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CURVATURE TENSOR FIELDS ON HOMOGENEOUS SPACES

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Abstract. In this paper, we make a minute and detailed proof of a part which is omitted in the process of obtaining the value of the curvature tensor for an invariant affine connection at the point $\{H\}$ of a reductive homogeneous space G/H in the paper 'Invariant affine connections on homogeneous spaces' by K. Nomizu.

1. Introduction

The theory of a reductive homogeneous space is well known (cf. [1,2,6,7]). The study on a reductive homogeneous space begin with the study on invariant affine connections on homogeneous spaces.

The study (cf. [4,5]) on curvature tensor fields for affine connections in geometry is important. There is a part which is omitted in the process of obtaining the curvature tensor for an invariant affine connection at the point $\{H\}$ of a reductive homogeneous space G/H in the paper 'invariant affine connections on homogeneous spaces' by K. Nomizu (cf. [3]).

In this paper, we make a minute and detailed proof of the part which is omitted in the process of obtaining the value of the curvature tensor for an invariant affine connection at the point $\{H\}$ of the reductive homogeneous space G/H.

The proof of this omitted part is very important in the study on a reductive homogeneous space. But a circumstantial proof of the part is rarely ever seen.

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2. Reductive homogeneous spaces

A homogeneous space G/H of a connected Lie group G is called *reductive* if the following condition is satisfied: in the Lie algebra $\mathfrak{g}$ of G there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ (direct sum of vector spaces) and $\mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m}$ for all $h \in H$, where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ corresponding to the identity component H_o of H and $\mathrm{Ad}(h)$ denotes the adjoint representation of H in $\mathfrak{g}$.

From now on, we shall always assume a homogeneous space to be reductive and shall consider a fixed decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ of the Lie algebra satisfying the above condition.

We denote by p_o the point represented by the coset $\{H\} \in G/H$. The subspace $\mathfrak{m}$ can be identified with the tangent space at p_o. Let $X_1, X_2, \cdots, X_n$ be a base of $\mathfrak{g}$ such that the first m elements span $\mathfrak{m}$ and the last $n-m$ elements span $\mathfrak{h}$.

The following lemma (cf. [1,2,6]) is well known.

Lemma 2.1. *For the decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, there exist bounded, open, connected neighborhoods N_1 and N_2 of the zero vector 0 in $\mathfrak{m}$ and $\mathfrak{h}$, respectively, such that the mapping $\Phi : (A, B) \to \exp A \exp B$ is a diffeomorphism of $N_1 \times N_2$ onto an open neighborhood of e in G, where $\alpha(t) = \exp tX$ $(t \in \mathbb{R},\ X \in \mathfrak{g})$ is a 1-parameter subgroup of G.*

If we denote by π the canonical projection of the Lie group G onto G/H, π is a differential homeomorphism of $\exp N$ onto a neighborhood $\pi(\exp N)$ of p_o in G/H.

For the sake of simplicity, we put $\exp N_1 = C_1$ and $\pi(C_1) = {C_1}^*$ Then, $\pi(c) = \tau(c)p_o$ for $c \in C_1$, where $\tau(c)$ denotes the transformation of G/H which is induced by c.

For each element X of $\mathfrak{m}$, we define a C^∞-vector field X^* on ${C_1}^*$ as follows:

$$X^*{}_{\tau(c)p_o} = \tau(c)_{*p_o} X \qquad (c \in C_1), \tag{2.1}$$

where $\tau(c)_*$ denotes the differential map of $\tau(c)$.

Let h be any element of H. Since $\tau(h)p_o = p_o$, $\tau(h)$ induces a linear map $\tau(h)_{*p_o}$ of the tangent space at p_o onto itself, which is the same as $\mathrm{Ad}(h)$ on $\mathfrak{m}$. We shall show how each vector field X^* $(X \in \mathfrak{m})$ on ${C_1}^*$ is transformed by $\tau(h)$. Take a subset ${C_1}'$ of C_1 such that $h{C_1}'h^{-1} \subset C_1$, and let ${C_1}'^* = \{\tau(c)p_o |\ c \in {C_1}'\}$. Then, $\tau(h){C_1}'^* \subset {C_1}^*$ and

$$\tau(h)_* X^* = (\mathrm{Ad}(h)X)^* \quad \text{on} \quad \tau(h){C_1}'^*. \tag{2.2}$$

In fact, for any $c \in {C_1}'$,

$$\begin{aligned}((\tau(h))_* X^*)_{\tau(h)\tau(c)p_o} &= \tau(h)_*(X^*_{\tau(c)p_o}) = \tau(hc)_* X \\ &= \tau(hch^{-1})_*(\tau(h)_* X) = \tau(hch^{-1})_*(\mathrm{Ad}(h)X) \\ &= (\mathrm{Ad}(h)X)^*_{\tau(hch^{-1})p_o} = (\mathrm{Ad}(h)X)^*_{\tau(h)\tau(c)p_o}.\end{aligned}$$

For later use, we have

Lemma 2.2. *Let $X, Y \in \mathfrak{g}$ and $f \in C^\infty(G)$. Then,*

$$[X,Y]_e(f) = \lim_{s,t\to 0} \frac{1}{st}\{f(x(t)y(s)) - f(y(s)x(t))\}.$$

Proof. Let $x(t)$ and $y(s)$ be the 1-parameter subgroup of G generated by X and Y respectively. For any function f on G,

$$(Yf)_{x(t)} = \lim_{s\to 0}(1/s)\{f(x(t)y(s)) - f(x(t))\},$$

and

$$\begin{aligned}X_e(Yf) &= \lim_{t\to 0}(1/t)\{(Yf)(x(t)) - (Yf)(e)\} \\ &= \lim_{t\to 0}(1/t)[\lim_{s\to 0}(1/s)\{f(x(t)y(s)) - f(x(t)) \\ &\qquad - f(y(s)) + f(e)\}].\end{aligned}$$

So,

$$X_e(Yf) - Y_e(Xf) = \lim_{s,t\to 0} \frac{1}{st}\{f(x(t)y(s)) - f(y(s)x(t))\}.$$

Thus the proof of this lemma is completed. □

Lemma 2.3. *For any X and Y in $\mathfrak{m}$, $[X^*, Y^*]_{p_o} = [X,Y]_{\mathfrak{m}}$, where $X_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of the element $X \in \mathfrak{g} = \mathfrak{m} + \mathfrak{h}$.*

Proof. Let $x(t)$ and $y(s)$ be the 1-parameter subgroup of G generated by X and Y respectively. For any function f in ${C_1}^*$,

$$(Y^*f)_{x^*(t)} = \lim_{s\to 0}(1/s)\{(f\circ\pi)(x(t)y(s)) - (f\circ\pi)(x(t))\},$$

where $x^*(t)$ denotes the image of $x(t)$ by π, and

$$\begin{aligned}X^*_{p_o}(Y^*f) &= \lim_{t\to 0}(1/t)\{(Y^*f)(x^*(t)) - (Y^*f)(p_o)\} \\ &= \lim_{t\to 0}(1/t)[\lim_{s\to 0}(1/s)\{(f\circ\pi)(x(t)y(s)) - (f\circ\pi)(x(t)) \\ &\qquad - (f\circ\pi)(y(s)) + (f\circ\pi)(e)\}].\end{aligned}$$

Thus from Lemma 2.2

$$\begin{aligned}X^*_{p_o}(Y^*f) - Y^*_{p_o}(X^*f) &= \lim_{s,t\to 0}\frac{1}{st}\{(f\circ\pi)(x(t)y(s)) - (f\circ\pi)(y(s)x(t))\} \\ &= [X,Y]_e(f\circ\pi) = [X,Y]_{\mathfrak{m}}(f).\end{aligned}$$

Thus the proof of this lemma is completed. □

LEMMA 2.4. ([3]). *Let G/H be a reductive homogeneous space with a fixed decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$. There exists a one-to-one correspondence between the set of all invariant affine connection on G/H and the set of all bilinear function α on $\mathfrak{m} \times \mathfrak{m}$ with values in $\mathfrak{m}$ which are invariant by $\mathrm{Ad}(H)$, that is, $\mathrm{Ad}(h)(\alpha(X,Y)) = \alpha(\mathrm{Ad}(h)X, \mathrm{Ad}(h)Y)$ for all $X, Y \in \mathfrak{m}$ and $h \in H$. The correspondence is given by*

$$\alpha(X,Y) = (D_{X^*}Y^*)_{p_o}. \tag{2.3}$$

3. Curvature tensor fields of invariant affine connections

We keep the notation of the preceding section. Let D be the invariant affine connection on G/H which is determined by a connection function α on $\mathfrak{m} \times \mathfrak{m}$ which is appeared in Lemma 2.4.

Let X^* and Y^* be the C^∞ vector fields on ${C_1}^*$ which correspond to X and Y in $\mathfrak{m}$ respectively. The torsion tensor field T on ${C_1}^*$ is given as follows:

$$T(X^*, Y^*) = D_{X^*}Y^* - D_{Y^*}X^* - [X^*, Y^*]. \tag{3.1}$$

By virtue of Lemmas 2.3, 2.4 and (3.1), we get the following expression for the value at p_o of the torsion tensor field:

$$T(X,Y) = \alpha(X,Y) - \alpha(Y,X) - [X,Y]_{\mathfrak{m}}. \tag{3.2}$$

As for the curvature tensor field R on ${C_1}^*$

$$\begin{aligned} R(X^*,Y^*)Z^* = D_{X^*}D_{Y^*}Z^* &- D_{Y^*}D_{X^*}Z^* \\ &- D_{[X^*,Y^*]}Z^*, \ (X,Y,Z \in \mathfrak{m}). \end{aligned} \tag{3.3}$$

In order to calculate this, let $X_1, X_2, \cdots, X_m$ be a base of $\mathfrak{m}$. We set for any $p = \tau(c)p_o \quad (c \in C_1)$

$$(\tau(c^{-1})_* {X_i}^*)_q = \sum_{\mu=1}^m \phi_{\mu i}(p,q)({X_\mu}^*)_q, \quad q \in {C_1}^*, \quad i = 1, 2, \cdots, m, \tag{3.4}$$

where $\phi_{\mu i}$ are differentiable functions in p and q which satisfy the following condition:

$$\phi_{\mu i}(p,\ p_o) = \phi_{\mu i}(p_o,\ p) = \delta_{\mu i} \quad \text{for any } p, q \in {C_1}^*. \tag{3.5}$$

Then we have from Lemma 2.4 and (3.5)

$$\begin{aligned}
(D_{X_k^*}X_l^*)_p &= \tau(c)_*(D_{\tau(c^{-1})_*X_k{}^*}\ \tau(c^{-1})_*X_l{}^*)_{p_o}\\
&= \tau(c)_*(D_{\sum_\mu \phi_{\mu k}X_\mu^*}\ \sum_\nu \phi_{\nu l}X_\nu^*)_{p_o}\\
&= \tau(c)_*\{\sum_{\mu\nu}\phi_{\mu k}(p,p_o)\ \phi_{\nu l}(p,p_o)\ \alpha(X_\mu, X_\nu)\\
&+ \sum_{\mu,\nu}\phi_{\mu k}(p,p_o)\ (X_\mu^*\phi_{\nu l}(p,p_o))\ X_\nu\}\\
&= (\alpha(X_k, X_l))_p^* + \sum_\nu\ (X_k^*\phi_{\nu l})(p,p_o)\ (X_\nu^*)_p,
\end{aligned}$$

where $(X_k^*\phi_{\nu l})(p,p_o)$ is the function of p which is the result of applying X_k^* to the second variable of $\phi_{\nu l}(p,q)$ at $q=p_o$.

By a similar computation as above we get

$$\begin{aligned}
(D_{X_j^*}D_{X_k^*}X_l^*)_{p_o} &= \alpha(X_j, \alpha(X_k, X_l)) + \sum_\nu (X_j^*)_{p_o}(X_k^*\phi_{\nu l})(p,p_o)X_\nu\\
&+ \sum_\nu (X_k^*\phi_{\nu l})(p_o,p_o)\alpha(X_j, X_\nu)\\
&= \alpha(X_j, \alpha(X_k, X_l)) + \sum_\nu (X_j^*)_{p_o}(X_k^*\phi_{\nu l})(p,p_o)X_\nu,
\end{aligned}$$

since $(X_k^*\phi_{\nu l})(p_o,p_o)=0$ by (3.5). In the last expression, $(X_j^*)_{p_o}((X_k^*\phi_{\nu l})(p,p_o))$ denotes the result applying X_j^* at $p=p_o$ to the function $(X_k^*\phi_{\nu l})(p,p_o)$. Hence we get

$$\begin{aligned}
&(D_{X_j^*}D_{X_k^*}X_l^*)_{p_o} - (D_{X_k^*}D_{X_j^*}X_l^*)_{p_o}\\
&\quad = \alpha(X_j, \alpha(X_k, X_l)) - \alpha(X_k, \alpha(X_j, X_l))\\
&\quad + \sum_\nu \{(X_j^*)_{p_o}\ (X_k^*\phi_{\nu l})(p,p_o)\ -\ (X_k^*)_{p_o}(X_j^*\phi_{\nu l})(p,p_o)\}\ X_\nu.
\end{aligned} \tag{3.6}$$

We shall show that the last term is equal to $-[[X_j, X_k]_{\mathfrak{h}}, X_l]$, where $X_{\mathfrak{h}}$ denotes the $\mathfrak{h}$-component of the element $X \in \mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. The proof of this part in [3] is omitted.

Let $p=\tau(c)p_o$ and $q=\tau(c')p_o$ with $c, c' \in C_1$. Then, from Lemma 2.1, (2.1) and (2.2),

$$\begin{aligned}(\tau(c^{-1})_*X_l^*)_q &= \tau(c^{-1})_*(X_l)^*_{\tau(cc')p_o}\\ &= \tau(c^{-1})_*(\tau(c'')_*X_l)\\ &= \tau(c')_*(\tau(h^{-1})_*X_l)\\ &= \tau(c')_*(\mathrm{Ad}(h^{-1})X_l),\end{aligned}$$

where elements $c'' \in C_1$ and $h \in C_2 := \exp N_2$ are uniquely determined such that $cc' = c''h$. Therefore, we have

$$(3.7)\quad (\tau(c^{-1})_*{X_l}^*)_q = \textstyle\sum_\nu \phi_{\nu l}(p,q)({X_\nu}^*)_q = (\mathrm{Ad}(h^{-1})X_l)^*_q\ .$$

In the expression $(X_j^*)_{p_o}(X_k^*\phi_{\nu l})(p,p_o)$, we get

$$(3.8)\quad \begin{aligned}&(X_k^*\phi_{\nu l})(\exp tX_jp_o,\ p_o)\\ &\quad= \lim_{s\to 0}(1/s)\{\phi_{\nu l}(\exp tX_jp_o, \exp sX_kp_o) - \phi_{\nu l}(\exp tX_jp_o,\ p_o)\}\\ &\quad= \lim_{s\to 0}(1/s)\{\phi_{\nu l}(\exp tX_jp_o, \exp sX_kp_o) - \delta_{\nu l}\},\end{aligned}$$

and

$$\begin{aligned}&(X_j^*)_{p_o}(X_k^*\phi_{\nu l})(p,p_o)\\ &= \lim_{t\to 0}(1/t)\{(X_k^*\phi_{\nu l})(\exp tX_jp_o,p_o) - (X_k^*\phi_{\nu l})(p_o,p_o)\}\\ &= \lim_{s,t\to 0}\frac{1}{st}\{\phi_{\nu l}(\exp tX_jp_o, \exp sX_kp_o) - \delta_{\nu l}\}.\end{aligned}$$

Similarly, we obtain

$$(3.9)\quad \begin{aligned}&(X_k^*)_{p_o}(X_j^*\phi_{\nu l})(p,p_o)\\ &= \lim_{s,t\to 0}\frac{1}{st}\{\phi_{\nu l}(\exp sX_kp_o, \exp tX_jp_o) - \delta_{\nu l}\}.\end{aligned}$$

Hence, we have

$$(3.10)\quad \begin{aligned}&(X_j^*)_{p_0}(X_k^*\phi_{\nu l})(p,p_o) - (X_k^*)_{p_o}(X_j^*\phi_{\nu l})(p,p_o)\\ &= \lim_{s,t\to 0}\frac{1}{st}\{\phi_{\nu l}(\exp tX_jp_o, \exp sX_kp_o) - \phi_{\nu l}(\exp sX_kp_o, \exp tX_jp_o)\}.\end{aligned}$$

We put $p_1(t) = \tau(c_1(t))p_o$, $q_1(s) = \tau(c_1'(s))p_o$, and $c_1(t)c_1'(s) = \exp tX_j\exp sX_k = c_1''(t,s)h_1(t,s) \in C_1C_2$. Similarly, we put $p_2(s) = \tau(c_2(s))p_o$, $q_2(t) = \tau(c_2'(t))p_o$, and $c_2(s)c_2'(t) = \exp sX_k\exp tX_j = c_2''(s,t)h_2(s,t) \in C_1C_2$.

Moreover we get from Lemma 2.2

$$\begin{aligned} &\lim_{s,t\to 0}\frac{1}{st}\{\exp tX_j \exp sX_k - \exp sX_k \exp tX_j\} \\ (3.11)\qquad &= [X_j,\ X_k]_e \\ &= d\exp t[X_j, X_k]/dt\rfloor_{t=0} \\ &= d\exp t([X_j, X_k]_{\mathfrak{m}} + [X_j, X_k]_{\mathfrak{h}})/dt\rfloor_{t=0}\ . \end{aligned}$$

By virtue of (3.10) and (3.11), we have from (3.7)

$$\begin{aligned} &\sum_\nu \{(X_j^*)_{p_o}(X_k^*\phi_{\nu l})(p,p_o) - (X_k^*)_{p_o}(X_j^*\phi_{\nu l})(p,p_o)\}X_\nu \\ (3.12)\qquad &= -[[X_j, X_k]_{\mathfrak{h}}, X_l]. \end{aligned}$$

Finally the term $(D_{[X_j^*,X_k^*]}X_l^*)_{p_o}$ is equal to $\alpha([X_j, X_k]_{\mathfrak{m}}, X_l)$ due to Lemmas 2.3 and 2.4.

Summing up, we obtain

THEOREM 3.1. *Let G/H be a reductive homogeneous space with a fixed decomposition $\mathfrak{g} = \mathfrak{m}+\mathfrak{h}$, $(\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m})$. Let D be a $G-$invariant affine connection on G/H and α the connection function on $\mathfrak{m}\times\mathfrak{m}$ which is corresponding to D. Then, the curvature tensor field at p_o is given as follows:*

$$\begin{aligned} R(X,Y)Z = &\ \alpha(X, \alpha(Y,Z)) - \alpha(Y, \alpha(X,Z)) \\ &- \alpha([X,Y]_{\mathfrak{m}}, Z) - [[X,Y]_{\mathfrak{h}}, Z], \qquad (X,Y,Z \in \mathfrak{m}). \end{aligned}$$

REMARK 3.2. The process to derive (3.12) from (3.7) in [3] is omitted. The proof of this omitted part is very important in the study on a reductive homogeneous space. But a circumstantial proof of the part is rarely ever seen. The proof for (3.12) in this paper is in full detail.

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