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SOME TOEPLITZ OPERATORS AND THEIR DERIVATIVES

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Abstract. We prove that Toeplitz operators with symbols in RW are bounded and we calculate some upper bounds of the norm of these Toeplitz operators. We also analyze n-th derivative of Toeplitz operators and get some local estimates.

1. Introduction

Let dA denote the normalized area measure on the unit disk $\mathbb{D}$. For any real number α with $\alpha > -1$, $\int_{\mathbb{D}} (1-|z|^2)^{\alpha} dA(z) = \frac{1}{\alpha+1}$ and hence $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^{\alpha} dA(z)$ is a probability measure on $\mathbb{D}$. For $p \geq 1$, the weighted Bergman space L^p_a consists of analytic fuctions on $\mathbb{D}$ which are also in $L^p(\mathbb{D}, dA_\alpha)$. Since L^2_a is a closed subspace of $L^2(\mathbb{D}, dA_\alpha)$, for each $z \in \mathbb{D}$, there exists a fuction K^α_z in L^2_a such that $f(z) =< f, K^\alpha_z >_\alpha$ for every $f \in L^2_a$. The function K^α_z is called the Bergman kernel and we define $k^\alpha_z = \frac{K^\alpha_z}{||K^\alpha_z||_{2,\alpha}}$ which is called the normalized Bergman kernel, where $|| \cdot ||_{p,\alpha}$ is the norm in the space $L^p(\mathbb{D}, dA_\alpha)$ and $< \cdot, \cdot >_\alpha$ is the inner product in the space $L^2(\mathbb{D}, dA_\alpha)$. As is well known ([1],[5]),

$$K^\alpha_z(w) = \frac{1}{(1-\overline{z}w)^{2+\alpha}} \text{ and } k^\alpha_z(w) = \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}}.$$

For a linear operator S on L^2_a, S induces a function $\widetilde{S}$ on $\mathbb{D}$ given by

$$\widetilde{S}(z) = < Sk^\alpha_z, k^\alpha_z >_\alpha, z \in \mathbb{D}.$$

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The function $\widetilde{S}$ is called the Berezin transform of S.

For $u \in L^1(\mathbb{D}, dA)$, the Toeplitz operator T_u^α with symbol u is the operator on L_a^2 defined by $T_u^\alpha(f) = P_\alpha(uf)$, where P_α is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto L_a^2.

Since $L^\infty(\mathbb{D}, dA)$ is dense in $L^1(\mathbb{D}, dA)$ if the Toeplitz operator T_u^α with symbol u in $L^\infty(\mathbb{D}, dA)$ is bounded, then the Toeplitz operator T_u^α with symbol u in $L^1(\mathbb{D}, dA)$ is densely defined on L_a^2 and the Berezin transform $\widetilde{u}$ of a function is defined to be the Berezin transform of T_u^α.

Let $\mathrm{Aut}(\mathbb{D})$ denote the set of all bianalytic maps of $\mathbb{D}$ onto $\mathbb{D}$. By Schwarz's lemma, each element of $\mathrm{Aut}(\mathbb{D})$ is a linear fractional transformation of the form $\lambda\varphi_z$, $|\lambda| = 1$, where $\varphi_z(w) = \dfrac{z-w}{1-\overline{z}w}$.

For $z \in \mathbb{D}$, let $U_z^\alpha : L_a^2 \to L_a^2$ be defined by $U_z^\alpha f = (f \circ \varphi_z)(\varphi_z')^{1+\frac{\alpha}{2}}$. Since $(\varphi_z'(\varphi_z(w)))^{1+\frac{\alpha}{2}} \varphi_z(w)^{1+\frac{\alpha}{2}} = 1$, $U_z^\alpha \circ U_z^\alpha$ is the identity function on L_a^2. Since $\int_{\mathbb{D}} |f \circ \varphi_z(\lambda)|^2 \times$
$\times |\varphi_z'(\lambda)|^{2+\alpha} dA_\alpha(\lambda) = \int_{\mathbb{D}} |f(w)|^2 \times (1-|w|^2)^\alpha dA(w) = ||f||_{2,\alpha}$, U_z^α is an isometry. Thus U_z^α is a self-adjoint unitary operator.

We also define the conjugation operator S_z given by $S_z = U_z^\alpha S U_z^\alpha$, where S is an operator on L_a^2.

We consider the problem to determine when a Toeplitz operator is bounded on L_a^2. Miao and Zheng ([2]) proved that Toeplitz operators with symbols in BT are bounded. Let RW=$\{f \in L^1(\mathbb{D}, dA) : ||f||_{RW} = \sup_{z\in\mathbb{D}} ||fk_z^\alpha||_{s,\alpha} < +\infty$ for some $s \in (2, \infty)\}$. Clearly RW is closed under the formation of conjugations.

In this paper, we will show that Toeplitz operators with symbols in RW are bounded and for $u \in RW$, $|u|dA_\alpha$ is a Carleson measure. In Section 3, we evaluate some upper bounds for the Toeplitz operator norm. Section 4 deals with n-th derivatives of Toeplitz operators and we get some local estimates.

Throughout the paper, we use p' to denote the conjugate of p, that is, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

2. Boundedness of some Toeplitz operators

Since dA_α is a probability measure on $\mathbb{D}$, whenever $0 < p < q < +\infty$, $L_a^q \subseteq L_a^p$. Thus for any $f \in RW$, $\sup_{z\in\mathbb{D}} ||fk_z^\alpha||_{2,\alpha} \le ||f||_{RW}$.

Since $|\widetilde{T_f^\alpha}(z)|=|<T_f^\alpha k_z^\alpha, k_z^\alpha>_\alpha| \leq ||T_f^\alpha k_z^\alpha||_{2,\alpha} \leq ||f k_z^\alpha||_{2,\alpha}$, $\sup_{z\in\mathbb{D}}|\widetilde{T_f^\alpha}(z)| \leq ||f||_{RW}$ and $\widetilde{|f|}(z) \leq ||f||_{RW}$.

Let μ be a finite positive Borel measure on $\mathbb{D}$ and let $1 \leq p < +\infty$. The closed Graph Theorem shows that L_a^p is contained in $L^p(\mathbb{D}, d\mu)$ if and only if the inclusion map $i_p : L_a^p \to L^p(\mathbb{D}, d\mu)$ is bounded. We say that μ is a Carleson measure for L_a^p if the inclusion map from L_a^p to $L^p(\mathbb{D}, d\mu)$ is bounded. We notice that $\widetilde{\mu}$ is the Berezin symbol of μ, that is, $\widetilde{\mu}(z) = \int_{\mathbb{D}} |k_z^\alpha(w)|^2 d\mu(w)$, $z \in \mathbb{D}$.

The following lemma comes from [5].

Lemma 2.1. *Suppose μ is a positive Borel measure on $\mathbb{D}$ and $1 \leq p < +\infty$. Then the followings are equivalent :*

(a) $\sup\left\{\dfrac{\mu(D(z,r))}{(1-|z|^2)^{2+\alpha}} : z \in \mathbb{D}\right\} < +\infty$;

(b) $\sup\{\widetilde{\mu}(z) : z \in \mathbb{D}\} < +\infty$;

(c) μ is a Carleson measure on $\mathbb{D}$;

(d) $\sup\left\{\dfrac{\int_{\mathbb{D}}|f|^p d\mu}{\int_{\mathbb{D}}|f|^p dA_\alpha} : f \in L_a^p\right\} < +\infty.$

Here, $D(z,r) = \{w \in \mathbb{D} : \beta(w,z) < r\}$ is the Bergman disk.

Proposition 2.2. *Suppose $f \in RW$. Then*

(1) $|f|dA_\alpha$ is a Carleson measure on $\mathbb{D}$;

(2) for $1 < p < +\infty$, T_f^α is bounded on L_a^p and there is a constant C such that $||T_f^\alpha||_p \leq C||f||_{RW}$,

where $||T_f^\alpha||_p$ is the operator norm on L_a^p.

Proof. For $z \in \mathbb{D}$, $\widetilde{|f|}(z)=\int_{\mathbb{D}}|k_z^\alpha(w)|^2|f(w)|dA_\alpha(w) \leq ||fk_z^\alpha||_{2,\alpha}$. This implies that (b) in Lemma 2.1 is finite. Thus $|f|dA_\alpha$ is a Carleson measure. To show (2), suppose $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, $g \in L_a^p$ and $h \in L_a^{p'}$. Since $|f|dA_\alpha$ is a Carleson measure, $|<T_f^\alpha g, h>| \leq (\int_{\mathbb{D}}|g|^p|f|dA_\alpha)^{\frac{1}{p}}$ $(\int_{\mathbb{D}}|h|^{p'}|f|dA_\alpha)^{\frac{1}{p'}} \leq C||f||_{BW}||g||_{p,\alpha}||h||_{p',\alpha}$ for some constant C. Thus $||T_f^\alpha||_p \leq C||f||_{RW}$. □

Example 2.3. Suppose $2 < s < +\infty$. For $0 \leq x \leq 1$, let

$$f(x) = \begin{cases} 2^{\frac{k}{s}} & \text{if } \frac{1}{2^k} - (\frac{1}{2^{k+1}})^2 \leq x \leq \frac{1}{2^k}; \\ 0 & \text{otherwise} . \end{cases}$$

We define $f(z) = f(|z|)$ for all $z \in \mathbb{D}$, that is, f is a radial function. Since $\lim_{k\to\infty} \Big(\frac{1}{2^k} - (\frac{1}{2^{k+1}})^2\Big)=0$, f is not in L^∞. Since $k_z^\alpha(w)=\frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}}$, $|k_z^\alpha(w)| \leq \frac{1}{(\frac{1}{2})^{2+\alpha}}=2^{2+\alpha}$ for all $|w| \leq \frac{1}{2}$ and hence $\int_{\mathbb{D}} |f(w)k_z^\alpha(w)|^s dA_\alpha(w)$ $\leq 2^{(2+\alpha)s+\alpha} \int_0^{\frac{1}{2}} |f(t)|^s dt < +\infty$. Thus L^∞ is a proper subset of RW.

For $f \in L_a^2$ and $w \in \mathbb{D}$, we have

$$\begin{aligned} (T_u^\alpha f)(w) &= < T_u^\alpha f, K_w^\alpha >_\alpha \\ &= < f, (T_u^\alpha)^* K_w^\alpha >_\alpha \\ &= \int_{\mathbb{D}} f(z)\overline{((T_u^\alpha)^* K_w^\alpha)(z)} dA_\alpha(z) \\ &= \int_{\mathbb{D}} f(z)(T_u^\alpha K_z^\alpha)(w) dA_\alpha(z). \end{aligned}$$

That is, T_u^α is the integral operators with kernel $T_u^\alpha K_z^\alpha(w)$. Since for any $u \in RW$, $|u|dA_\alpha$ is a Carleson measure on L_a^p, T_u^α is a bounded linear operator on L_a^p. In the following section, we will evaluate some upper bounds of the Toeplitz operator norms.

3. Some upper bounds

In order to find some upper bounds, we need the following lemma which is a special case of Lemma 3.10 in [5].

LEMMA 3.1. *Suppose $a < \alpha + 1$. If $a + b - \alpha < 2$ then*

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \frac{dA_\alpha(w)}{(1-|w|^2)^a |1-\overline{z}w|^b} < +\infty.$$

PROPOSITION 3.2. *Suppose $u \in L^1(\mathbb{D}, dA)$ and $0 < a < 1$. Then there exists t in $(1, \frac{2+\alpha}{2-a+\alpha})$ and there exists a constant C such that*

$$\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|w|^2)^a} dA_\alpha(w) \leq \frac{C||(T_u^\alpha)_z 1||_{t',\alpha}}{(1-|z|^2)^\alpha}$$

for all $z \in \mathbb{D}$ *and*

$$\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|z|^2)^a} dA_\alpha(z) \leq \frac{C||(T_u^\alpha)_z 1||_{t',\alpha}}{(1-|w|^2)^\alpha}$$

for all $w \in \mathbb{D}$.

Proof. Take any z in $\mathbb{D}$. Since

$$T_u^\alpha K_z^\alpha = \frac{T_u^\alpha U_z^\alpha 1}{(|z|^2-1)^{1+\frac{\alpha}{2}}} = \frac{(T_u^\alpha)_z 1 \circ \varphi_z (\varphi_z')^{1+\frac{\alpha}{2}}}{(|z|^2-1)^{1+\frac{\alpha}{2}}},$$

put $w{=}\varphi_z(\lambda)$ to obtain the following :

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{|T_u^\alpha K_z^\alpha(w)|}{(1-|w|^2)^a} dA_\alpha(w) \\
= \ & \frac{1}{(1-|z|^2)^{1+\frac{\alpha}{2}}} \int_{\mathbb{D}} \frac{|((T_u^\alpha)_z 1)(\varphi_z(w))||\varphi_z'(w)|^{1+\frac{\alpha}{2}}}{(1-|w|^2)^a} (1-|w|^2)^\alpha dA(w) \\
= \ & \frac{1}{(1-|z|^2)^a} \int_{\mathbb{D}} \frac{|(T_u^\alpha)_z 1(\lambda)|}{(1-|\lambda|^2)^a |1-\overline{z}\lambda|^{2-2a+\alpha}} dA_\alpha(\lambda) \\
\leq \ & \frac{||(T_u^\alpha)_z 1||_{t',\alpha}}{(1-|z|^2)^a} \Big(\int_{\mathbb{D}} \frac{dA_\alpha(\lambda)}{(1-|\lambda|^2)^{at} |1-\overline{z}\lambda|^{(2-2a+\alpha)t}} \Big)^{\frac{1}{t}}.
\end{aligned}$$

Since we can pick t in $\Big(1, \dfrac{2+\alpha}{1-a+\alpha}\Big)$ so that $at+(2-2a+\alpha)t-\alpha<2$, by Lemma 3.1, the above integral is finite. Put

$$C^t = \int_{\mathbb{D}} \frac{dA_\alpha(\lambda)}{(1-|\lambda|^2)^{at} |1-\overline{z}\lambda|^{(2-2a+\alpha)t}}.$$

Then we get the first inequality. Since $T_u^\alpha K_z^\alpha(w) =< T_u^\alpha K_z^\alpha, K_w^\alpha > =< K_z^\alpha, (T_u^\alpha)^* K_w^\alpha > = \overline{(T_u^\alpha)^* K_w(z)}$ and $(T_u^\alpha)^* {=} T_{\overline{u}}^\alpha$, we obtain the second inequality. $\square$

LEMMA 3.3. *Suppose* $0<a<1$ *and* $u \in RW$, *that is,* $\sup ||uk_z^\alpha||_{s,\alpha} < +\infty$ *for some* $s>2$. *If* $\dfrac{2+\alpha}{a} < s$ *then there exists a constant* C *such that*

$$\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|w|^2)^a} dA_\alpha(w) \leq \frac{C||u||_{RW}}{(1-|z|^2)^a}$$

for all $z \in \mathbb{D}$ *and*

$$\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^a)(w)|}{(1-|z|^2)^a} dA_\alpha(z) \leq \frac{C||u||_{RW}}{(1-|w|^2)^a}$$

for all $w \in \mathbb{D}$.

Proof. Since $\dfrac{2+\alpha}{a} < s$, there exists t in $\Big(\dfrac{2+\alpha}{a}, s\Big)$ such that $1 < t' < \dfrac{2+\alpha}{2-a+\alpha}$. Put $C^{t'} = \displaystyle\int_{\mathbb{D}} \frac{dA_\alpha(\lambda)}{(1-|\lambda|^2)^{at'}|1-\overline{z}\lambda|^{(2-2a+\alpha)t'}}$. Since $(2-a+\alpha)t'-\alpha < 2$, Lemma 3.1 implies that C is finite. By Proposition 3.2, $\displaystyle\int_{\mathbb{D}} \frac{|(T_u^\alpha K_z^\alpha)(w)|}{(1-|w|^2)^a} dA_\alpha(w) \le \frac{C||(T_u^\alpha)_z 1||_{t,\alpha}}{(1-|z|^2)^a}$. Since $t < s$, $||(T_u^\alpha)_z 1||_{t,\alpha} \le ||u||_{RW}$. Thus we get the results. □

We notice that for every $u \in RW$, T_u^α is the integral operator with kernel $T_u^\alpha K_z^\alpha(w)$. In order to find an upper bound of the operator norm $||T_u^\alpha||_p$, we need Schur's theorem ([5]).

Theorem 3.4. *Suppose K is a nonnegative measurable function on $X \times X$, T is the integral operator with kernel K and $1 < p < +\infty$. If there exist positive constants C_1 and C_2 and a positive measurable function h on X such that*

$$\int_X K(x,y)h(y)^{p'} d\mu(y) \le C_1 h(x)^{p'}$$

for almost every x in X and

$$\int_X K(x,y)h(x)^{p} d\mu(x) \le C_2 h(y)^{p}$$

for almost every y in X, then T is a bounded linear operator on $L^p(X, d\mu)$ with norm less than or equal to $C_1{}^{\frac{1}{p'}} C_2{}^{\frac{1}{p}}$.

Theorem 3.5. *Suppose $1 < p < +\infty$ and $\sup_z ||uk_z^\alpha||_{s,\alpha} < +\infty$ for some $2 < s$, where $u \in L^1(\mathbb{D}, dA)$. If $q(2+\alpha) < s$, where $q = \max\{p, p'\}$ then there is a constant C such that $||T_u^\alpha||_p \le C||u||_{RW}$.*

Proof. Let $h(z) = \Big(\dfrac{1}{1-|z|^2}\Big)^{\frac{1}{pp'}}$. Since $q(2+\alpha) < s$, there is t such that $q(2+\alpha) < t < s$ and $1 < t' < \dfrac{2+\alpha}{2-\frac{1}{q}+\alpha}$. Let $C^{t'} = \displaystyle\int_{\mathbb{D}} \frac{dA_\alpha(\lambda)}{(1-|\lambda|^2)^{\frac{t'}{q}}|1-\overline{z}\lambda|^{(2-\frac{2}{q}+\alpha)t'}}$. Since $1 < t' < \dfrac{2+\alpha}{1-\frac{1}{q}+\alpha}$, $\dfrac{t'}{q} + (2 - \dfrac{2}{q} + \alpha)t' < 2+\alpha$ and hence C is finite. Since $t < s$, $||(T_u^\alpha)_z 1||_{t,\alpha}$

and $||(T^{\alpha}_{\overline{u}})_z 1||_{t,\alpha}$ are less than or equal to $||u||_{RW}$. This completes the proof. □

4. Some operators

In this section, we will assume that u is an element of RW. Since $||f||_{RW}$=$||\overline{f}||_{RW}$, RW is closed under the formation of conjugations. For any $f \in L^1(\mathbb{D}, dA)$, $(T^{\alpha}_f)^* = T^{\alpha}_{\overline{f}}$ and hence one has the following :

Proposition 4.1. *For any $u \in RW$, the commutator $(T^{\alpha}_u)^* T^{\alpha}_u - T^{\alpha}_u (T^{\alpha}_u)^* = T^{\alpha}_{\overline{u}} T^{\alpha}_u - T^{\alpha}_u T^{\alpha}_{\overline{u}}$ is a bounded linear operator.*

Let $1 \leq p < +\infty$. If $f \in L^p_a$ then $P_\alpha(f) = f$ and hence for $u \in RW$ and $h \in L^2_a$, $T^{\alpha}_f T^{\alpha}_u h = f T^{\alpha}_u h$. Since H^∞ is dense in L^2_a, given a function f in $L^2(\mathbb{D}, dA_\alpha)$. H^{α}_f is densely defined which is given by $H^{\alpha}_f g$ $= (I - P_\alpha)(fg)$ for all $g \in L^2_a$.

By the definitions of Hankel and Toeplitz operators, we get the following identity

$$(H^{\alpha}_u)^* H^{\alpha}_u \quad = \quad T^{\alpha}_{|u|^2} - T^{\alpha}_{\overline{u}} T^{\alpha}_u.$$

Suppose $|u|^2$ and u are in RW. Since $T^{\alpha}_{|u|^2}$ and $T^{\alpha}_{\overline{u}} T^{\alpha}_u$ are bounded, $(H^{\alpha}_u)^* H^{\alpha}_u$ is also bounded. If $f \in H^\infty$ then $||H^{\alpha}_u(f)||_{2,\alpha} \leq ||uf||_{2,\alpha} + ||P_\alpha(uf)||_{2,\alpha} \leq ||f||_\infty (||u||_{2,\alpha} + ||u||_{RW})$ and hence H^{α}_u is bounded.

Proposition 4.2. *If $u \in RW$ then*

$$|(T^{\alpha}_{\overline{u}} h)(w)| \leq \frac{1}{(1-|w|^2)^{1+\frac{\alpha}{2}}} ||h||_{2,\alpha} ||u||_{RW}$$

for every $h \in L^2_a$ and every $w \in \mathbb{D}$.

Proof. Since $u \in RW$, $||u||_{RW} = \sup_{z \in \mathbb{D}} ||u k^{\alpha}_z||_{s,\alpha} < +\infty$ for some $s \in (2, \infty)$. Suppose $w \in \mathbb{D}$ and $h \in L^2_a$. Then $(T^{\alpha}_{\overline{u}} h)(w) = < T^{\alpha}_{\overline{u}} h, K^{\alpha}_w >_\alpha = < h, u K^{\alpha}_w >_\alpha = \dfrac{1}{(1-|w|^2)^{1+\frac{\alpha}{2}}} \times$
$\times < h, u k^{\alpha}_w >_\alpha$. Hölder's inequality implies that $| < h, u k^{\alpha}_w >_\alpha | \leq ||h||_{s',\alpha} ||u k^{\alpha}_w||_{s,\alpha}$. Since $||h||_{s',\alpha} \leq ||h||_{2,\alpha}$, one has the result. □

For real numbers a, b, c, we define integral operators as following :

$$S_{a,b,c} f(w) \quad = \quad (1-|w|^2)^a \int_{\mathbb{D}} \frac{(1-|z|^2)^b}{|1-\overline{z}w|^c} f(z) dA(z),$$

and

$$T_{a,b,c}f(w) \quad = \quad (1-|w|^2)^a \int_{\mathbb{D}} \frac{(1-|z|^2)^b}{(1-\overline{z}w)^c} f(z)dA(z).$$

This is Theorem 3.11 of Zhu[5].

Proposition 4.3. *Suppose $p > 1$. If c is nether 0 nor a negative integer, then the following are equivalent.*
(a) $S_{a,b,c}$ is bounded on $L^p(\mathbb{D}, dA_\alpha)$.
(b) $T_{a,b,c}$ is bounded on $L^p(\mathbb{D}, dA_\alpha)$.
(c) $c \le 2+a+b$ and $-pa < \alpha+1 < p(b+1)$.

Suppose $u \in L^1(\mathbb{D}, dA)$. Then $\widetilde{u}(w) = \widetilde{T^\alpha_u}(w) = < T^\alpha_u k^\alpha_w, k^\alpha_w >_\alpha = \int_{\mathbb{D}} \frac{(1-|w|^2)^{2+\alpha}}{|1-w\overline{z}|^{4+2\alpha}} \times u(z)dA_\alpha(z)$ and if $f, h \in L^2_a$ then $T^\alpha_f T^\alpha_u h = fT^\alpha_u h$.

Proposition 4.4. *Suppose $||u||_{RW} = \sup_{z\in\mathbb{D}} ||uk^\alpha_z||_{s,\alpha} < +\infty$ for some $s \in (3,\infty)$ and $w \in \mathbb{D}$. If $h^{s'} \in L^2_a$ then*

$$|(T^\alpha_{\overline{u}}h)'(w)| \le \frac{(2+\alpha)c}{(1-|w|)^{\frac{2+\alpha}{s'}+\frac{1}{s}}} (\widetilde{|u|^s}(w))^{\frac{1}{s'}} ||h^{s'}||^{\frac{1}{s'}}_{2,\alpha}$$

for some constant c.

Proof. Since $(T^\alpha_{\overline{u}}h)(w) = \int_{\mathbb{D}} \frac{\overline{u}(z)h(z)}{(1-\overline{z}w)^{2+\alpha}} dA_\alpha(z)$, $(T^\alpha_{\overline{u}}h)'(w) = (2+\alpha)\int_{\mathbb{D}} \frac{\overline{z}\,\overline{u}(z)h(z)}{(1-\overline{z}w)^{4+2\alpha}} \times (1-\overline{z}w)^{1+\alpha}\ dA_\alpha(z)$ and hence

$$|(T^\alpha_{\overline{u}}h)'(w)| \le (2+\alpha)\Big(\int_{\mathbb{D}} \frac{|u(z)|^s(1-|w|^2)^{2+\alpha}}{|1-\overline{z}w|^{4+2\alpha}} dA_\alpha(z)\Big)^{\frac{1}{s}}$$

$$\times\Big(\int_{\mathbb{D}} \frac{|h(z)|^{s'}|1-\overline{z}w|^{(1+\alpha)s'}}{|1-\overline{z}w|^{4+2\alpha}(1-|w|^2)^{(2+\alpha)\frac{s'}{s}}} dA_\alpha(z)\Big)^{\frac{1}{s'}}.$$

Since $|1-\overline{z}w|$ and $1-|w|^2$ are greater than $1-|w|$ and $2+\alpha-(1+\alpha)s' < 2+\alpha-(1+\alpha)\frac{s'}{s}$, $\int_{\mathbb{D}} \frac{|h(z)|^{s'}|1-\overline{z}w|^{(1+\alpha)s'} dA_\alpha(z)}{|1-\overline{z}w|^{4+2\alpha}(1-|w|^2)^{(2+\alpha)\frac{s'}{s}}} \le \frac{c_1}{(1-|w|)^{2+\alpha+\frac{s'}{s}}} ||h^{s'}||_{2,\alpha}$ for some constant c_1. This completes the proof. $\square$

LEMMA 4.5. *Suppose* $||u||_{RW} = \sup_{z\in\mathbb{D}} ||uk_z^\alpha||_{s,\alpha}$ *for some* $s \in (3, +\infty)$ *and* $w \in \mathbb{D}$. *If* $h^{s'} \in L_a^2$ *then there is a constant* C *such that* $|(T_{\overline{u}}^\alpha h)''(w)| \leq \dfrac{C(2+\alpha)(3+\alpha)}{(1-|w|)^{\frac{2+\alpha}{s'}+\frac{1}{s}+1}} (\widetilde{|u|^s}(w))^{\frac{1}{s}} \times ||h^{s'}||_{2,\alpha}^{\frac{1}{s'}}$.

Proof. Since $(T_{\overline{u}}^\alpha h)'(w) = (2+\alpha)\int_{\mathbb{D}} \frac{\overline{z}\,\overline{u}(z)h(z)}{(1-\overline{z}w)^{3+\alpha}} dA_\alpha(z)$, $(T_{\overline{u}}^\alpha h)''(w) = (2+\alpha)\times(3+\alpha)\int_{\mathbb{D}} \frac{\overline{z}^2\overline{u}(z)h(z)}{(1-\overline{z}w)^{4+\alpha}} dA_\alpha(z)$ and hence $|(T_{\overline{u}}^\alpha h)''(w)| \leq (2+\alpha)(3+\alpha)\times\Big(\int_{\mathbb{D}} \frac{|u(z)|^s(1-|w|^2)^{2+\alpha}}{|1-\overline{z}w|^{4+2\alpha}} dA_\alpha(z)\Big)^{\frac{1}{s}} \Big(\int_{\mathbb{D}} \frac{|h(z)|^{s'}|1-\overline{z}w|^{\alpha s'}}{|1-\overline{z}w|^{4+2\alpha}(1-|w|^2)^{(2+\alpha)\frac{s'}{s}}} dA_\alpha(z)\Big)^{\frac{1}{s'}}$

$$\leq (2+\alpha)(3+\alpha)(\widetilde{|u|^s}(w))^{\frac{1}{s}}$$
$$\times\Big(\frac{1}{(1-|w|)^{2+\alpha+\frac{s'}{s}+s'}}\int_{\mathbb{D}} \frac{|h(z)|^{s'}\, dA_\alpha(z)}{|1-\overline{z}w|^{2+\alpha-(1+\alpha)s'}(1-|w|^2)^{(1+\alpha)\frac{s'}{s}}}\Big)^{\frac{1}{s'}}$$
$$\leq \frac{C(2+\alpha)(3+\alpha)}{(1-|w|)^{\frac{2+\alpha}{s'}+\frac{1}{s}+1}} (\widetilde{|u|^s}(w))^{\frac{1}{s}} ||h^{s'}||_{2,\alpha}^{\frac{1}{s'}}$$ for some constant C.

THEOREM 4.6. *Suppose* $||u||_{RW} = \sup_{z\in\mathbb{D}} ||uk_z^\alpha||_{s,\alpha} < +\infty$ *for some* $s \in (3, +\infty)$ *and* $w \in \mathbb{D}$. *If* $h^{s'} \in L_a^2$ *and* $n \geq 3$ *then there is* C *such that* $|(T_{\overline{u}}^\alpha)^{(n)}(w)| \leq \dfrac{C\Gamma(n+2+\alpha)}{\Gamma(2+\alpha)(1-|w|)^{\frac{2+\alpha}{s'}+\frac{1}{s}+n-1}} (\widetilde{|u|^s}(w))^{\frac{1}{s}} ||h^{s'}||_{2,\alpha}^{\frac{1}{s'}}$.

Proof. Since $(T_{\overline{u}}^\alpha)''(w) = (2+\alpha)(3+\alpha)\int_{\mathbb{D}} \frac{\overline{z}^2\overline{u}(z)h(z)}{(1-\overline{z}w)^{4+\alpha}} dA_\alpha(z)$,

$(T_{\overline{u}}^\alpha h)'''(w) = (2+\alpha)\times(3+\alpha)(4+\alpha)\int_{\mathbb{D}} \frac{\overline{z}^3\overline{u}(z)h(z)}{(1-\overline{z}w)^{5+\alpha}} dA_\alpha(z)$ and hence

$$|(T_{\overline{u}}^\alpha h)'''(w)| \leq \frac{(2+\alpha)(3+\alpha)(4+\alpha)}{(1-|w|)} \times \int_{\mathbb{D}} \frac{|u(z)||1-\overline{z}w|^\alpha|h(z)|}{|1-\overline{z}w|^{4+2\alpha}} dA_\alpha(z)$$

$$\leq \frac{(2+\alpha)(3+\alpha)(4+\alpha)}{(1-|w|)} (\widetilde{|u|^s}(w))^{\frac{1}{s}} \frac{C}{(1-|w|)^{\frac{2+\alpha}{s'}+\frac{1}{s}+1}} \times ||h^{s'}||_{2,\alpha}^{\frac{1}{s'}}$$

$$= \frac{C(2+\alpha)(3+\alpha)(4+\alpha)}{(1-|w|)^{\frac{2+\alpha}{s'}+\frac{1}{s}+2}} ||h^{s'}||_{2,\alpha}^{\frac{2}{s'}}$$ for some constant C.

This implies that for $n \geq 3$, $|(T_{\overline{u}}^{\alpha}h)^{(n)}(w)| \leq$
$$\frac{C\Gamma(n+2+\alpha)}{\Gamma(2+\alpha)(1-|w|)^{\frac{2+\alpha}{s'}+\frac{1}{s}+n-1}}(\widetilde{|u|^s}(w))^{\frac{1}{s}}||h^{s'}||_{2,\alpha}^{\frac{1}{s'}}.$$
□

COROLLARY 4.7. *Suppose* $||u||_{RW} = \sup_{z\in\mathbb{D}} ||uk_z^{\alpha}||_{s,\alpha} < +\infty$ *for some* $s \in (2, +\infty)$, $w \in \mathbb{D}$ *and* $1 < p < s-1$. *If* $h^{s'} \in L_a^p$ *and* $n \geq 3$ *then there is a constant* c *such that*
$$|(T_u^{\alpha}h)^{(n)}(w)| \leq \frac{c\Gamma(n+2+\alpha)}{\Gamma(2+\alpha)(1-|w|)^{\frac{2+\alpha}{s'}+\frac{1}{s}+n-1}}(\widetilde{|u|^s}(w))^{\frac{1}{s}}||h^{s'}||_{p,\alpha}^{\frac{1}{s'}}.$$

Proof. If follows from the fact that $p(1+\alpha)\dfrac{s'}{s} = p(1+\alpha)\dfrac{1}{s-1} < 1+\alpha < p(1+\alpha)$ and Theorem 4.6. □

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