

INTEGRAL DOMAINS WHICH ARE t -LOCALLY NOETHERIAN

HWANKOO KIM* AND TAE IN KWON**

ABSTRACT. In this note, a module-theoretic characterization of t -locally Noetherian domains is given. We also give some characterizations of strong Mori domains via t -locally Noetherian domains.

1. Introduction

Strong Mori domains were introduced by Wang and McCasland in [11]. It is well known that every strong Mori domain is t -locally Noetherian ([12]). However, as Example 2.1 shows, the converse is not true in general. The aim of this note is to give a module-theoretic characterization of t -locally Noetherian domains and some characterizations of strong Mori domains via t -locally Noetherian domains. In order to do so, we first review some notions and terminologies.

Throughout, let R be an integral domain with quotient field K . Let $\mathbf{F}(R)$ be the set of nonzero fractional ideals of R . For an $I \in \mathbf{F}(R)$, define $I^{-1} = \{x \in K \mid xI \subseteq R\}$. The v -operation on R is a mapping on $\mathbf{F}(R)$ defined by $I \mapsto I_v = (I^{-1})^{-1}$. The t -operation on R is defined by $I \mapsto I_t = \bigcup \{J_v \mid J \text{ is a nonzero finitely generated fractional subideal of } I\}$. Recall that an ideal J of R is called a *Glaz-Vasconcelos ideal* (GV-ideal) if J is finitely generated and $J^{-1} = R$. We denote the set of GV-ideals by $GV(R)$. The w -operation on R is a mapping on $\mathbf{F}(R)$ defined by $I \mapsto I_w = \{x \in K \mid Jx \subseteq I \text{ for some } J \in GV(R)\}$. An $I \in \mathbf{F}(R)$ is said to be a v -ideal (resp., t -ideal, w -ideal) if $I_v = I$ (resp.,

Received September 23, 2011; Accepted November 18, 2011.

2010 Mathematics Subject Classification: Primary 13A15; Secondary 13F05.

Key words and phrases: t -locally Noetherian, strong Mori domain.

Correspondence should be addressed to Hwankoo Kim, hkkim@hoseo.edu

*This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0011996).

**This research is financially supported by Changwon National University in 2009-2010.

$I_t = I, I_w = I$). A t -ideal (resp., w -ideal) M of R is called a *maximal t -ideal* (resp., *maximal w -ideal*) if M is maximal among proper integral t -ideals (resp., w -ideals) of R . Let $t\text{-Max}(R)$ (resp., $w\text{-Max}(R)$) be the set of maximal t -ideals (resp., maximal w -ideals) of R . It was shown that $t\text{-Max}(R) = w\text{-Max}(R)$ and the notion of t -invertibility is precisely same as that of w -invertibility. An integral domain R is a *strong Mori domain* (SM domain) if it satisfies the ascending chain condition (ACC) on integral w -ideals of R ([12]).

Let M be a module over an integral domain R . Set $\mathfrak{r}(M) := \{x \in M \mid \text{ann}_R(x)_w = R\}$. Following [8], the w -envelope of M is defined by $M_w = p^{-1}(\mathfrak{r}(E(M)/M))$, where $E(M)$ denotes the *injective envelope* (or *injective hull*) of M and $p : E(M) \rightarrow E(M)/M$ is the canonical projection. Then it is easy to see that $M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in GV(R)\}$. M is said to be *co-semi-divisorial* if $\mathfrak{r}(M) = 0$; equivalently, if whenever $Jx = 0$ for some $J \in GV(R)$ and $x \in M$, we have that $x = 0$. Any undefined terminology is standard, as in [4].

2. Main results

As the t -theoretic analog, an integral domain R is said to be *t -locally Noetherian* if R_P is a Noetherian domain for all maximal t -ideals P of R . It is easy to verify that R is t -locally Noetherian if and only if R_P is Noetherian for all prime w -ideals P of R . Some ring-theoretic characterizations of t -locally Noetherian domains are given in [2, Theorem 1,4]: R is t -locally Noetherian, if and only if $R[X]$ is t -locally Noetherian, if and only if $R[X]_{N_v}$ is t -locally Noetherian, if and only if $R[X]_{N_v}$ is locally Noetherian, where $N_v = \{f \in R[X] \mid C(f)_v = R\}$ and $C(f)$ is the content ideal generated by the coefficients of f .

Every SM domain is t -locally Noetherian. However, as the following example shows, the converse is not true in general. Following [7, Section IV], an integral domain R is called *t -almost Dedekind domain* if R_M is a rank-one discrete valuation domain for each maximal t -ideal M of R .

EXAMPLE 2.1. Let R be a t -almost Dedekind domain which is not a Krull domain. Then R is t -locally Noetherian, but not an SM domain since a Krull domain is exactly an integrally closed SM domain.

Let M be an R -module. If M has a non-zero element c with the following condition (*), then we call M a *cocyclic R -module*.

(*) For every R -module N , every R -homomorphism $\phi : M \rightarrow N$ with $c \notin \ker(\phi)$ is monic.

Then it is easily seen that M is a cocyclic R -module if and only if every non-zero submodule of M contains c , i.e. Rc is the smallest submodule of M .

PROPOSITION 2.1. *A co-semi-divisorial R -module M is cocyclic if and only if M is an essential extension of R/P for some maximal w -ideal P of R .*

Proof. This follows from a slight modification of the proof of [10, Proposition 1]. \square

LEMMA 2.2. *Let R be a t -locally Noetherian domain and M be a cocyclic co-semi-divisorial R -module. Then M has the structure of an R_P -module for some maximal w -ideal P of R .*

Proof. This follows from a slight modification of the proof of [10, Proposition 2] using Proposition 2.1. \square

THEOREM 2.3. *The following statements are equivalent for an integral domain R .*

- (1) R is a t -locally Noetherian domain.
- (2) Every cocyclic co-semi-divisorial R -module satisfies the DCC on submodules.
- (3) For every maximal w -ideal P of R , $E(R/P)$ satisfies the DCC on submodules.

Proof. (1) \Rightarrow (2) Let M be a cocyclic co-semi-divisorial R -module. Then there exists a maximal w -ideal P of R such that $R/P \subseteq M \subseteq E := E(R/P)$. By Lemma 2.2, M has an R_P -module structure. Since R_P is Noetherian, by [9, Proposition 3], M satisfies the DCC on submodules as an R_P -module, and hence as an R -module.

(2) \Rightarrow (3) This is trivial.

(3) \Rightarrow (1) Let P be a maximal w -ideal of R . By [10, Lemma 1], $E(R/P)$ has an R_P -module structure and hence $E(R/P)$ satisfies the DCC on submodules as an R_P -module. It follows from [6, Corollary 3.2] that $E(R/P)$ is faithfully injective as an R_P -module. Thus by [6, Theorem 4.1], R_P satisfies the ACC on ideals, i.e. R_P is Noetherian. \square

Let P be a prime ideal of R containing an ideal A of R . Then we call P a Nagata prime of A if there exists a multiplicative system S of R such that $S \cap A = \emptyset$ and such that $PR_S \supseteq AR_S$ and PR_S is maximal with respect to being contained in the set of zero-divisors mod AR_S . It is clear that any prime ideal of R which is minimal with respect to the

property of containing A is a Nagata prime of A and such prime ideals are called *minimal prime divisor of A* .

In order to characterize SM domains via t -locally Noetherian domains, we need a couple of lemmas.

LEMMA 2.4. *Let R be an integral domain which satisfies the ACC on prime w -ideals and suppose that each w -finite ideal of R has only finitely many minimal prime divisors. If P is any prime w -ideal of R , then P is the unique minimal prime divisor of some w -finite ideal A of R .*

Proof. This follows from a slight modification of the proof of [1, Lemma 1.1]. \square

COROLLARY 2.5. *Let R be an integral domain which satisfies the ACC on prime w -ideals. Then R is an SM domain if and only if each w -finite ideal A of R has only finitely many minimal prime divisors and \sqrt{A} is w -finite.*

Proof. (\Rightarrow) By [11, Theorem 4.9 and Proposition 1.6].

(\Leftarrow) Let P be a prime w -ideal of R . Then by Lemma 2.4, P is the unique minimal prime of some w -finite ideal A of R . Thus $P = \sqrt{A}$ and P is w -finite. Thus by [11, Theorem 4.3], R is an SM domain. \square

LEMMA 2.6. *If R is a t -locally Noetherian domain, then R satisfies the ACC on prime w -ideals (and hence on prime t -ideals).*

Proof. Let $P_1 \subseteq P_2 \subseteq \cdots$ be a chain of prime w -ideals of R and set $P := \bigcup_{i=1}^{\infty} P_i$. Then P is a prime w -ideal of R (since w is of finite character) and R_P is Noetherian. Since $P_1 R_P \subseteq P_2 R_P \subseteq \cdots \subseteq P R_P$ is a chain of prime ideals of R_P , there exists an integer n such that $P R_P = P_{n+i} R_P$ for each nonnegative integer i . Hence $P = P_{n+i}$ for each nonnegative integer i . Therefore, R satisfies the ACC on prime w -ideals. \square

Let P be a prime ideal of R containing an ideal A of R . Then P is called a *Bourbaki prime* (resp., *Zariski-Samuel prime*) of A if $P = A :_R x$ (resp., $P = \sqrt{A :_R x}$) for some $x \in R$. Finally we call P a *weak-Bourbaki prime of A* if there exists $x \in R$ such that P is a minimal prime ideal of $A :_R x$.

THEOREM 2.7. *The following conditions are equivalent for a t -locally Noetherian domain R .*

- (1) R is an SM domain.

- (2) Each w -finite ideal of R may be expressed as a finite intersection of primary w -ideals of R .
- (3) Each w -finite ideal of R has only finitely many Bourbaki primes.
- (4) Each w -finite ideal of R has only finitely many Zariski-Samuel primes.
- (5) Each w -finite ideal of R has only finitely many weak-Bourbaki primes.
- (6) Each w -finite ideal I of R has only finitely many minimal prime divisors and \sqrt{I} is w -finite.

Proof. (1) \Rightarrow (2) [11, Theorem 4.11].

(2) \Rightarrow (3) Let I be a w -ideal of R , and let $I = Q_1 \cap \dots \cap Q_n$ be a primary decomposition. It is obvious that if P is a prime ideal of R minimal over I , then $P = \sqrt{Q_i}$ for some Q_i . Thus the number of such P 's is at most finite. Moreover $P = I :_R x$ for some $x \in R$ ([11, Theorem 4.9])

(3) \Rightarrow (4) \Rightarrow (5) This follows from the fact that (weak-Bourbaki prime) \Rightarrow (Zariski-Samuel prime) \Rightarrow (weak-Bourbaki prime) ([5, p.279]).

(5) \Rightarrow (1) This follows from a slight modification of the proof of (vi) \Rightarrow (i) in [3, Theorem 4.5].

(1) \Leftrightarrow (6) This follows from Corollary 2.5 and Lemma 2.6. □

Acknowledgements

We would like to thank the referee for his/her very helpful comments and suggestions which resulted in an improved version of the paper.

References

- [1] J. T. Arnold and J. W. Brewer, *Commutative rings which are locally Noetherian*, J. Math. Kyoto Univ. 11(1971), 45-49.
- [2] G. W. Chang, *Strong Mori domains and the ring $D[X]_{N_v}$* , J. Pure Appl. Algebra **197** (2005), 293-304.
- [3] G. Fusacchia, *Strong semistar Noetherian domains*, Houston J. Math., to appear in Houston J. Math.
- [4] R. Gilmer, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics, 90, Queen's University, Kingston, Ontario, 1992.
- [5] W. Heinzer and J. Ohm, *Locally Noetherian commutative rings*, Trans. Amer. Math. Soc. **158** (1971), 273-284.
- [6] T. Ishikawa, *Faithfully exact functors and their applications to projective modules and injective modules*, Nagoya Math. J. **24** (1964), 29-42.
- [7] B. G. Kang, *Prüfer v -multiplication domains and the ring $R[X]_{N_v}$* , J. Algebra **123** (1989), 151-170.

- [8] H. Kim, *Module-theoretic characterizations of t -linkative domains*, Comm. Algebra **36** (2008), 1649-1670.
- [9] E. Matlis, *Modules with descending chain condition*, Trans. Amer. Math. Soc. **97** (1960), 495-508.
- [10] H. Uda, *On a characterization of almost Dedekind domains*, Hiroshima Math. J. **2** (1972), 339-344.
- [11] F. Wang and R. L. McCasland, *On w -modules over strong Mori domains*, Comm. Algebra **25** (1997), 1285-1306.
- [12] F. Wang and R. L. McCasland, *On strong Mori domains*, J. Pure Appl. Algebra **135** (1999), 155-165.

*

Department of Information Security
Hoseo University
Asan 336-795, Republic of Korea
E-mail: hkkim@hoseo.edu

**

Department of Applied Mathematics
Changwon National University
Changwon, 641-773, Republic of Korea
E-mail: taekwon@changwon.ac.kr