SOME APPLICATIONS OF THE UNION OF
STAR-CONFIGURATIONS IN $\mathbb{P}^n$

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ABSTRACT. It has been proved that if $\mathbb{X}^{(s,s)}$ is the union of two linear star-configurations in $\mathbb{P}^2$ of type $s \times s$, then $(I_{\mathbb{X}^{(s,s)}})_s \neq \{0\}$ for $s = 3, 4, 5$, and $(I_{\mathbb{X}^{(s,s)}})_s = \{0\}$ for $s \geq 6$. We extend $\mathbb{P}^2$ to $\mathbb{P}^n$ and show that if $\mathbb{X}^{(s,s)}$ is the union of two linear star-configurations in $\mathbb{P}^n$, then $(I_{\mathbb{X}^{(s,s)}})_s = \{0\}$ for $n \geq 3$ and $s \geq 3$. Using this generalization, we also prove that the secant variety $\mathrm{Sec}_1(\mathrm{Split}_s(\mathbb{P}^n))$ has the expected dimension $2ns + 1$ for $n \geq 3$ and $s \geq 3$.

1. Introduction

We are interested in the *secant variety* to the variety $\mathbb{X} \subset \mathbb{P}^n$ and the *tangent space ideal at a point* in $\mathbb{X} \subset \mathbb{P}^n$, where $\mathbb{X}$ is a non-degenerate, reduced, and irreducible variety of dimension d. We are also interested in the *dimension* of the *secant variety* to determine if the *secant variety* is not defective. Recent papers studied the secant varieties ([1, 3, 4, 5, 7, 8, 9, 10, 12, 13]).

In [13], the author showed that if $\mathbb{X}^{(t,s)}$ is the union of two linear star-configurations in $\mathbb{P}^2$ of type $t \times s$ with $3 \leq t \leq 9$ and $s \geq t$, then $R/I_{\mathbb{X}^{(t,s)}}$ has generic Hilbert function, and $(I_{\mathbb{X}^{(s,s)}})_s = \{0\}$ for $s \geq 6$. With these two results, the author also showed that when $n = 2$, the secant variety $\mathrm{Sec}_1(\mathrm{Split}_s(\mathbb{P}^2))$ has the expected dimension $4s + 1$ for $s \geq 6$ and that the ideal $I_{\mathbb{X}^{(s,s)}}$ has the following property:

$$\dim_k(I_{\mathbb{X}^{(s,s)}})_s = \begin{cases} 3, 3, 1 & \text{for } s = 3, 4, 5, \text{ respectively}, \\ 0, & \text{for } s \geq 6. \end{cases}$$

In this paper we attempt to generalize this result and find an answer to the following question.

QUESTION 1.1. What is $\dim_k(I_{\mathbb{X}})_s$ when $\mathbb{X} := \mathbb{X}^{(s,s)}$ is the union of two linear star-configurations in $\mathbb{P}^n$ of type $s \times s$, $n \geq 3$ and $s \geq 3$?

In [3] the author showed that when $3(s-1) \leq n$ and $s > 2$, the secant variety $\mathrm{Sec}_{r-1}(\mathrm{Split}_s(\mathbb{P}^n))$ has the expected dimension using Terracini's Lemma, which will be introduced in the next section. We will however use the ideal of the union of two linear star-configurations $\mathbb{X}$ in $\mathbb{P}^n$ instead of Terracini's Lemma to find first the dimension, $\dim_k(I_{\mathbb{X}})_s$, and then the secant variety $\mathrm{Sec}_{r-1}(\mathrm{Split}_s(\mathbb{P}^n))$.

Our goal is to find an answer to Question 1.1 and its applications. In Section 2, we briefly review some definitions, notations, and preliminary results of the secant varieties $\mathrm{Sec}_{r-1}(\mathrm{Split}_s(\mathbb{P}^n))$. In Section 3, we show that if $\mathbb{X} := \mathbb{X}^{(s,s)}$ is the union of two linear star-configurations in $\mathbb{P}^n$ with $n \geq 3$ and $s \geq 3$, then

$$(I_{\mathbb{X}})_s = \{0\},$$

which is the key element to the complete answer to Question 1.1. With this result, we introduce another method to prove that the secant variety

$$\mathrm{Sec}_1(\mathrm{Split}_s(\mathbb{P}^n))$$

has the expected dimension $2ns + 1$ for $n \geq 3$ and $s \geq 3$.

2. Preliminary results and definitions

First, we recall definitions of Hilbert function, the secant varieties $\mathrm{Sec}_{r-1}(\mathrm{Split}_s(\mathbb{P}^n))$, and irreducible varieties respectively. Let $R = k[x_0, x_1, \ldots, x_n]$ be an $(n+1)$-variable polynomial ring over a field k of characteristic 0, R_d its homogeneous part of degree d, and $\mathbb{P}^n$ the projective n-space over a field k. With these notations, $\mathbb{P}(R_d) := \mathbb{P}^{\binom{n+d}{d}-1}$ is naturally identified with the set of hypersurfaces of degree d in $\mathbb{P}^n$. Recall that if I is a homogeneous ideal in R or the ideal of a subscheme $\mathbb{X}$ in $\mathbb{P}^n$, then $R/I = \bigoplus_{t \geq 0} R_t/I_t$ is a graded ring. In this situation the *Hilbert function of* $\mathbb{X}$ (or R/I) is the function of the subscheme $\mathbb{X}$ (or of the ring R/I) as follows:

$$\mathbf{H}_{\mathbb{X}}(t) = \mathbf{H}(R/I, t) := \dim_k R_t - \dim_k I_t.$$

The first difference of the Hilbert function $\mathbf{H}$ is defined by

$$\Delta\mathbf{H}(0) = 1 \text{ and } \Delta\mathbf{H}(t) = \mathbf{H}(t) - \mathbf{H}(t-1) \text{ for } t > 0.$$

Let $\lambda \vdash d$ denote a *partition* of the integer d, i.e.

$$\lambda = (\lambda_1, \ldots, \lambda_r) \text{ where } \lambda_1 \geq \cdots \geq \lambda_r \geq 1 \text{ and } \sum_{i=1}^{r} \lambda_i = d.$$

We associate a variety, denoted by $\mathbb{X}_{\lambda,n}$, to $R = k[x_0, x_1, \ldots, x_n]$ and λ, which is defined by

$$\mathbb{X}_{\lambda,n} := \{[F] \in \mathbb{P}(R_d) \mid F = F_1 \cdots F_r, \ \deg F_i = \lambda_i\},$$

and we omit the n if it is clear from the context. Such varieties are called *varieties of reducible forms*. If λ is the d-tuple $(1, \ldots, 1)$, then the variety is often referred to as the variety of *completely decomposable forms* or *split* forms. In this case, $\mathbb{X}_{\lambda,n}$ is denoted by $\mathrm{Split}_d(\mathbb{P}^n)$.

Let $\mathbb{X}_1, \ldots, \mathbb{X}_r$ all be non-degenerate, reduced and irreducible varieties in $\mathbb{P}^n$ with $\dim \mathbb{X}_i = d_i$.

DEFINITION 2.1 (Definition 2.1, [1]).   (a) Choose points $P_i \in \mathbb{X}_i$ such that $\{P_1, \ldots, P_r\}$ are linearly independent (and so $r \leq n$). The *join* of $\{P_1, \ldots, P_r\}$ is the linear space spanned by the points, i.e.,

$$\Lambda(P_1, \ldots, P_r) := \langle P_1, \ldots, P_r \rangle \simeq \mathbb{P}^{r-1}.$$

(b) The *join* of $\mathbb{X}_1, \ldots, \mathbb{X}_r$ is

$$\Lambda(\mathbb{X}_1, \ldots, \mathbb{X}_r) = \overline{\bigcup \Lambda(P_1, \ldots, P_r)}$$

for all $P_1, \ldots, P_r$ linearly independent with $P_i \in \mathbb{X}_i$.

(c) If $\mathbb{X}_1 = \cdots = \mathbb{X}_r = \mathbb{X}$ with $\dim \mathbb{X} = d$, then we write

$$\Lambda(\mathbb{X}_1, \ldots, \mathbb{X}_r) = \mathrm{Sec}_{r-1}(\mathbb{X})$$

and call it the $(r-1)$-st *secant variety* to $\mathbb{X}$.

The number of parameters shows that the upper bound of the dimension of the join is

$$\dim \Lambda(\mathbb{X}_1, \ldots, \mathbb{X}_r) \leq \min \left\{ n, \sum_{i=1}^{r} d_i + (r-1) \right\},$$

and thus

$$\dim \mathrm{Sec}_{r-1}(\mathbb{X}) \leq \min\{n, dr + (r-1)\}.$$

We now introduce Terracini's Lemma, which is useful to find the dimensions of both joins and secant varieties.

LEMMA 2.2 (Terracini's Lemma, [14]). *Let $\mathbb{X}_1, \ldots, \mathbb{X}_r$ be as above and let P_i be a generic point on $\mathbb{X}_i$. Let P be generic points in $\Lambda(P_1, \ldots, P_r)$. Then the projective tangent space to $\Lambda = \Lambda(\mathbb{X}_1, \ldots, \mathbb{X}_r)$ at P is*

$$T_{P,\Lambda} = \langle T_{P_1, \mathbb{X}_1}, \ldots, T_{P_r, \mathbb{X}_r} \rangle$$

i.e., the linear space of the tangent spaces at the given points.

DEFINITION 2.3. Let $\mathbb{X} \subset \mathbb{P}^n$ be a projective variety of dimension d. Then the *expected dimension* of the secant variety $\mathrm{Sec}_{r-1}(\mathbb{X})$ to $\mathbb{X}$ is defined by

$$\mathrm{expdim}(\mathrm{Sec}_{r-1}(\mathbb{X})) = \min\{n, dr + (r-1)\}.$$

However, the expected dimension of $\mathrm{Sec}_{r-1}(\mathbb{X})$ is not always the same as $\dim \mathrm{Sec}_{r-1}(\mathbb{X})$. When $\delta_{r-1} = \mathrm{expdim}(\mathrm{Sec}_{r-1}(\mathbb{X})) - \dim \mathrm{Sec}_{r-1}(\mathbb{X}) > 0$, we say that the secant variety $\mathrm{Sec}_{r-1}(\mathbb{X})$ to $\mathbb{X}$ is *defective* and δ_{r-1} is called *defect*.

Since we are interested in secants to the varieties of reducible forms, we introduce another important result (in view of Terracini's Lemma) in [7] to find a description of the tangent space at a generic point of those varieties.

PROPOSITION 2.4 ([7]). *Let $\lambda \vdash d$, $\lambda = (\lambda_1, \ldots, \lambda_r)$ and let $\mathbb{X}_{\lambda,n} \subset \mathbb{P}^{\binom{d+n}{n}-1}$. Let $P = [F_1 \cdots F_r]$ be a generic point of $\mathbb{X}_{\lambda,n}$ where $\deg F_i = \lambda_i$, $i = 1, \ldots, r$. Then*

$$T_{P,\mathbb{X}_{\lambda,n}} = \mathbb{P}(V_P)$$

where V_P is the subspace of $R_d = k[x_0, \ldots, x_n]_d$ defined by

$$V_P := \sum_{i=1}^{r} (F_1 \ldots \hat{F}_i \ldots F_r) R_{\lambda_i},$$

where $\hat{*}$ means that we omit $*$.

When we wish to find the dimension of the secant variety $\mathrm{Sec}_{r-1}(\mathbb{X}_{\lambda,n})$ to $\mathbb{X}_{\lambda,n}$, Terracini's Lemma clearly suggests that we choose first generic points, $P_1, \ldots, P_r$ on $\mathbb{X}_{\lambda,n}$, and then find the dimension of the subspace

$$V_{P_1} + \cdots + V_{P_r} \subset k[x_0, \ldots, x_n]_d.$$

We try to place this problem in a more general context.

DEFINITION 2.5. The *tangent space ideal of $\mathbb{X}_{\lambda,n}$ at the point P* is the unique saturated ideal, $\mathcal{T}_P$, in $R = k[x_0, \ldots, x_n]$, with the property that

$$(\mathcal{T}_P)_d = V_P.$$

The following corollary shows the dimension of the secant variety, by which we can decide whether or not the dimension and the expected dimension of the secant variety are the same. In other words, we can determine if the given secant variety is not defective.

COROLLARY 2.6 ([7]). *Let* $\lambda \vdash d$, $\lambda = (\lambda_1, \ldots, \lambda_r)$ *and let* $\mathbb{X}_{\lambda,n} \subset$ $\mathbb{P}^{\binom{d+n}{n}-1}$. *Let* $P_1, \ldots, P_r$ *be* r *generic points on* $\mathbb{X}_{\lambda,n}$. *Then*

$$\dim \mathrm{Sec}_{r-1}(\mathbb{X}_{\lambda,n}) = \left[\binom{d+n}{n} - \mathbf{H}(A, d) \right] - 1 = \dim_k I_d - 1$$

where $A = R/I$ *and* $I = \mathcal{T}_{P_1} + \cdots + \mathcal{T}_{P_r}$.

Finally we introduce a *star-configuration* and a *linear star-configuration* in $\mathbb{P}^n$.

PROPOSITION 2.7 ([1]). *Let* $F_1, F_2, \ldots, F_r$ *be general forms in* $R = k[x_0, x_1, \ldots, x_n]$ *with* $r \geq 3$. *Then*

$$\bigcap_{1 \leq i < j \leq r} (F_i, F_j) = \sum_{i=1}^{r} (F_1 \cdots \hat{F}_i \cdots F_r).$$

DEFINITION 2.8. With notations in Proposition 2.7, the variety $\mathbb{X}$ in $\mathbb{P}^n$ of the ideal $\bigcap_{1 \leq i < j \leq r} (F_i, F_j) = \sum_{i=1}^{r} (F_1 \cdots \hat{F}_i \cdots F_r)$ is called a *star-configuration* in $\mathbb{P}^n$ of type r. In particular, if $\deg F_i = 1$ for every $i = 1, \ldots, r$, we call $\mathbb{X}$ a *linear star-configuration* in $\mathbb{P}^n$ of type r. Furthermore, if $\mathbb{X}$ is the union of two star-configurations in $\mathbb{P}^n$ defined by s general linear forms and t general linear forms, respectively, then we call $\mathbb{X}$ the union of two linear star-configurations $\mathbb{P}^n$ of type $s \times t$, denoted by $\mathbb{X} := \mathbb{X}^{(s,t)}$.

REMARK 2.9.   (a) If $\mathbb{X}$ is a star-configuration in $\mathbb{P}^n$, then $\mathbb{X}$ is an arithmetically Cohen-Macaulay subscheme in $\mathbb{P}^n$ of codimension 2 (see Remark 2.2, [2]).

(b) Let $R = k[x_0, \ldots, x_n]$, $\mathbb{X}$ be a star-configuration in $\mathbb{P}^n$ with $n \geq 3$, and L be a general linear form in $R/I_{\mathbb{X}}$. Since $\mathbb{X}$ is an arithmetically Cohen-Macaulay subscheme in $\mathbb{P}^n$ of codimension 2, L is a non-zero divisor of $R/I_{\mathbb{X}}$. Thus $R/(I_{\mathbb{X}}, L)$ is also a Cohen-Macaulay ring of codimension 2. In other words, $(I_{\mathbb{X}}, L)/(L)$ is also the ideal of a star-configuration in $\mathbb{P}^{n-1}$.

3. Secant varieties $\mathrm{Sec}_1(\mathrm{Split}_s(\mathbb{P}^n))$

As mentioned in the introduction, in [3] they proved that the secant variety $\mathrm{Sec}_{r-1}(\mathrm{Split}_s(\mathbb{P}^n))$ has the expected dimension for $3(s-1) \leq n$ and $s > 2$ using Terracini's Lemma. As these two conditions indicate, it has been unknown for $n = 2$. In [13], the author showed that the secant

variety $\mathrm{Sec}_1(\mathrm{Split}_s(\mathbb{P}^2))$, where $n = 2$, has the expected dimension for $s > 2$.

In this section, we shall find the dimension of the ideal of the union of two linear star-configurations in $\mathbb{P}^n$ of type $s \times s$ in degree s. With this result, we shall give another method to prove that $\mathrm{Sec}_1(\mathrm{Split}_s(\mathbb{P}^n))$ has the expected dimension for $n \geq 3$ and $s > 2$.

The following lemma is immediately from Proposition 2.7 and Remark 2.9 (a), (see also Corollary 2.5 in [2]).

LEMMA 3.1. *Let L_i be general linear forms in $k[x_0, x_1, x_2, x_3]$ for $i = 1, \ldots, s$ with $s \geq 3$ and let*

$$I^{[s]} \quad := \quad \sum_{i=1}^{s} (L_1 \cdots \hat{L}_i \cdots L_s)$$

be the ideal of a linear star-configuration in $\mathbb{P}^3$. Then the Hilbert function of $R/I^{[s]}$ is

$$\mathbf{H}(R/I^{[s]}, t) \quad = \quad \begin{cases} \binom{3+t}{3}, & 0 \leq t \leq s-2, \\ \binom{3+(s-2)}{3} + (t - (s-2))\binom{2+(s-2)}{2}, & t \geq s-1. \end{cases}$$

Proof. Let L be a general linear form in R. By Remark 2.9 (b), the Hilbert function of $R/(I^{[s]}, L)$ is the same as the Hilbert function of the linear star-configuration in $\mathbb{P}^2$ of type s. Thus the first difference of the Hilbert function of $R/I^{[s]}$ is

$$\Delta \mathbf{H}(R/I^{[s]}, t) = \begin{cases} \binom{2+t}{2}, & 0 \leq t \leq s-2, \\ \binom{2+(s-2)}{2}, & t \geq s-1. \end{cases}$$

This implies that

$$\mathbf{H}(R/I^{[s]}, t) \quad = \quad \begin{cases} \binom{3+t}{3}, & 0 \leq t \leq s-2, \\ \binom{3+(s-2)}{3} + (t - (s-2))\binom{2+(s-2)}{2}, & t \geq s-1, \end{cases}$$

as we wished. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

REMARK 3.2. Let n and s be positive integers. By induction on n, we can easily obtain the following equation, and so we omit the proof.

$$\binom{s+n}{n} = \binom{(s-1)+n}{(s-1)} + \binom{(s-1)+(n-1)}{(s-1)} + \cdots + \binom{(s-1)+1}{(s-1)} + \binom{(s-1)+0}{(s-1)}.$$

The following proposition shows the dimension of the ideal of the linear star-configurations in $\mathbb{P}^n$. Using Corollary 2.5 in [2], one can obtain the following proposition. However, an elementary proof (which we now give) is also possible.

PROPOSITION 3.3. *Let $R = k[x_0, x_1, \ldots, x_n]$ and L_i be general linear forms in R for $i = 1, \ldots, s$ with $s \geq 3$ and $n \geq 3$. Let*

$$I^{[s]} \quad := \quad \sum_{i=1}^{s} (L_1 \cdots \hat{L}_i \cdots L_s)$$

be the ideal of a linear star-configuration in $\mathbb{P}^n$. Then

$$\dim_k I_s^{[s]} = ns + 1.$$

Proof. We shall prove this by induction on n with $n \geq 3$.
First, let $n = 3$. Then, by Lemma 3.1, the statement holds for this case.
Now assume $n > 3$. By Remark 2.9 (b), the first difference of the Hilbert function of $R/I^{[s]}$ is the Hilbert function of a linear star-configuration in $\mathbb{P}^{n-1}$ of type s. Hence, by induction on n, we have

$$
\begin{aligned}
&\mathbf{H}(R/I^{[s]}, s) - \mathbf{H}(R/I^{[s]}, s-1) \\
=\ & \Delta\mathbf{H}(R/I^{[s]}, s) \\
=\ & \binom{(s-2)+(n-1)}{(n-1)} + 2 \cdot \binom{(s-2)+(n-2)}{(n-2)} + \cdots + \\
& ((n-1)-2) \cdot \binom{(s-2)+3}{3} + ((n-1)-1) \cdot \binom{(s-2)+2}{2}.
\end{aligned}
$$

This implies that
(3.1)

$$
\begin{aligned}
&\mathbf{H}(R/I^{[s]}, s) \\
=\ & \Delta\mathbf{H}(R/I^{[s]}, s) + \mathbf{H}(R/I^{[s]}, s-1) \\
=\ & \left[\binom{(s-2)+(n-1)}{(n-1)} + 2 \cdot \binom{(s-2)+(n-2)}{(n-2)} + \cdots + ((n-1)-2) \cdot \binom{(s-2)+3}{3} + \right. \\
& \left. ((n-1)-1) \cdot \binom{(s-2)+2}{2} \right] + \left[\dim_k R_{s-1} - \dim_k I_{s-1}^{[s]} \right] \\
=\ & \left[\binom{(s-2)+(n-1)}{(n-1)} + 2 \cdot \binom{(s-2)+(n-2)}{(n-2)} + \cdots + ((n-1)-2) \cdot \binom{(s-2)+3}{3} + \right. \\
& \left. ((n-1)-1) \cdot \binom{(s-2)+2}{2} \right] + \left[\binom{(s-1)+n}{n} - s \right] \\
=\ & \left[\binom{(s-2)+(n-1)}{(n-1)} + 2 \cdot \binom{(s-2)+(n-2)}{(n-2)} + \cdots + \right. \\
& \left. ((n-1)-2) \cdot \binom{(s-2)+3}{3} + ((n-1)-1) \cdot \binom{(s-2)+2}{2} \right] \\
& + \left[\binom{(s-2)+n}{n} + \binom{(s-2)+(n-1)}{(n-1)} + \cdots + \binom{(s-2)+2}{2} + \right. \\
& \left. \binom{(s-2)+1}{1} + \binom{(s-2)+0}{0} - s \right] \quad \text{(by Remark 3.2)} \\
=\ & \left[\binom{(s-2)+(n-1)}{(n-1)} + 2 \cdot \binom{(s-2)+(n-2)}{(n-2)} + \cdots + \right. \\
& \left. ((n-1)-2) \cdot \binom{(s-2)+3}{3} + ((n-1)-1) \cdot \binom{(s-2)+2}{2} \right] + \\
& \left[\binom{(s-2)+n}{n} + \binom{(s-2)+(n-1)}{(n-1)} + \cdots + \binom{(s-2)+2}{2} + \binom{(s-2)+1}{1} \right]
\end{aligned}
$$

$$
\begin{aligned}
= \ & \Bigl[\tbinom{(s-2)+(n-1)}{(n-1)} + 2 \cdot \tbinom{(s-2)+(n-2)}{(n-2)} + \cdots + \\
& ((n-1)-2) \cdot \tbinom{(s-2)+3}{3} + ((n-1)-1) \cdot \tbinom{(s-2)+2}{2}\Bigr] + \\
& \Bigl[\tbinom{(s-2)+n}{n} + \tbinom{(s-2)+(n-1)}{(n-1)} + \cdots + \tbinom{(s-2)+2}{2} + \tbinom{(s-2)+1}{1}\Bigr] \\
= \ & \tbinom{(s-2)+n}{(s-2)} + 2 \cdot \tbinom{(s-2)+(n-1)}{(s-2)} + \cdots + \\
& (n-2) \cdot \tbinom{(s-2)+3}{(s-2)} + (n-1) \cdot \tbinom{(s-2)+2}{(s-2)},
\end{aligned}
$$

as we wanted. Furthermore, it is from Remark 3.2 that

$$
\begin{aligned}
\tbinom{s+n}{n} &= \tbinom{(s-1)+n}{n} + \tbinom{(s-1)+(n-1)}{(n-1)} + \cdots + \tbinom{(s-1)+1}{1} + \tbinom{(s-1)+0}{0}, \\
\tbinom{(s-1)+n}{n} &= \tbinom{(s-2)+n}{n} + \tbinom{(s-2)+(n-1)}{(n-1)} + \cdots + \\
&\quad \tbinom{(s-2)+2}{2} + \tbinom{(s-2)+1}{1} + \tbinom{(s-2)+0}{0}, \\
\tbinom{(s-1)+(n-1)}{(n-1)} &= \tbinom{(s-2)+(n-1)}{(n-1)} + \tbinom{(s-2)+(n-2)}{(n-2)} + \cdots + \\
&\quad \tbinom{(s-2)+2}{2} + \tbinom{(s-2)+1}{1} + \tbinom{(s-2)+0}{0}, \\
&\ \ \vdots \qquad\qquad\qquad\qquad \vdots \\
\tbinom{(s-1)+3}{3} &= \tbinom{(s-2)+3}{3} + \tbinom{(s-2)+2}{2} + \tbinom{(s-2)+1}{1} + \tbinom{(s-2)+0}{0}, \\
\tbinom{(s-1)+2}{2} &= \tbinom{(s-2)+2}{2} + \tbinom{(s-2)+1}{1} + \tbinom{(s-2)+0}{0}.
\end{aligned}
$$

This indicates that

$$
\begin{aligned}
& \dim_k R_s \\
= \ & \tbinom{s+n}{n} \\
= \ & \tbinom{(s-2)+n}{n} + 2 \cdot \tbinom{(s-2)+(n-1)}{(n-1)} + \cdots + (n-2) \cdot \tbinom{(s-2)+3}{3} + \\
& (n-1) \cdot \tbinom{(s-2)+2}{2} + \Bigl[\tbinom{(s-1)+1}{1} + \tbinom{(s-1)+0}{0}\Bigr] + \\
& (n-1)\Bigl[\tbinom{(s-2)+1}{1} + \tbinom{(s-2)+0}{0}\Bigr] \\
= \ & \tbinom{(s-2)+n}{n} + 2 \cdot \tbinom{(s-2)+(n-1)}{(n-1)} + \cdots + (n-2) \cdot \tbinom{(s-2)+3}{3} + \\
& (n-1) \cdot \tbinom{(s-2)+2}{2} + ns + 1 \\
= \ & \mathbf{H}(R/I^{[s]}, s) + ns + 1.
\end{aligned}
$$

(3.2)

From equation (3.2), we have

$$
\dim_k I_s^{[s]} = \dim_k R_s - \mathbf{H}(R/I^{[s]}, s) = ns + 1,
$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We now find the dimension of the ideal of the union of two linear star-configurations in $\mathbb{P}^3$ of type $s \times s$ in degree s. This lemma is a bridge to the main theorem (see Theorem 3.7).

LEMMA 3.4. *Let $R = k[x_0, x_1, x_2, x_3]$ and L_i, M_i be general linear forms in R for $i = 1, 2, \ldots, s$ with $s \geq 3$ and let*

$$
\begin{aligned}
I^{[s]} &:= \sum_{i=1}^{s} (L_1 \cdots \hat{L}_i \cdots L_s), \\
J^{[s]} &:= \sum_{i=1}^{s} (M_1 \cdots \hat{M}_i \cdots M_s),
\end{aligned}
$$

i.e., the ideals of linear star-configurations in $\mathbb{P}^3$ of type s defined by linear forms $L_1, \ldots, L_s$ and $M_1, \ldots, M_s$, respectively. Then, for $3 \leq s \leq 5$,

$$
\dim_k(I^{[s]} \cap J^{[s]})_s = 0.
$$

Proof. We shall prove this lemma with 3 cases for $s = 3, 4$, and 5, respectively.

Case 1. Let $s = 3$.
Define the ideal $I^{[2]} = (x_0, x_1)$. Without loss of generality, we may assume that

$$
M_1 = x_2, M_2 = x_3, M_3 = ax_0 + bx_1 + cx_2 + dx_3,
$$

where $a, b, c, d \in k - \{0\}$. Consider the following exact seqeuence.

$$
(3.3) \qquad 0 \;\to\; J^{[3]} \cap I^{[2]} \;\to\; J^{[3]} \;\to\; J^{[3]}/(J^{[3]} \cap I^{[2]}) \;\to\; 0.
$$

Since $J^{[3]}/(J^{[3]} \cap I^{[2]}) \simeq (J^{[3]} + I^{[2]})/I^{[2]}$, we can rewrite equation (3.3) as

$$
(3.4) \qquad 0 \;\to\; J^{[3]} \cap I^{[2]} \;\to\; J^{[3]} \;\to\; (J^{[3]} + I^{[2]})/I^{[2]} \;\to\; 0.
$$

Since the dimesnion of $(J^{[3]} + I^{[2]})/I^{[2]}$ in degree 2 is represented by $\dim_k(R/(x_0, x_1))_2$:

$$
\begin{aligned}
&\dim_k((J^{[3]} + I^{[2]})/I^{[2]})_2 \\
=\;&\dim_k\left(((M_1 M_2, M_1 M_3, M_2 M_3) + (x_0, x_1))/(x_0, x_1)\right)_2 \\
=\;&\dim_k\big(((x_2 x_3, x_2(ax_0 + bx_1 + cx_2 + dx_3), \\
&\quad x_3(ax_0 + bx_1 + cx_2 + dx_3)) + (x_0, x_1))/(x_0, x_1)\big)_2 \\
=\;&\dim_k\left(((x_2 x_3, cx_2^2, dx_3^2) + (x_0, x_1))/(x_0, x_1)\right)_2 \\
=\;&\dim_k\langle \bar{x}_2^2, \bar{x}_2 \bar{x}_3, \bar{x}_3^2 \rangle_2 \\
=\;&\dim_k(R/(x_0, x_1))_2,
\end{aligned}
$$

we get that

$$
\dim_k((J^{[3]} + I^{[2]})/I^{[2]})_3 = \dim_k(R/(x_0, x_1))_3 = 4.
$$

Note that the Hilbert function of $R/J^{[3]}$ is $3t + 1$ for $t \geq 0$. It is from equation (3.4) that

$$
\begin{aligned}
\text{(3.5)} \qquad \dim_k(J^{[3]} \cap I^{[2]})_3 &= \dim_k J_3^{[3]} - \dim_k(J^{[3]} + I^{[2]})/I^{[2]})_3 \\
&= 10 - 4 = 6.
\end{aligned}
$$

Now consider two ideals $I^{[3]}$ and $J^{[3]}$ and we assume

$$
L_1 = x_0, L_2 = x_1, L_3 = x_2.
$$

Define

$$
\begin{aligned}
J^{[3,1]} &= (M_1, M_2) \cap (M_1, M_3) \cap (M_2, M_3) \cap (x_1, x_2), \quad \text{and} \\
J^{[3,2]} &= J^{[3,1]} \cap (x_0, x_2) \\
&= J^{[3]} \cap (x_0, x_2) \cap (x_1, x_2).
\end{aligned}
$$

Since M_1, M_2, and M_3 are general linear forms,

$$
\begin{aligned}
\text{(3.6)} \qquad &\dim_k((J^{[3,1]} + I^{[2]})/I^{[2]})_3 \\
&= \dim_k((\overline{M_1 M_2}, \overline{M_1 M_3}, \overline{M_2 M_3}) \cap (\overline{x_2}))_3 \\
&= \dim_k((\overline{x_2 M_1 M_2}, \overline{x_2 M_1 M_3}, \overline{x_2 M_2 M_3}))_3 \\
&= 3.
\end{aligned}
$$

Moreover, it is from equation (3.5) that

$$
\begin{aligned}
\dim_k J_3^{[3,1]} &= \dim_k((M_1, M_2) \cap (M_1, M_3) \cap (M_2, M_3) \cap (x_1, x_2))_3 \\
&= \dim_k((M_1, M_2) \cap (M_1, M_3) \cap (M_2, M_3) \cap (x_0, x_1))_3 \\
&= \dim_k(J^{[3]} \cap I^{[2]})_3 = 6.
\end{aligned}
$$

So, for every $t \geq 0$,

$$
\begin{aligned}
\text{(3.7)} \qquad \dim_k J_t^{[3,2]} &= \dim_k(J^{[3]} \cap (x_1, x_2) \cap (x_0, x_2))_t \\
&= \dim_k(J^{[3]} \cap (x_1, x_2) \cap (x_0, x_1))_t \\
&= \dim_k(J^{[3,1]} \cap I^{[2]})_t.
\end{aligned}
$$

Using equation (3.7) and the following exact sequence

$$
0 \rightarrow J^{[3,1]} \cap I^{[2]} \rightarrow J^{[3,1]} \rightarrow (J^{[3,1]} + I^{[2]})/I^{[2]} \rightarrow 0,
$$

we obtain

$$
\begin{aligned}
\text{(3.8)} \qquad \dim_k J_3^{[3,2]} &= \dim_k(J^{[3,1]} \cap I^{[2]})_3 \\
&= \dim_k J_3^{[3,1]} - \dim_k((J^{[3,1]} + I^{[2]})/I^{[2]})_3 \\
&= 3.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
J^{[3,2]} &= J^{[3,1]} \cap (x_0, x_2) \\
&= J^{[3]} \cap (x_1, x_2) \cap (x_0, x_2) \\
&= J^{[3]} \cap (x_0 x_1, x_2),
\end{aligned}
$$

and so

$$
\begin{aligned}
&\dim_k((J^{[3,2]} + I^{[2]})/I^{[2]})_3 \\
= \ &\dim_k((J^{[3,2]} + (x_0, x_1))/(x_0, x_1))_3 \\
= \ &\dim_k((J^{[3]} \cap (x_0 x_1, x_2) + (x_0, x_1))/(x_0, x_1))_3 \\
= \ &\dim_k(\overline{x_2 M_1 M_2}, \overline{x_2 M_1 M_3}, \overline{x_2 M_2 M_3})_3 \\
= \ &3.
\end{aligned}
$$

Note that

(3.9)
$$
\begin{aligned}
J^{[3]} \cap I^{[3]} \ &= \ J^{[3]} \cap (x_0, x_1) \cap (x_1, x_2) \cap (x_0, x_2) \\
&= \ \left(J^{[3]} \cap (x_1, x_2) \cap (x_0, x_2)\right) \cap (x_0, x_1) \\
&= \ J^{[3,2]} \cap I^{[2]}.
\end{aligned}
$$

Using equation (3.9) and the following exact sequence

$$
0 \ \to \ J^{[3,2]} \cap I^{[2]} \ \to \ J^{[3,2]} \ \to \ (J^{[3,2]} + I^{[2]})/I^{[2]} \ \to \ 0,
$$

we have that

(3.10)
$$
\begin{aligned}
\dim_k(J^{[3]} \cap I^{[3]})_3 &= \dim_k J_3^{[3,2]} - \dim_k\left((I^{[3,2]} + I^{[2]})/I^{[2]}\right)_3 \\
&= 3 - 3 = 0.
\end{aligned}
$$

Case 2. Let $s = 4$.
Without loss of generality, assume that

$$
L_1 = x_0, L_2 = x_1, L_3 = x_2, L_4 = x_3.
$$

Since all the M_i are general linear forms, we have that

(3.11)
$$
\begin{aligned}
\dim_k((J^{[4]} + I^{[2]}]/I^{[2]})_3 \ &= \ \dim_k\left(\sum_{i=1}^{4}(\overline{M_1 \cdots \hat{M}_i \cdots M_4})\right)_3 \\
&= \ 4 \\
&= \ \dim_k(R/(x_0, x_1))_3.
\end{aligned}
$$

Thus

$$
\dim_k((J^{[4]} + I^{[2]})/I^{[2]})_4 = 5.
$$

Using Proposition 3.3 and the following exact sequence

$$
0 \ \to \ J^{[4]} \cap I^{[2]} \ \to \ J^{[4]} \ \to \ (J^{[4]} + I^{[2]})/I^{[2]} \ \to \ 0,
$$

we obtain that

(3.12)
$$
\begin{aligned}
\dim_k(J^{[4]} \cap I^{[2]})_4 &= \dim_k J_4^{[4]} - \dim_k((J^{[4]} + I^{[2]})/I^{[2]})_4 \\
&= 13 - 5 = 8.
\end{aligned}
$$

Define

$$
\begin{aligned}
J^{[4,1]} \ &:= \ J^{[4]} \cap (x_1, x_2), \quad \text{and} \\
J^{[4,2]} \ &:= \ J^{[4]} \cap (x_1, x_2) \cap (x_2, x_3).
\end{aligned}
$$

By equation (3.12),

$$\begin{aligned}
\dim_k J_4^{[4,1]} &= \dim_k[J^{[4]} \cap (x_1, x_2)]_4 \\
&= \dim_k[J^{[4]} \cap (x_0, x_1)]_4 \quad (\text{since } M_i \text{ are general}) \\
&= \dim_k(J^{[4]} \cap I^{[2]})_4 \\
&= 8.
\end{aligned}$$

(3.13)

Note that

$$\begin{aligned}
&\dim_k((J^{[4,1]} + I^{[2]})/I^{[2]})_4 \\
&= \dim_k(J^{[4]} \cap (x_1, x_2) + (x_0, x_1))/(x_0, x_1))_4 \\
&= \dim_k(\overline{J^{[4]}} \cap (\bar{x}_2))_4 \\
&= \dim_k(\overline{J^{[4]}})_3 \\
&= \dim_k J_3^{[4]} \\
&= 4, \qquad\qquad (\text{by equation (3.11)}), \text{ and} \\
\dim_k J_t^{[4,2]} &= \dim_k(J^{[4]} \cap (x_1, x_2) \cap (x_2, x_3))_t \\
&= \dim_k(J^{[4]} \cap (x_1, x_2) \cap (x_0, x_1))_t \\
&\quad (\text{since } M_i \text{ are general linear forms}) \\
&= \dim_k(J^{[4,1]} \cap I^{[2]})_t \quad \text{for every } t \geq 0.
\end{aligned}$$

(3.14)

Using equation (3.14) and the following exact sequence

$$0 \;\rightarrow\; J^{[4,1]} \cap I^{[2]} \;\rightarrow\; J^{[4,1]} \;\rightarrow\; (J^{[4,1]} + I^{[2]})/I^{[2]} \;\rightarrow\; 0,$$

we get that

$$\begin{aligned}
\dim_k J_4^{[4,2]} &= \dim_k(J^{[4,1]} \cap I^{[2]})_4 \\
&= \dim_k(J^{[4,1]})_4 - \dim_k((J^{[4,1]} + I^{[2]})/I^{[2]})_4 \\
&= 8 - 4 \\
&= 4.
\end{aligned}$$

Using the same method as in equation (3.14), we obtain that

$$\begin{aligned}
&\dim_k((J^{[4,2]} + I^{[2]})/I^{[2]})_4 \\
&= \dim_k((J^{[4,2]} + (x_0, x_1))/(x_0, x_1))_4 \\
&= \dim_k(J^{[4]} \cap (x_1, x_2) \cap (x_2, x_3) + (x_0, x_1))/(x_0, x_1))_4 \\
&= \dim_k(J^{[4]} \cap (x_2) \cap (x_2, x_3) + (x_0, x_1))/(x_0, x_1))_4 \\
&= \dim_k(J^{[4]} \cap (x_2) + (x_0, x_1))/(x_0, x_1))_4 \\
&= \dim_k(\bar{J}^{[4]} \cap (\bar{x}_2))_4 \\
&= \dim_k \bar{J}_3^{[4]} \\
&= 4.
\end{aligned}$$

(3.15)

Note that

$$\begin{aligned}
J^{[4]} \cap I^{[4]} &\subseteq (J^{[4]} \cap (x_1, x_2) \cap (x_2, x_3)) \cap (x_0, x_1) \\
&= J^{[4,2]} \cap I^{[2]}.
\end{aligned}$$

(3.16)

Using equations (3.15) and (3.16), and the following exact sequence

$$0 \quad \to \quad J^{[4,2]} \cap I^{[2]} \quad \to \quad J^{[4,2]} \quad \to \quad ((J^{[4,2]} + I^{[2]})/I^{[2]} \quad \to \quad 0,$$

we have

$$\begin{aligned}
\dim_k(J^{[4]} \cap I^{[4]})_4 \quad &\le \quad \dim_k J^{[4,2]} \cap I^{[2]} \\
&= \quad \dim_k(J^{[4,2]})_4 - \dim_k(((J^{[4,2]} + I^{[2]})/I^{[2]})_4 = 0.
\end{aligned}$$

Case 3. Let $s = 5$.
Define

$$L_1 = x_0, L_2 = x_1, L_3 = x_2, L_4 = x_3.$$

Since the M_i are general linear forms, we have that

$$\begin{aligned}
\dim_k((J^{[5]} + I^{[2]}]/I^{[2]})_4 \quad &= \quad \left(\sum_{i=1}^{5} \overline{(M_1 \cdots \hat{M}_i \cdots M_5)} \right)_4 \\
&= \quad 5 \\
&= \quad \dim_k(R/(x_0, x_1))_4.
\end{aligned}$$

Thus we know

$$\dim_k((J^{[5]} + I^{[2]})/I^{[2]})_5 = 6,$$

and by Proposition 3.3,

$$\dim_k J_5^{[5]} = 16.$$

Using the following exact sequence

$$0 \quad \to \quad J^{[5]} \cap I^{[2]} \quad \to \quad J^{[5]} \quad \to \quad (J^{[5]} + I^{[2]})/I^{[2]} \quad \to \quad 0,$$

we obtain that

$$\begin{aligned}
(3.17) \qquad \dim_k(J^{[5]} \cap I^{[2]})_5 &= \dim_k J_5^{[5]} - \dim_k((J^{[5]} + I^{[2]})/I^{[2]})_5 \\
&= 16 - 6 = 10.
\end{aligned}$$

Define

$$\begin{aligned}
J^{[5,1]} \quad &:= \quad J^{[5]} \cap (x_1, x_2), \quad \text{and} \\
J^{[5,2]} \quad &:= \quad J^{[5]} \cap (x_1, x_2) \cap (x_2, x_3).
\end{aligned}$$

By equation (3.17),

$$\begin{aligned}
(3.18) \qquad \dim_k J_5^{[5,1]} \quad &= \quad \dim_k(J^{[5]} \cap (x_1, x_2))_5 \\
&= \quad \dim_k(J^{[5]} \cap (x_0, x_1))_5 \\
&= \quad \dim_k(J^{[5]} \cap I^{[2]})_5 \\
&= \quad 10.
\end{aligned}$$

Moreover, note that

$$
\begin{aligned}
&\dim_k((J^{[5,1]} + I^{[2]})/I^{[2]})_5 \\
=\ &\dim_k((J^{[5,1]} + (x_0, x_1))/(x_0, x_1))_5 \\
=\ &\dim_k(J^{[5]} \cap (x_1, x_2) + (x_0, x_1))/(x_0, x_1))_5 \\
=\ &\dim_k(\overline{J^{[5]}} \cap (\bar{x}_2))_5 \\
=\ &\dim_k(\overline{J^{[5]}})_4 \\
=\ &\dim_k J_4^{[5]} \\
=\ &5, \quad \text{and} \\
\dim_k J_5^{[5,2]} =\ &\dim_k(J^{[5]} \cap (x_1, x_2) \cap (x_2, x_3))_5 \\
=\ &\dim_k(J^{[5]} \cap (x_1, x_2) \cap (x_0, x_1))_5 \\
&(\text{since } M_i \text{ are general linear forms}) \\
=\ &\dim_k(J^{[5,1]} \cap I^{[2]})_5.
\end{aligned}
$$

(3.19)

Using equations (3.18), (3.19) and the following exact sequence

$$
0 \ \rightarrow\ J^{[5,1]} \cap I^{[2]} \ \rightarrow\ J^{[5,1]} \ \rightarrow\ (J^{[5,1]} + I^{[2]})/I^{[2]} \ \rightarrow\ 0,
$$

we obtain that

(3.20)
$$
\begin{aligned}
\dim_k J_5^{[5,2]} &= \dim_k(J^{[5,1]})_5 - \dim_k((J^{[5,1]} + I^{[2]})/I^{[2]})_5 \\
&= 10 - 5 = 5.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\dim_k((J^{[5,2]} + I^{[2]})/I^{[2]})_5 \\
=\ &\dim_k(J^{[5]} \cap (x_1, x_2) \cap (x_2, x_3) + (x_0, x_1))/(x_0, x_1))_5 \\
=\ &\dim_k(J^{[5]} \cap (x_2) \cap (x_2, x_3) + (x_0, x_1))/(x_0, x_1))_5 \\
=\ &\dim_k(J^{[5]} \cap (x_2) + (x_0, x_1))/(x_0, x_1))_5 \\
=\ &\dim_k(\overline{J^{[5]}} \cap (\overline{x_2}))_5 \\
&(\text{since } M_i \text{ are general linear forms}) \\
=\ &\dim_k(\overline{J^{[5]}})_4 \\
=\ &\dim_k(J^{[5]})_4 \\
=\ &5, \quad \text{and} \\
&\dim_k(J^{[5]} \cap I^{[5]})_5 \\
\leq\ &\dim_k(J^{[5]} \cap I^{[4]})_5 \\
\leq\ &\dim_k(J^{[5]} \cap (x_1, x_2) \cap (x_2, x_3) \cap (x_0, x_1))_5 \\
=\ &\dim_k(J^{[5,2]} \cap I^{[2]})_5.
\end{aligned}
$$

(3.21)

Using equations (3.20) and (3.21), and the following exact sequence

$$
0 \ \rightarrow\ J^{[5,2]} \cap I^{[2]} \ \rightarrow\ J^{[5,2]} \ \rightarrow\ (J^{[5,2]} + I^{[2]})/I^{[2]} \ \rightarrow\ 0,
$$

we have

$$
\begin{aligned}
\dim_k(J^{[5]} \cap I^{[5]})_5 &\leq \dim_k(J^{[5,2]} \cap I^{[2]})_5 \\
&= \dim_k(J^{[5,2]})_5 - \dim_k((J^{[5,2]} + (x_0, x_1))/(x_0, x_1))_5 \\
&= 0,
\end{aligned}
$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now we are ready to prove the main theorem. We first introduce the following theorem and proposition in [13].

THEOREM 3.5 ([13]). *Let* $R = k[x_0, x_1, x_2] = \bigoplus_{i=0}^n R_i$ *Let* $\mathbb{X} := \mathbb{X}^{(t,s)}$ *be the union of two linear star-configurations in* $\mathbb{P}^2$ *of type* $t \times s$ *with* $3 \leq t \leq 9$ *and* $s \geq t$. *Then* $R/I_{\mathbb{X}}$ *has generic Hilbert function.*

PROPOSITION 3.6 (Proposition 4.1, [13]). *Let* $\mathbb{X} := \mathbb{X}^{(s,s)}$ *be the union of two linear star-configurations in* $\mathbb{P}^2$ *of type* $s \times s$ *with* $s \geq 6$. *Then*

$$
(I_{\mathbb{X}})_s = \{0\}.
$$

THEOREM 3.7. *Let* $R = k[x_0, x_1, \ldots, x_n]$ *with* $n \geq 3$ *and* L_i, M_i *be general linear forms in* R *for* $i = 1, 2, \ldots, s$ *with* $s \geq 3$. *Let*

$$
\begin{aligned}
I^{[s]} &:= \sum_{i=1}^{s}(L_1 \cdots \hat{L}_i \cdots L_s), \\
J^{[s]} &:= \sum_{i=1}^{s}(M_1 \cdots \hat{M}_i \cdots M_s).
\end{aligned}
$$

Then

$$
\dim_k(I^{[s]} \cap J^{[s]})_s = 0.
$$

Proof. We shall prove this theorem by induction on $n \geq 3$. First, by Lemma 3.4, the statement holds for $n = 3$ and $3 \leq s \leq 5$.

Now assume $n > 3$ and $3 \leq s \leq 5$. By Remark (b), the union of two star-configurations in $\mathbb{P}^n$ is also a subscheme in $\mathbb{P}^n$ of codimension 2, and so we may assume $L = x_0$ is a nonzero divisor of $R/I^{[s]} \cap J^{[s]}$. Define

$$
\begin{aligned}
(I^{[s]} \cap J^{[s]}, L)/(L) &:= \overline{I^{[s]} \cap J^{[s]}} \\
&\subseteq R/(x_0) \simeq S = k[x_1, \ldots, x_n], \\
\bar{L}_i &:= (L_i + (x_0))/(x_0), \\
\bar{M}_i &:= (M_i + (x_0))/(x_0), \\
\bar{I}^{[s]} &:= \sum_{i=1}^{s}(\bar{L}_1 \cdots \hat{\bar{L}}_i \cdots \bar{L}_s), \quad \text{and} \\
\bar{J}^{[s]} &:= \sum_{i=1}^{s}(\bar{M}_1 \cdots \hat{\bar{M}}_i \cdots \bar{M}_s).
\end{aligned}
$$

Since $\overline{I^{[s]} \cap J^{[s]}}$ is not saturated in general, for $3 \leq s \leq 5$

(3.22)
$$\dim_k(\overline{I^{[s]} \cap J^{[s]}})_s \leq \dim_k(\bar{I}^{[s]} \cap \bar{J}^{[s]})_s = 0$$
$$\text{(by Lemma 3.4 and induction on } n)$$
$$\Rightarrow \quad \dim_k(\overline{I^{[s]} \cap J^{[s]}})_s = 0.$$

Furthermore, since L is not a zero divisor of $I^{[s]} \cap J^{[s]}$, we get that

$$(I^{[s]} \cap J^{[s]})_s = \{0\}$$

for such s.

Now consider the case for $n \geq 3$ and $s \geq 6$. With the same notations as above, by Proposition 3.6,

$$(\bar{I}^{[s]} \cap \bar{J}^{[s]})_s = \{0\} \quad \text{for } n = 3 \text{ and } s \geq 6.$$

By the same arguments as in equation (3.22),

$$(I^{[s]} \cap J^{[s]})_s = \{0\} \quad \text{for } n = 3 \text{ and } s \geq 6.$$

Therefore, by induction on n, we show that

$$(I^{[s]} \cap J^{[s]})_s = \{0\} \quad \text{for } n \geq 3 \text{ and } s \geq 6,$$

which completes the proof. $\square$

As an immediate consequence of Proposition 3.3, Lemma 3.4, and Theorem 3.5 with Corollary 4.3 in [13], we obtain the following corollary.

COROLLARY 3.8.
$$\mathrm{Sec}_1(\mathrm{Split}_s(\mathbb{P}^n))$$
has the expected dimension for $n \geq 2$ and $s \geq 3$. In particular,

$$\dim \mathrm{Sec}_1(\mathrm{Split}_s(\mathbb{P}^n)) = \mathrm{expdim}\, \mathrm{Sec}_1(\mathrm{Split}_s(\mathbb{P}^n)) = 2ns + 1,$$

for $n \geq 3$ and $s \geq 3$.

Proof. First, by Corollary 4.3 in [13], the statement holds for $n = 2$ and $s \geq 3$.
Now suppose $n \geq 3$ and $s \geq 3$. Let $\mathbb{X} := \mathbb{X}^{(s,s)}$ be the union of two linear star-configurations $\mathbb{X}_1$ and $\mathbb{X}_2$ in $\mathbb{P}^n$ of type s, and let $I := I_{\mathbb{X}_1} + I_{\mathbb{X}_2}$. By Theorem 3.7 and the following exact sequence

$$0 \quad \to \quad I_{\mathbb{X}} \quad \to \quad I_{\mathbb{X}_1} \oplus I_{\mathbb{X}_2} \quad \to \quad I \quad \to \quad 0,$$

we have that

$$\dim_k I_s = \dim_k(I_{\mathbb{X}_1})_s + \dim_k(I_{\mathbb{X}_2})_s = 2\dim_k(I_{\mathbb{X}_1})_s = 2ns + 2,$$

and hence

$$
\begin{aligned}
&\text{expdim}\,\text{Sec}_1(\text{Split}_s(\mathbb{P}^n))\\
=\;&\min\{2\times\dim(\underbrace{\mathbb{P}(R_1)\times\cdots\times\mathbb{P}(R_1)}_{s\text{-times}})+1,\dim\mathbb{P}(R_s)\}\\
=\;&\min\left\{2ns+1,\binom{s+n}{n}-1\right\}\\
=\;&2ns+1\qquad(\text{since } n\geq 3 \text{ and } s\geq 3)\\
=\;&\dim_k I_s-1\\
=\;&\dim\text{Sec}_1(\text{Split}_s(\mathbb{P}^n))\quad(\text{by Corollary 2.6}),
\end{aligned}
$$

as we wished. □

REMARK 3.9. In [3], they showed that the secant variety

$$\text{Sec}_{r-1}(\text{Split}_s(\mathbb{P}^n))$$

has the expected dimension for $3(s-1)\leq n$ and $s>2$ using Terracini's Lemma (see [3] and [14]). Their results however do not cover the case of reducible plane curves. For this case, the author in [13] showed that the secant line varieties $\text{Sec}_1(\text{Split}_s(\mathbb{P}^2))$ still have the expected dimension. In Corollary 3.8 of this paper, we introduced another way (algebraic method) to prove that the secant line varieties $\text{Sec}_1(\text{Split}_s(\mathbb{P}^n))$ have the expected dimension for $n\geq 3$.

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