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A NEW FAMILY OF NEGATIVE QUADRANT DEPENDENT BIVARIATE DISTRIBUTIONS WITH CONTINUOUS MARGINALS

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Abstract. In this paper, we study a family of continuous bivariate distributions that possesses the negative quadrant dependence property and the generalized negatively quadrant dependent F-G-M copula. We also develop the partial ordering of this new parametric family of negative quadrant dependent distributions.

1. Introduction

In statistical analysis, the assumption that random variables are independent is seldom valid in practice. For example, two components in a reliability structure may share the reverse load or are subjected to the reverse set of stresses. This will tend to cause two lifetime random variables to be negatively dependent.

In other words, our consideration to dependence structures for random variables are aroused.

Random variables X and Y are called negatively quadrant dependent(NQD) if

$$P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y)$$

or, equivalently,

$$P(X > x, Y > y) \leq P(X > x)P(Y > y)$$

for all x and y. This notion was introduced by Lehmann(1965).

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Negative quadrant dependence is shown to be a stronger notion of dependence than negative (Pearson) correlation but weaker than the "negative association" introduced by Joag-Dev and Proschan(1983).

Negative quadrant dependence is a qualitative form of negative dependence and indicates whether or not a pair of random variables exhibits negative dependence. However, for many purposes, in addition to the knowledge of the nature of negative dependence, it is also important to know the degree of negative quadrant dependence.

Ahmed et al.(1979) have studied very extensively the partial ordering of positive quadrant dependence which permits us to compare pairs of positive quadrant dependent bivariate random variables of interest with specified marginals as to their degree of positive quadrant dependence. Quite in the same spirit, Ebrahimi(1982) studied the degree of negative quadrant dependence.

Lai and Xie(2000) showed conditions for having positive quadrant dependence and studied a class of bivariate uniform distributions having positive quadrant dependence property by generalizing the uniform representation of a well-known Farlie-Gumbel-Morgenstern distribution. By a simple transformation, they also obtained families of bivariate distributions with pre-specified marginals. In Section 2, quite in the same spirit of Lai and Xie(2000), we study a family of continuous bivariate distributions that possesses the negative quadrant dependence and in Section 3 we consider a generalized Farlie-Gumbel-Morgenstern copula having negative quadrant dependence property. In Section 4, we also discuss the negative quadrant dependence ordering of the new negative quadrant dependent family.

In particular, we extend the notions of Lai and Xie(2000) to the negative quadrant dependence.

2. Conditions for negative quadrant dependence

Let $H(x, y)$ denote the bivariate distribution function of (X, Y) having continuous marginal cdfs $F(x)$ and $G(y)$ with marginal pdfs $f(x) = \frac{d}{dx}F(x)$ and $g(y) = \frac{d}{dy}G(y)$. The joint distribution function $H(x, y)$ may be written as

$$H(x, y) = F(x)G(y) + w(x, y) \ \textit{for all } x \textit{ and } y, \tag{2.1}$$

where $w(x, y)$ satisfies the following conditions:

$$w(x, \infty) = 0,\ w(\infty, x) = 0,\ w(x, -\infty) = 0,\ w(-\infty, x) = 0 \tag{2.2}$$

for all x and y,

$$\frac{\partial^2 w(x,y)}{\partial x \partial y} + f(x)g(y) \geq 0 \; for\ all\ x\ and\ y. \tag{2.3}$$

From the definition of negative quadrant dependence, $H(x,y)$ is negatively quadrant dependent if and only if, for all x and y

$$w(x,y) \leq 0. \tag{2.4}$$

REMARK 2.1. Note that (2.1) is equivalent to

$$\bar{H}(x,y) = \bar{F}(x)\bar{G}(y) + w(x,y) \; for\ all\ x\ and\ y, \tag{2.5}$$

where $\bar{H}(x,y) = P(X > x, Y > y)$, $\bar{F}(x) = P(X > x)$ and $\bar{G}(x) = P(Y > y)$.

Proof.

$$\begin{aligned}
F(x,y) &= P(X > x, Y > y) \\
&= P(X > x) - P(X > x, Y \leq y) \\
&= P(X > x) - P(Y \leq y) - P(X \leq x, Y \leq y) \\
&= P(X > x) - P(Y \leq y) + H(x,y) \\
&= F(x)G(y) + w(x,y) + \bar{F}(x) - 1 + \bar{F}(y) \\
&= (1 - \bar{F}(x))(1 - \bar{G}(y)) + w(x,y) + \bar{F}(x) + \bar{F}(y) - 1 \\
&= \bar{F}(x)\bar{G}(y) + w(x,y).
\end{aligned}$$

□

EXAMPLE 2.2. *Morgenstern(1956), Farlie(1960) and Gumbel(1955) have discussed families of bivariate distributions of the form*

$$H(x,y) = F(x)G(y)[1 + \rho(1 - F(x))(1 - G(y))], \; -1 \leq \rho \leq 1, \tag{2.6}$$

which is called F-G-M bivariate distribution. It is easy to verify that X and Y are negatively quadrant dependent if $-1 \leq \rho \leq 0$. Clearly, for $-1 \leq \rho \leq 0$,

$$w(x,y) = \rho F(x)G(y)(1 - F(x))(1 - G(y)) \leq 0, \tag{2.7}$$

and (2.6) implies

$$h(x,y) = f(x)g(y)[1 + \rho(1 - 2F(x))(1 - 2G(y))] \geq 0, \tag{2.8}$$

where $h(x,y)$ is joint probability density function of X and Y.

REMARK 2.3. For $-1 \leq \rho \leq 0$, $w(x,y)$ satisfies the conditions (2.2),(2.3) and (2.4).

EXAMPLE 2.4. *Consider a special case of the negatively quadrant dependent F-G-M system where both marginals are exponential. The joint distribution function is then of the form, see for example, Johnson and Kotz(1972, pp262-263)*

$$H(x,y) = (1 - e^{\lambda_1 x})(1 - e^{\lambda_2 y})[1 + \rho e^{-\lambda_1 x - \lambda_2 y}], \ -1 \le \rho \le 0, \ \lambda_1, \lambda_2 > 0.$$

Clearly, for $-1 \le \rho \le 0$,

$$w(x,y) = \rho e^{-\lambda_1 x - \lambda_2 y}(1 - e^{\lambda_1 x})(1 - e^{\lambda_2 y})$$

satisfies conditions (2.2), (2.3) and (2.4), that is, for $-1 \le \rho \le 0$ $H(x,y)$ *is negatively quadrant dependent.*

REMARK 2.5. Note that for $\lambda_1, \ \lambda_2 > 0$ and $-1 \le \rho \le 0$

$$\bar{H}(x,y) = e^{-\lambda_1 x - \lambda_2 y}[1 + \rho(1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 y})].$$

In the following theorem we establish preservation of negative quadrant dependence under combination.

THEOREM 2.6. *Every convex combination of two bivariate distribution functions of the form (2.1), which have fixed continuous marginals* $F(x)$ *and* $G(y)$ *and satisfy (2.2)-(2.4), still satisfies (2.2)-(2.4).*

Proof. Let

$$H_1(x,y) = F(x)G(y) + w_1(x,y), \ H_2(x,y) = F(x)G(y) + w_2(x,y),$$

where $w_1(x,y)$ and $w_2(x,y)$ satisfy conditions (2.2)-(2.4).

Let $H(x,y) = \alpha H_1(x,y) + (1-\alpha)H_2(x,y), \ 0 \le \alpha \le 1$.

Then

$$H(x,y) = F(x)G(y) + w(x,y)$$

where $w(x,y) = \alpha w_1(x,y) + (1-\alpha)w_2(x,y)$.

Clearly, $w(x,y)$ satisfies (2.2)-(2.4). Hence, $H(x,y)$ is a negatively quadrant dependent bivariate distribution such that $w(x,y)$ satisfies (2.2)-(2.4). □

EXAMPLE 2.7. *Let*

$$H_0(x,y) = F(x)G(y) \ for \ all \ x \ and \ y, \tag{2.9}$$

and

$$H^*(x,y) = \max(0, F(x) + G(y) - 1) \text{ for all } x \text{ and } y. \tag{2.10}$$

Then

$$H_\alpha(x,y) = \alpha H_0(x,y) + (1-\alpha)H^*(x,y)$$

satisfies (2.2)-(2.4) and thus $H_\alpha(x,y)$ *is negatively quadrant dependent bivariate distribution. First note that* $H^*(x,y) = \max(0, F(x) + G(y) -$

1) *is negatively quadrant dependent and* $w(x,y) = (\alpha-1)[F(x)G(y) - \max(0, F(x)+G(y)-1)]$ *satisfies conditions (2.2)-(2.4).*

Proof. It is clear that for $0 < \alpha < 1$,

$$
\begin{aligned}
H_\alpha(x,y) &= \alpha F(x)G(y) \ if \ F(x)+G(y) \le 1 \\
&\quad F(x)G(y) - (1-\alpha)(1-F(x))(1-G(y)) \\
&\quad\ if \ F(x)+G(y) \ge 1
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
w(x,y) &= -(1-\alpha)F(x)G(y) \ if \ F(x)+G(y) \le 1 \\
&\quad -(1-\alpha)(1-F(x))(1-G(y)) \ if \ F(x)+G(y) \ge 1.
\end{aligned}
$$

Hence, $w(x,y)$ satisfies (2.2)-(2.4) and thus $H_\alpha(x,y)$ is negatively quadrant dependent. □

The F-G-M bivariate distribution has been studied extensively. It has several applications in various contexts, for example, in competing risk problems(Tolley and Norman(1979)), in joint occurrence of certain trace elements in water(Cook and Johnson(1986)) and a robustness(Delahorra and Fernandez(1985)). For more details on copulas see, for example, Nelsen(1999). In the next section, we study a generalized F-G-M bivariate distribution having negative quadrant dependence property.

3. A generalized negatively quadrant dependent F-G-M copula

For simplicity, let F and G be absolutely continuous and let X and Y be dependent according to a copula $C(u,v)$ for $0 \le u \le 1,\ 0 \le v \le 1$. Thus the joint distribution $H(x,y)$ of X and Y is given by

$$H(x,y) = C(F(x), G(y)), \tag{3.1}$$

see, e.g. Nelsen(2006, p.15). Let $U = F(X)$ and $V = G(Y)$, so that they are two uniform random variables following the joint distribution $C(u,v) = H(F^{-1}(u), G^{-1}(v))$, where $F^{-1} = \inf\{x : F(x) = u\}$, i.e., $F^{-1}(u)$ is either the inverse function of F or the inverse set function. G^{-1} is defined similarly.

It is well known that the negatively quadrant dependent F-G-M bivariate distribution discussed in Section 2 has copula given by

$$C(u,v) = uv\{1+\rho(1-u)(1-v)\},\ -1 \le \rho \le 0, \tag{3.2}$$

where $w(u,v) = \rho uv(1-u)(1-v)$.

Let $w(u,v)$ be generalized further to a bivariate beta function, i.e.,

$$w(u,v) = \rho u^b v^b (1-u)^a (1-v)^a,\ a,b \geq 1,\ -1 \leq \rho \leq 0. \tag{3.3}$$

THEOREM 3.1. *Let $C(u,v) = uv + w(u,v) = uv + \rho u^b v^b (1-u)^a (1-v)^a$. Then, $C(u,v)$ is the distribution function of a bivariate uniform distribution having the negative quadrant dependence property.*

Proof. It is clear that conditions (2.2) and (2.4) are satisfied. The condition (2.3) is also satisfied (see Appendix). □

From Theorem 3.1 we automatically obtain the following result.

COROLLARY 3.2. *Suppose that $w(u,v)$ has a form*

$$w(u,v) = \rho uv(1-u)^a(1-v)^a,\ a \geq 1,\ -1 \leq \rho \leq 0.$$

Then,

$$C(u,v) = uv\{1 + \rho(1-u)^a(1-v)^a\} \tag{3.4}$$

gives rises to a negative quadrant dependent bivariate distribution.

We establish preservation of negative quadrant dependence under convex combination.

THEOREM 3.3. *Let $C_1(u,v) = uv + \rho_1 u^{b_1} v^{b_1} (1-u)^{a_1} (1-v)^{a_1},\ a_1 \geq 1, b_1 \geq 1,\ -1 \leq \rho_1 \leq 0$ and $C_2(u,v) = uv + \rho_2 u^{b_2} v^{b_2} (1-u)^{a_2} (1-v)^{a_2},\ a_2 \geq 1, b_2 \geq 1,\ -1 \leq \rho_2 \leq 0$ be the distributions of bivariate uniform distribution, and let the convex combination $C(u,v) = \alpha C_1(u,v) + (1-\alpha)C_2(u,v),\ 0 < \alpha < 1$. Then, $C(u,v)$ is the distribution of a bivariate uniform distribution having negative quadrant dependence property.*

Proof. (i) The case $a_1 = a_2 = a,\ b_1 = b_2 = b$; Let

$$\begin{aligned} C(u,v) &= \alpha C_1(u,v) + (1-\alpha)C_2(u,v) \\ &= uv + [\alpha\rho_1 + (1-\alpha)\rho_2] u^b v^b (1-u)^a (1-v)^a \\ &= uv + \rho u^b v^b (1-u)^a (1-v)^a, \end{aligned} \tag{3.5}$$

where $\rho = \alpha\rho_1 + (1-\alpha)\rho_2$.

It is clear that $-1 \leq \min(\rho_1, \rho_2) \leq \rho \leq 0$. Hence, by Theorem 3.1 (3.5) is negatively quadrant dependent.

(ii) The case $a_1 \neq a_2,\ b_1 \neq b_2$; Let

$$\begin{aligned} C(u,v) &= \alpha C_1(u,v) + (1-\alpha)C_2(u,v) \\ &= uv + \alpha\rho_1 u^{b_1} v^{b_1} (1-u)^{a_1} (1-v)^{a_1} \\ &\qquad + (1-\alpha)\rho_2 u^{b_2} v^{b_2} (1-u)^{a_2} (1-v)^{a_2} \\ &= uv + w(u,v), \end{aligned} \tag{3.6}$$

where $w(u,v) = \alpha\rho_1 u^{b_1} v^{b_1}(1-u)^{a_1}(1-v)^{a_1} + (1-\alpha)\rho_2 u^{b_2} v^{b_2}(1-u)^{a_2}(1-v)^{a_2}$. It is clear that $w(u,v)$ satisfies (2.2) and (2.4). By Appendix, for $-1 \leq \rho_1 \leq 0$ and $-1 \leq \rho_2 \leq 0$,

$$\frac{\partial^2}{\partial u \partial v}[\alpha\rho_1 u^{b_1} v^{b_1}(1-u)^{a_1}(1-v)^{a_1}] \geq -\alpha, \tag{3.7}$$

$$\frac{\partial^2}{\partial u \partial v}[(1-\alpha)\rho_2 u^{b_2} v^{b_2}(1-u)^{a_2}(1-v)^{a_2}] \geq -(1-\alpha). \tag{3.8}$$

It follows from (3.7) and (3.8) that $\frac{\partial^2 w(u,v)}{\partial u \partial v} \geq -1$ which satisfies (2.3).

Hence, (3.6) is negatively quadrant dependent. □

Remark 3.4. When $-1 \leq \rho_1 \leq 0$ and $-1 \leq \rho_2 \leq 0$, take $a = \min(a_1, a_2)$ and $b = \min(b_1, b_2)$ and $\rho = \max(\rho_1, \rho_2)$. Then, we also have

$$\begin{aligned} C(u,v) &\leq uv + (\alpha\rho_1 + (1-\alpha)\rho_2)[u^b v^b (1-u)^a (1-v)^a] \\ &\leq uv + \rho[u^b v^b (1-u)^a (1-v)^a] \leq uv. \end{aligned}$$

Hence $C(u,v)$ is negatively quadrant dependent.

4. The ordering of negative quadrant dependence

It would be of some interest to study bivariate negative dependence orderings of this new parametric family of distributions we obtained earlier. Fundamentally, we consider the concept of one pair of random variables being more negatively dependent than another pair.

Let $\beta = \beta(F, G)$ denote the class of bivariate joint distribution functions H on $\mathbb{R}^2$ having specified marginal distribution functions F and G. Let $\bar{\beta}$ denote the subclass of β where H is negative quadrant dependent.

Definition 4.1 (Ebrahimi(1982)). Let H_1 and H_2 belong to $\bar{\beta}$. The bivariate distribution H_2 is said to be more negatively quadrant dependent than H_1 if

$$H_2(x,y) \leq H_1(x,y) \textit{ for all } x \textit{ and } y. \tag{4.1}$$

Remark 4.2. Note that the requirement of specified marginals is important because we can alter the degree of negative quadrant dependence changing the scale.

Remark 4.3. An equivalent form of (4.1) is

$$\bar{H}_2(x,y) \leq \bar{H}_1(x,y) \textit{ for all } x \textit{ and } y. \tag{4.2}$$

where $\bar{H}(x,y) = P(X > x, Y > y)$.

THEOREM 4.4. *Suppose both $H_1(x,y)$ and $H_2(x,y)$ belong to $\bar{\beta}$ and $H_1(x,y)$ and $H_2(x,y)$ can be written as (2.1) i.e., let*

$$H_1(x,y) = F(x)G(y) + w_1(x,y)$$

and

$$H_2(x,y) = F(x)G(y) + w_2(x,y),$$

where $w_1(x,y)$ and $w_2(x,y)$ satisfy the conditions (2.2)-(2.4).

Then

$$w_2(x,y) \leq w_1(x,y) \Leftrightarrow H_2(x,y) \leq H_1(x,y).$$

Now we define a negative quadrant dependence family of bivariate distributions H which is increasing in λ(decreasing in λ) and give some interesting examples of such families.

DEFINITION 4.5 (Ebrahimi(1982)). A family of distributions $\{H = H_\lambda(x,y) : \lambda \in \Lambda \text{ and } H \in \bar{\beta}\}, \Lambda \in \mathbb{R}$, is said to be increasing negative quadrant dependent(decreasing negative quadrant dependent) in λ if and only if

$$\lambda^{'} > \lambda \Rightarrow H_{\lambda'} < H_\lambda (H_{\lambda'} > H_\lambda).$$

EXAMPLE 4.6. *Let $H(x,y) = F(x)G(y) + w(x,y)$, where $w(x,y)$ satisfies (2.2)-(2.4). For $\lambda, \lambda' > 0$ let*

$$\begin{aligned} H_\lambda &= F(x)G(y) + \lambda w(x,y), \\ H_{\lambda'} &= F(x)G(y) + \lambda^{'} w(x,y). \end{aligned}$$

Then $0 < \lambda^{'} < \lambda \Rightarrow H_\lambda < H_{\lambda'}$.

EXAMPLE 4.7. *Consider the bivariate F-M-G family of distributions with df*

$$\begin{aligned} H_\alpha(x,y) &= F(x)G(y)[1 + \alpha(1 - F(x))(1 - G(y))], \ -1 \leq \alpha \leq 0. \\ &= F(x)G(y) + w_\alpha(x,y). \end{aligned}$$

Then for $-1 < \alpha_1 < \alpha_2 \leq 0$,

$$\begin{aligned} & \alpha_1(1 - F(x))(1 - G(y)) < \alpha_2(1 - F(x))(1 - G(y)) \\ \Rightarrow \ & w_{\alpha_1}(x,y) < w_{\alpha_2}(x,y) \\ \Rightarrow \ & H_{\alpha_1}(x,y) < H_{\alpha_2}(x,y). \end{aligned}$$

Hence, H_α is decreasing negatively quadrant dependent in α.

EXAMPLE 4.8. *For* $-1 \leq \rho_1, \rho_2 \leq 0,\ \lambda_1 > 0,\ \lambda_2 > 0$, *let*

$$H_1(x,y) = (1 - e^{\lambda_1 x})(1 - e^{\lambda_2 y})[1 + \rho_1 e^{-\lambda_1 x - \lambda_2 y}]$$

and

$$H_2(x,y) = (1 - e^{\lambda_1 x})(1 - e^{\lambda_2 y})[1 + \rho_2 e^{-\lambda_1 x - \lambda_2 y}].$$

Then,

$$\rho_1 \leq \rho_2 \text{ implies } H_1 \leq H_2.$$

Moreover, let

$$C_\rho(u,v) = uv + \rho u^b v^b (1-u)^a (1-v)^a,\ a, b \geq 1,\ -1 \leq \rho \leq 0. \tag{4.3}$$

Then

$$\rho_1 < \rho_2 \text{ implies } C_{\rho_1} \leq C_{\rho_2}.$$

Hence, $C_\rho(u,v)$ *is decreasing negative quadrant dependent in* ρ.

THEOREM 4.9. *Define* $C_\rho(u,v)$ *as above in (4.3). Then, the correlation coefficient is decreasing in* ρ.

Proof. The joint product moments are

$$\begin{aligned} EU^iV^i &= \frac{1}{i+1} \times \frac{1}{j+1} \\ &+ \rho B(i+b,a)B(j+b,a)[b - \frac{(a+b)(i+b)}{i+a+b}][b - \frac{(a+b)(j+b)}{i+a+b}]. \end{aligned}$$

The covariance is

$$Cov(U,V) = \rho[B(b+1,a+1)]^2$$

and the correlation coefficient is

$$Corr(U,V) = 12\rho[B(b+1,a+1)]^2.$$

Hence, the correlation coefficient is decreasing in $\rho(-1 \leq \rho \leq 0)$. $\square$

Appendix

Lemma A Let $w(u,v) = \rho u^b v^b (1-u)^a (1-v)^a,\ a, b \geq 1$ and $-1 \leq \rho \leq 0$. Then

$$\frac{\partial^2 w(u,v)}{\partial u \partial v} \geq -1 \text{ for all } u \geq 1 \text{ and } v \geq 1.$$

Proof. The proof is similar to the idea in Appendix in Lai and Xie(2000). But for the completeness we repeat it here.

From partial differentiation of $w(u,v) = \rho u^b v^b (1-u)^a (1-v)^a$, we have

$$\frac{\partial w(u,v)}{\partial u} = \rho v^b (1-v)^b g(u),\ \frac{\partial^2 w(u,v)}{\partial u \partial v} = \rho g(u) g(v),$$

where

$$(A.1) \qquad g(u) = u^{b-1}(1-u)^{a-1}[b-(a+b)u].$$

It is enough to show that

$$\frac{\partial^2 w(u,v)}{\partial u \partial v} = \rho g(u)g(v) \geq -1, \text{ for } -1 \leq \rho \leq 0.$$

$$(A.2) \qquad g^{'}(u) = b(b-1) - 2u[b(a+b)-1] + (a+b)(a+b-1)u^2$$

and $g^{'}(u^*) = 0$ at the point

$$(A.3) \qquad u^* = \frac{b(a+b-1) + \sqrt{ab(a+b-1)}}{(a+b)(a+b-1)}.$$

From (A.3) we have

$$[b-(a+b)u^*] = -\sqrt{\frac{ab}{a+b-1}}.$$

Note that for $0 \leq u \leq b/(a+b), g(u) \geq 0$ and for $b/(a+b) \leq u \leq 1, g(u) \leq 0$. It follows from (A.3) that $b/(a+b) \leq u^* \leq 1$ and thus $g(u^*) \leq 0$.

Rewrite u^* given in (A.3), i.e.,

$$u^* = \frac{b}{a+b} + \frac{1}{a+b} \times \sqrt{\frac{ab}{a+b-1}}.$$

For $a, b \geq 1, (b-1) \geq (b-1)/a$ implies $ab > a+b-1$, which yields $u^* \geq (b+1)/(a+b)$.

Since $(1-u)^{a-1} \geq 0$ we only consider $h_1(b) = u^{b-1}[b-(a+b)u]$, where $g(u) = (1-u)^{a-1}h_1(b)$.

Now $\frac{\partial h_1(b)}{\partial b} = u^{b-1}\{\log u \times [b-(a+b)u] + (1-u)\} > 0$ for $u \geq b/(a+b)$ which shows that g is an increasing function in u for $u \geq b/(a+b)$.

Similarly, by the fact that $u^{b-1} \geq 0$, we only consider $h_2(a) = (1-a)^{a-1}[b-(a+b)u]$, where $g(u) = u^{b-1}h_2(a)$. Then $\frac{\partial h_2(a)}{\partial a} = (1-u)^{a-1}\{\log(1-u) \times [b-(a+b)u] - u\} \geq 0$ if $\{\log(1-u) \times [b-(a+b)u] - u\} \geq 0$.

Since $\log(1-u) < -u$, it follows that $\log(1-u)[b-(a+b)u] - u > -u[b-(a+b)u] - u > 0$ for $u \geq (b+1)/(a+b)$. In other words, $h_2^{'}(a) \geq 0$ and thus g is increasing in a for $u \geq (b+1)/(a+b)$.

Since $(b+1)/(a+b) \leq u^* \leq 1$, $g(u^*)$ is increasing in both a and b.

When $a = 1$, and $b = 1$ $g(u^*) = -1$, which yields $g(u) \geq -1$ for all $a \geq 1$ and $b \geq 1$. Also $g(u) < 1$, for all $a \geq 1$, $b \geq 1$ and $0 \leq u \leq 1$.

Hence, $\frac{\partial^2 w(u,v)}{\partial u \partial v} = \rho g(u)g(v) \geq -1$ for $-1 \leq \rho \leq 0$, $0 \leq u \leq 1$. In other words $w(u,v)$ satisfies condition (2.3). $\square$

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