 2.10. *Let E be a reflexive and smooth Banach space. Then $\{x_1, \cdots, x_n\}$ is J-orthogonal if and only if $\{Jx_1, \cdots, Jx_n\}$ is J^*-orthogonal.*

Proof. If $i \neq j$, $< x_i, Jx_j >= 0$. Note that

$$\begin{aligned} < Jx_i, J^*(Jx_j) > &= < Jx_i, x_j > \\ &= \widehat{x_j}(Jx_i) = (Jx_i)(x_j) \\ &= < x_j, Jx_i >= 0. \end{aligned}$$

□

LEMMA 2.11. [3] *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

DEFINITION 2.12. Let S be any nonempty subset of a smooth Banach space E. The J-dual cone of S is the set

$$S_J^0 = \{x \in E :< y, Jx >\leq 0 \text{ for all } y \in S\}.$$

The J-orthogonal complement of S is the set

$$S_J^{\perp} = S_J^0 \cap (-S)_J^0 = \{x \in E :< y, Jx >= 0 \text{ for all } y \in S\}.$$

By the definition 2.0.12, we have some basic results about the J-dual cone and the J-orthogonal complement of a set S.

Theorem 2.13. *Let S be a nonempty subset of a smooth Banach space E. Then*

(1) S^0_J is a closed cone and $S^{\perp}_J$ is a closed cone.

(2) $S^0_J = (\overline{S})^0_J$ and $S^{\perp}_J = (\overline{S})^{\perp}_J$.

(3) $S^0_J = [con(S)]^0_J = \overline{[con(S)]}^0_J$ and $S^{\perp}_J = [span(S)]^{\perp}_J = \overline{[span(S)]}^{\perp}_J$ where

$con(S)$ is the convex hull of S and $span(S)$ is the subspace generated by S.

(4) $\overline{S} \subset (S^0_J)^0$ and $\overline{S} \subset (S^{\perp}_J)^{\perp}$.

(5) If C is a cone, then $(C-y)^0_J = C^0_J \bigcap y^{\perp}_J$ for each $y \in C$.

(6) If M is a subspace, then $M^0_J = M^{\perp}_J$.

Proof. (1) Let $x_n \in S^0_J$ and $x_n \to x$. Then for any $y \in S$

$$< y, Jx >= \lim_{n\to\infty} < y, Jx_n > \leq 0$$

implies $x \in S^0_J$ and S^0_J is closed.

Let $x \in S^0_J$ and $\alpha \geq 0$. Then, by Proposition 2.0.1, for all $y \in S$,

$$< y, J(\alpha x) > = < y, \alpha Jx > = \alpha < y, Jx > \leq 0.$$

Thus $\alpha x \in S^0_J$, so S^0_J is a cone. Since $S^{\perp}_J = (S^0_J) \cap (-S)^0_J$, $S^{\perp}_J$ is a closed cone.

(2) Since $S \subseteq \overline{S}$, $(\overline{S})^0_J \subset S^0_J$. If $x \in S^0_J$ and $y \in \overline{S}$, choose $y_n \in S$ such that $y_n \to y$. Then $< y, Jx >= \lim_{n\to\infty} < y_n, Jx > \leq 0$ implies $x \in (\overline{S})^0_J$. Thus $S^0_J = (\overline{S})^0_J$. Moreover, $S^{\perp}_J = (\overline{S})^{\perp}_J$.

(3) Since $S \subset con(S)$, $[con(S)]^0_J \subset S^0_J$. Let $x \in S^0_J$ and $y \in con(S)$. By the definition of $con(S)$, $y = \sum_{i=1}^n \rho_i y_i$ for some $y_i \in S$ and $\rho_i \geq 0$ with $\sum_{i=1}^n \rho_i = 1$. Then

$$< y, Jx >= \sum_{i=1}^n \rho_i < y_i, Jx > \leq 0$$

implies $x \in [con(S)]^0_J$, so $S^0_J \subset [con(S)]^0_J$. Thus $S^0_J = [con(S)]^0_J$. Moreover, $S^{\perp}_J = [span(S)]^{\perp}_J = \overline{[span(S)]}^{\perp}_J$.

(4) Let $x \in S$. Then for all $y \in S^0_J$, $< x, Jy > \leq 0$. So $x \in S^{00}_J$. Thus $S \subset S^{00}_J$. Since S^{00}_J is closed, $\overline{S} \subset S^{00}_J$.

(5) Now $x \in (C-y)^0_J$ if and only if $< c-y, Jx > \leq 0$ for all $c \in C$. Let $x \in (C-y)^0_J$. Then $< c-y, Jx > \leq 0$ for all $c \in C$. Taking $c = 0$ and $c = 2y$, we have $< y, Jx >= 0$ and $< c, Jx > \leq 0$ for all $c \in C$. Thus $x \in C^0_J \cap y^{\perp}_J$. Moreover, if $x \in C^0_J \cap y^{\perp}_J$, then $< c, Jx > \leq 0$ and

$< y, Jx >= 0$ for all $c \in C$. So $< c - y, Jx > \leq 0$ for all $c \in C$. Thus $x \in (C - y)^0_J$. Therefore,

$$(C - y)^0_J = C^0_J \cap y^{\perp}_J$$

for each $y \in C$.

(6) If M is a subspace, then $-M = M$ implies

$$M^0_J = M^0_J \cap (-M)^0_J = M^{\perp}_J.$$

$\square$

Generally, because J is not additive, S^0_J is not convex even though S is convex. Moreover, $M^{\perp}_J$ is not a subspace even though M is a subspace.

3. Characterization of The Generalized Best Approximations

Let C be a nonempty closed convex subset of E. Suppose that E is a reflexive, strictly convex and smooth Banach space. Let $x \in E$ be given. If there exists a point $x_0 \in C$ such that

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x) := \phi(C, x)$$

then x_0 is called the best J-approximation or the generalized best approximation of x from C. The mapping $P^J_C : E \to C$ defined by $P^J_C(x) = x_0$ is called the J-metric projection or the generalized metric projection. The generalized metric projection P^J_C is fixed in each point $y \in C$, so P^J_C is idempotent. Moreover P^J_C is monotone in E, that is,

$$< P^J_C(x) - P^J_C(y), Jx - Jy > \geq 0$$

for any $x, y \in E$. We can find more results in [1].

Let C be a nonempty closed convex subset of E. By Alber[1] or Kamimura and Takahashi[4], for each $x \in E$, there exists a unique best J-approximation of x from C. If E is a Hilbert space, then P^J_C is coincident with the metric projection from E onto C. We also know the following proposition.

PROPOSITION 3.1. [4, 6] *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = P^J_C(x)$ if and only if*

$$< x_0 - y, Jx - Jx_0 > \geq 0$$

for all $y \in C$.

Proof. Let $y \in C$ and let $\lambda \in (0,1)$. Then

$$\phi(x_0, x) \leq \phi((1-\lambda)x_0 + \lambda y, x).$$

So,

$$\begin{aligned}
0 &\leq ||(1-\lambda)x_0 + \lambda y||^2 - 2 < (1-\lambda)x_0 + \lambda y, Jx > + ||x||^2 \\
&\quad - ||x_0||^2 + 2 < x_0, Jx > - ||x||^2 \\
&= ||(1-\lambda)x_0 + \lambda y||^2 - ||x_0||^2 - 2\lambda < y - x_0, Jx > \\
&\leq 2\lambda < y - x_0, J((1-\lambda)x_0 + \lambda y) > -2\lambda < y - x_0, Jx > \\
&= 2\lambda < y - x_0, J((1-\lambda)x_0 + \lambda y) - Jx > .
\end{aligned}$$

Since $2\lambda < x_0 - y, J((1-\lambda)x_0 + \lambda y_0) > \leq ||x_0||^2 - ||(1-\lambda)x_0 + \lambda y||^2$

$$< y - x_0,\ J((1-\lambda)x_0 + \lambda y) - Jx > \geq\ 0.$$

Taking the limit $\lambda \downarrow 0$, we obtain

$$< y - x_0,\ Jx_0 - Jx > \geq\ 0$$

since J is norm-to-weak* continuous. Thus

$$< x_0 - y,\ Jx - Jx_0 > \geq 0$$

for all $y \in C$.

Conversely, for any $y \in C$, we have

$$\begin{aligned}
\phi(y, x) - \phi(x_0, x) &= ||y||^2 - 2 < y, Jx > + ||x||^2 -\ ||x_0||^2 \\
&\qquad + 2 < x_0, Jx > - ||x||^2 \\
&= ||y||^2 -\ ||x_0||^2 - 2 < y - x_0,\ Jx > \\
&\geq 2 < y - x_0,\ Jx_0 > -2 < y - x_0,\ Jx > \\
&=\ 2 < y - x_0,\ Jx_0 - Jx > \\
&\geq\ 0.
\end{aligned}$$

Thus $x_0 = P_C^J(x)$. □

Corollary 3.2. *Let C be a closed convex subset of the innner product space E, $x \in E$ and $y_0 \in C$. Then $x_0 \in P_C(x)$ if and only if*

$$< x - x_0,\ y - x_0 > \leq\ 0$$

for all $y \in C$.

By the previous proposition, we have the characterization of generalized best approximation for a subspace.

Proposition 3.3. *Let M be a closed subspace of a reflexive, strictly convex and smooth Banach space E, $x \in E$ and $x_0 \in M$. Then $x_0 = P_M^J(x)$ if and only if*

$$< m,\, Jx - Jx_0 >= 0$$

for all $m \in M$.

Proof. Suppose that $x_0 = P_M^J(x)$. Since M is a subspace, $x_0 - m \in M$ for all $m \in M$. By Proposition 3.0.14,

$$< x_0 - (x_0 - m),\, Jx - Jx_0 >=< m,\, Jx - Jx_0 > \geq\ 0$$

for all $m \in M$. Similarly, we have

$$< x_0 - (x_0 + m),\, Jx - Jx_0 >=< -m,\, Jx - Jx_0 > \geq\ 0$$

for all $m \in M$. So,

$$< m,\, Jx - Jx_0 > \leq\ 0$$

for all $m \in M$. Thus,

$$< m,\, Jx - Jx_0 >= 0$$

for all $m \in M$.

Conversely, suppose that $< m,\, Jx - Jx_0 >= 0$ for all $m \in M$. Since $x_0 - m \in M$ for all $m \in M$, we have

$$< x_0 - m,\, Jx - Jx_0 > =\ 0$$

for all $m \in M$. So,

$$< x_0 - m,\, Jx - Jx_0 > \geq\ 0$$

for all $m \in M$. Thus $x_0 = P_M^J(x)$. □

Example 3.4. For $p \in (1, \infty)$, $\ell^p(2)$ is a uniformly convex and uniformly smooth Banach space. In $E = \ell^p(2)$, for each $x = (x_1, x_2) \in E$

$$J(x) = ||x||_p^{2-p}(x_1|x_1|^{p-2}, x_2|x_2|^{p-2}) \in \ell^q(2)$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Consider a closed subspace M of E which is generated by $(1, 0)$. By proposition 3.0.16, if $x = (x_1, x_2)$, then

$$x_0 = (x_0, 0) = P_M^J(x) \ \ \Leftrightarrow\ < (t, 0), Jx - Jx_0 > =\ 0$$

for all $t \in \mathbb{R}$.

$$\Leftrightarrow\ < (t, 0), ||x||_p^{2-p}(x_1|x_1|^{p-2}, x_2|x_2|^{p-2}) >$$
$$=< (t, 0), ||x_0||_p^{2-p}(x_0|x_0|^{p-2}, 0) >$$

for all $t \in \mathbb{R}$.

$$\Leftrightarrow\ \ ||x||_p^{2-p}x_1|x_1|^{p-2}t = ||x_0||_p^{2-p}x_0|x_0|^{p-2}t = x_0 t$$

for all $t \in \mathbb{R}$.

$$\Leftrightarrow \quad x_0 = ||x||_p^{2-p} x_1 |x_1|^{p-2}.$$

Hence $P_M^J(x) = P_M^J((x_1, x_2)) = (||x||_p^{2-p} x_1|x_1|^{p-2}, 0)$ for each $x \in E$.

Corollary 3.5. *Let M be a closed subspace of an inner product space E, $x \in E$ and $x_0 \in M$. Then $x_0 = P_M^J(x)$ if and only if*

$$< m, x - x_0 > = 0.$$

Corollary 3.6. *If M is a closed subspace of E, then $P_M^J(x) = 0$ if and only if $x \perp^J M$.*

Example 3.7. For $p \in (1, \infty)$, $\ell^p(2)(= \mathbb{R}_p^2)$ is a uniformly convex and uniformly smooth Banach space. Let $M = [(1, 0)]$ and $x \in E$. Then

$$P_M^J(x) = \{(0, 0)\} \quad \Leftrightarrow \quad x = [(0, 1)] \quad \Leftrightarrow \quad x \in M_J^{\perp}.$$

Corollary 3.8. *If M is a closed subspace of E, then P_M^J is homogeneous.*

Proof. Let $x_0 \in P_M^J(x)$. Then

$$< m, Jx - Jx_0 >= 0$$

for all $m \in M$. So for each real number α,

$$\begin{aligned} < m, J(\alpha x) - J(\alpha x_0) > &= < m, \alpha Jx - \alpha Jx_0 > \\ &= \alpha < \frac{m}{\alpha}, Jx - Jx_0 > \\ &= 0 \end{aligned}$$

for all $m \in M$. Thus $P_M^J(\alpha x) = \alpha P_M^J(x) = \alpha x_0$. □

In [6], Matsushita and Takahashi gave the following result.

Proposition 3.9. [**4**, **6**] *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, P_C^J(x)) + \phi(P_C^J(x), x) \leq \phi(y, x)$$

for all $y \in C$.

Proof. By proposition 3.0.14,

$$\begin{aligned} \phi(y,x) - \phi(y, P_C^J(x)) - \phi(P_C^J(x), x) &= ||y||^2 - 2 < y, Jx > + ||x||^2 - ||P_C^J(x)||^2 \\ &\quad + 2 < P_C^J(x), Jx > - ||x||^2 - ||y||^2 + 2 < y, JP_C^J(x) > - ||P_C^J(x)||^2 \\ &= -2 < y, Jx > + 2 < P_C^J(x), Jx > + 2 < y - P_C^J(x), JP_C^J(x) > \end{aligned}$$

$$= -2 < y - P_C^J(x), Jx > +2 < y - P_C^J(x), JP_C^J(x) >$$

for all $y \in C$.

By corollary 3.0.16, we have the following result for a closed subspace,

$$2 < y - P_C^J(x), JP_C^J(x) - Jx > \geq \ 0$$

$\square$

PROPOSITION 3.10. *Let E be a reflexive, strictly convex and smooth Banach space, let M be a nonempty closed subspace of E and let $x \in E$. Then*

$$\phi(y, P_M^J(x)) + \phi(P_M^J(x), x) = \phi(y, x)$$

for all $y \in M$.

Proof. By the definition of ϕ and proposition 3.0.16, we have

$$\begin{aligned}
&\phi(y, x) - \phi(P_M^J(x), x) - \phi(y, P_M^J(x)) \\
&\quad = ||y||^2 - 2 < y, Jx > +||x||^2 - ||P_M^J(x)||^2 \\
&\qquad +2 < P_M^J(x), Jx > -||x||^2 - ||y||^2 \\
&\qquad +2 < y, JP_M^J(x) > -||P_M^J(x)||^2 \\
&\quad = -2 < y, Jx > +2 < P_M^J(x), Jx > \\
&\qquad +2 < y, JP_M^J(x) > -2 < P_M^J(x), P_M^J(x) > \\
&\quad = 2 < y - P_M^J(x), JP_M^J(x) - Jx > \\
&\quad = 0
\end{aligned}$$

for all $y \in M$. Thus $\phi(y, P_M^J(x)) + \phi(P_M^J(x), x) = \phi(y, x)$ for all $y \in M$. $\square$

Now we verify corollary 3.0.23, in a example.

EXAMPLE 3.11. For $p \in (1, \infty)$, $\ell^p(2)$, is a uniformly convex and uniformly smooth Banach space. In example 3.0.17, we found the generalized best approximation of $x \in \ell^p(2)$. Note that

$$\begin{aligned}
\phi((t,0), P_M^J(x)) + \phi(P_M^J(x), x) &= \phi((t,0), (||x||_p^{2-p} x_1 |x_1|^{p-2}, 0)) \\
&\quad +\phi((||x||_p^{2-p} x_1 |x_1|^{p-2}, 0), (x_1, x_2)) \\
&= t^2 - 2||x||_p^{2-p} t x_1 |x_1|^{2-p} + 2(||x||_p^{2-p} x_1 |x_1|^{p-2})^2 \\
&\quad -2(||x||_p^{2-p} x_1 |x_1|^{p-2})^2 + ||x||_p^2 \\
&= \phi((t,0), x)
\end{aligned}$$

for all $(t, 0) \in M$.

We know the characterization for best approximation for a closed subspace in a normed linear space. Next we will give the characterization by use the normalized duality mapping.

THEOREM 3.12. *Let E be a normed linear space, let M be a subspace of E, $x \in E$ and $m_0 \in M$. Then $m_0 \in P_M(x)$ if and only if there exists $j(x-m_0) \in J(x-m_0)$ such that $< m, j(x-m_0) >= 0$ for all $m \in M$.*

Proof. Let $x \in E \backslash M$. Then $||x-m_0|| = d(x, M) > 0$. By Hahn-Banach Theorem, there exists $f_0 \in E^*$ such that $||f_0|| = 1, < m, f_0 >= 0$ for all $m \in M$ and $< x, f_0 >= ||x-m_0||$. Set $j(x-m_0) = ||x-m_0|| f_0$. Then $j(x-m_0) \in E^*$, $< m, j(x-m_0) >= 0$, for all $m \in M$ and $||j(x-m_0)|| = ||x-m_0||$.

Conversely, suppose that there exists $j(x-m_0) \in J(x-m_0)$ such that

$$< m, j(x-m_0) >= 0, \ \ ||j(x-m_0)|| = ||x-m_0||$$

for all $m \in M$. Then

$$\begin{aligned} ||x-m_0||^2 &=< x-m_0, j(x-m_0) > \\ &=< x-m, j(x-m_0) > \\ &\leq ||x-m|| \, ||j(x-m_0)|| \\ &= ||x-m|| \, ||x-m_0|| \end{aligned}$$

for all $m \in M$. So

$$||x-m_0|| \leq ||x-m||$$

for all $m \in M$. Thus $m_0 \in P_M(x)$. □

COROLLARY 3.13. *Let E be a normed linear space, M be a subspace of E, $x \in E \backslash M$ and $G \subset M$. Then $G \subset P_M(x)$ if and only if for each $g_0 \in G$ there exists an $j(x-g_0) \in J(x-g_0)$ such that $< m, j(x-g_0) >= 0$ for all $m \in M$.*

Proof. Suppose that $G \subset P_M(x)$. Then there exists $g_0 \in G$ such that $g_0 \in P_M(x)$. By the previous theorem, for each $g_0 \in G$, there exists an $j(x-g_0) \in J(x-g_0)$ such that $< m, j(x-g_0) >= 0$ for all $m \in M$.

Conversely, suppose that for each $g_0 \in G$ there exists an $j(x-g_0) \in J(x-g_0)$ such that $< m, j(x-g_0) >= 0$ for all $m \in M$. By the previous theorem, $g_0 \in P_M(x)$. Thus $G \subset P_M(x)$. □

REMARK 3.14. (1) If for some $m_0 \in P_M(x)$ there exists $j(x-m_0) \in J(x-m_0)$ such that $< m, j(x-m_0) >= 0$ for all $m \in M$, then $j(x-m_0) \in J(x-m)$ for all $m \in P_M(x)$.

Proof. Suppose that for some $m_0 \in P_M(x)$ there exists $j(x-m_0) \in J(x-m_0)$ such that $< m, j(x-m_0) >= 0$ for all $m \in M$. Then

$$\begin{aligned} ||x-m_0||^2 &= < x-m_0, j(x-m_0) > \\ &= < x-m,\, j(x-m_0) > \\ &\leq ||x-m||\, ||j(x-m_0)||, \end{aligned}$$

so $< x-m,\, j(x-m_0) >= ||x-m_0||^2 = ||x-m||^2$ for all $m \in P_M(x)$ and $j(x-m_0) \in J(x-m)$ for all $m \in P_M(x)$. □

(2) Because J is not additive when E is not a Hilbert space, we cannot give the characterization of generalized best approximation such as the above characterization of best approximation.

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