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DECOMPOSITION FORMULAS AND INTEGRAL REPRESENTATIONS FOR SOME EXTON HYPERGEOMETRIC FUNCTIONS

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Abstract. Generalizing the Burchnall-Chaundy operator method, the authors are aiming at presenting certain decomposition formulas for the chosen six Exton functions expressed in terms of Appell's functions F_3 and F_4, Horn's functions H_3 and H_4, and Gauss's hypergeometric function F. We also give some integral representations for the Exton functions X_i $(i = 6, 8, 14)$ each of whose kernels contains the Horn's function H_4.

1. Introduction and preliminaries

Almost seven decades ago, Burchnall and Chaundy (see [2, 3]) and Chaundy [4] systematically presented a number of expansion and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. Their method is based upon the following inverse pairs of symbolic operators:

$$\nabla_{xy}(h) := \frac{\Gamma(h)\,\Gamma(\delta_1+\delta_2+h)}{\Gamma(\delta_1+h)\,\Gamma(\delta_2+h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k\,(-\delta_2)_k}{(h)_k\,k!}, \tag{1.1}$$

$$\Delta_{xy}(h) := \frac{\Gamma(\delta_1+h)\,\Gamma(\delta_2+h)}{\Gamma(h)\,\Gamma(\delta_1+\delta_2+h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k\,(-\delta_2)_k}{(1-h-\delta_1-\delta_2)_k\,k!}, \tag{1.2}$$

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$$\begin{aligned}\nabla_{xy}(h)\,\Delta_{xy}(g) &:= \frac{\Gamma(h)\,\Gamma(\delta_1+\delta_2+h)}{\Gamma(\delta_1+h)\,\Gamma(\delta_2+h)}\frac{\Gamma(\delta_1+g)\,\Gamma(\delta_2+g)}{\Gamma(g)\,\Gamma(\delta_1+\delta_2+g)} \\ &= \sum_{k=0}^{\infty}\frac{(g-h)_k\,(g)_{2k}\,(-\delta_1)_k\,(-\delta_2)_k}{(g+k-1)_k\,(g+\delta_1)_k\,(g+\delta_2)_k\,k!} \\ &= \sum_{k=0}^{\infty}\frac{(h-g)_k\,(-\delta_1)_k\,(-\delta_2)_k}{(h)_k\,(1-g-\delta_1-\delta_2)_k\,k!} \quad \left(\delta_1 := x\frac{\partial}{\partial x};\ \delta_2 := y\frac{\partial}{\partial y}\right).\end{aligned} \tag{1.3}$$

In the definitions of operators (1.1) to (1.3) was used the well-known Gauss's summation theorem for the hypergeometric function ${}_2F_1 := F$ (see, e.g., [7, p. 104, Eq. (46)]):

$$F(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \quad \left(\Re(c-a-b)>0;\ c\in\mathbb{C}\setminus\mathbb{Z}_0^-\right), \tag{1.4}$$

where $\mathbb{C}$ is the set of complex numbers and $\mathbb{Z}_0^- := \{0, -1, -2, \ldots\}$.

By employing their symbolic operators, Burchnall and Chaundy obtained several decomposition formulas for Appell's hypergeometric function (see [1, p. 14, Eqs. (11)-(14)] and, also see [7, p. 225, Eqs. (6)-(9)]) and Humbert functions of two variables, which are particularly recorded in [7, p. 94, pp. 187-188, and pp. 234-235]. The presented operators decompose the double hypergeometric functions into the product of two one-dimensional functions. Therefore, an investigation of certain properties of the double hypergeometric functions can be made by means of one-dimensional functions. It is interesting to note that this method has been forgotten without any particular reason, though this method turned out to be useful in application of the other hypergeometric functions. We note that in monograph [25] was given the definitions of 205 hypergeometric functions with their regions of convergence. Furthermore, in the most recent works [5, 11, 12, 13, 14], aforementioned two-dimensional operators (1.1) to (1.3) were generalized to the following multi-dimensional inverse pairs of symbolic operators:

$$\begin{aligned}\tilde{\nabla}_{x_1;x_2,\ldots,x_r}(h) &:= \frac{\Gamma(h)\,\Gamma(\delta_1+\delta_2+\ldots+\delta_r+h)}{\Gamma(\delta_1+h)\,\Gamma(\delta_2+\ldots+\delta_r+h)} \\ &= \sum_{m_2,\ldots,m_r=0}^{\infty}\frac{(-\delta_1)_{m_2+\ldots+m_r}\,(-\delta_2)_{m_2}\cdots(-\delta_r)_{m_r}}{(h)_{m_2+\ldots+m_r}\,m_2!\cdots m_r!},\end{aligned} \tag{1.5}$$

$$\tilde{\Delta}_{x_1;x_2,\ldots,x_r}(h) := \frac{\Gamma(\delta_1+h)\,\Gamma(\delta_2+\cdots+\delta_r+h)}{\Gamma(h)\,\Gamma(\delta_1+\delta_2+\cdots+\delta_r+h)}$$

$$= \sum_{m_2,\cdots,m_r=0}^{\infty} \frac{(-\delta_1)_{m_2+\cdots+m_r}(-\delta_2)_{m_2}\cdots(-\delta_r)_{m_r}}{(1-h-\delta_1-\cdots-\delta_r)_{m_2+\cdots+m_r}\, m_2!\cdots m_r!}$$

$$= \sum_{m_2,\cdots,m_r=0}^{\infty} \frac{(-1)^{m_2+\cdots+m_r}}{(h+m_2+\cdots+m_r-1)_{m_2+\cdots+m_r}(h+\delta_1)_{m_2+\cdots+m_r}}$$

$$\cdot \frac{(h)_{2(m_2+\cdots+m_r)}(-\delta_1)_{m_2+\cdots+m_r}(-\delta_2)_{m_2}\cdots(-\delta_r)_{m_r}}{(h+\delta_2+\cdots+\delta_r)_{m_2+\cdots+m_r}\, m_2!\cdots m_r!} \tag{1.6}$$

$$\left(\delta_j := x_j\frac{\partial}{\partial x_j};\ j=1,2,...,r\right).$$

By means of generalized symbolic operators (1.5) and (1.6), decomposition formulas for Lauricella function (see [1, p. 114, Eqs. (1)-(4)]), Horn functions (see [1, p. 224, Eqs. (10)-(11)], and Srivastava's hypergeometric functions (see [23, 24], also see [25, p. 43, Eqs. (11)-(13)]) have been presented. It should be noted that the obtained decomposition formulas have then been used in the solution of boundary-value problems of degenerated differential equations of second order (see [9, 10, 15, 22]). An attentive reader may have observed that there is no inverse pairs of symbolic operators that can be used in one-dimensional case. For this purpose we define following operators:

$$H_{x_1,\ldots,x_l}(\alpha,\beta) := \frac{\Gamma(\beta)\,\Gamma(\alpha+\delta_1+\cdots+\delta_l)}{\Gamma(\alpha)\,\Gamma(\beta+\delta_1+\cdots+\delta_l)}$$

$$= \sum_{k_1,\cdots,k_l=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_l}(-\delta_1)_{k_1}\cdots(-\delta_l)_{k_l}}{(\beta)_{k_1+\cdots+k_l}\, k_1!\cdots k_l!} \tag{1.7}$$

and

$$\bar{H}_{x_1,\ldots,x_l}(\alpha,\beta) := \frac{\Gamma(\alpha)\,\Gamma(\beta+\delta_1+\cdots+\delta_l)}{\Gamma(\beta)\,\Gamma(\alpha+\delta_1+\cdots+\delta_l)}$$

$$= \sum_{k_1,\ldots k_l=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_l}(-\delta_1)_{k_1}\cdots(-\delta_l)_{k_l}}{(1-\alpha-\delta_1-\cdots-\delta_l)_{k_1+\cdots+k_l}\, k_1!\cdots k_l!} \tag{1.8}$$

$$\left(\delta_j := x_j\frac{\partial}{\partial x_j},\ j=1,...,l;\ \ l\in\mathbb{N}:=\{1,2,3,\ldots\}\right).$$

In order to obtain aforementioned operators, we have applied some known multiple hypergeometric summation formulas such as (see [2,

p. 117]):

$$F_D^{(r)}\left(\alpha;\beta_1,...,\beta_r;\gamma;1,...,1\right)=\frac{\Gamma\left(\gamma\right)\Gamma\left(\gamma-\alpha-\beta_1-\cdots-\beta_r\right)}{\Gamma\left(\gamma-\alpha\right)\Gamma\left(\gamma-\beta_1-\cdots-\beta_r\right)}$$

$$\left(\Re\left(\gamma-\alpha-\beta_1-\cdots-\beta_r\right)>0;\ \gamma\in\mathbb{C}\setminus\mathbb{Z}_0^-\right)$$

for Lauricella function $F_D^{(r)}$ in r variables (see [1, p. 117]).

Here, applying the inverse pairs of operators (1.7) and (1.8) to Exton functions X_i ($i = 2$, 6, 8, 14, 17, 20) recalled below among his twenty ones (see [8], also see [25, pp. 113–114]), we aim at presenting certain decomposition formulas for the chosen Exton functions expressed in terms of Appell's functions F_3 and F_4, Horn's functions H_3 and H_4, and Gauss's hypergeometric function F. We also give some integral representations for the Exton functions X_i ($i = 6$, 8, 14) each of whose kernels contains the Horn's function H_4.

We recall, for convenience, the six Exton functions:

$$X_2\left(a_1,a_2;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_{2m+2n+p}\left(a_2\right)_p}{\left(c_1\right)_m\left(c_2\right)_n\left(c_3\right)_p\,m!n!p!}x^my^nz^p \tag{1.9}$$

$$\left(\left\{2\sqrt{r}+2\sqrt{s}+1<1\right\},\ \ |x|\le r,\ \ |y|\le s,\ \ |z|\le t\right);$$

$$X_6\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_{2m+n+p}\left(a_2\right)_n\left(a_3\right)_p}{\left(c_1\right)_{m+n}\left(c_2\right)_p\,m!n!p!}x^my^nz^p \tag{1.10}$$

$$\left(\left\{t+2\sqrt{r}<1\wedge s<\frac{1}{2}\left(1-t\right)+\frac{1}{2}\sqrt{\left(1-t\right)^2-4r}\right\},\right.$$
$$\left.|x|\le r,\ \ |y|\le s,\ \ |z|\le t\right);$$

$$X_8\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_{2m+n+p}\left(a_2\right)_n\left(a_3\right)_p}{\left(c_1\right)_m\left(c_2\right)_n\left(c_3\right)_p\,m!n!p!}x^my^nz^p \tag{1.11}$$

$$\left(\left\{2\sqrt{r}+s+t<1\right\},\ \ |x|\le r,\ \ |y|\le s,\ \ |z|\le t\right);$$

$$X_{14}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{\left(a_1\right)_{2m+n}\left(a_2\right)_{n+p}\left(a_3\right)_p}{\left(c_1\right)_{m+n}\left(c_2\right)_p\,m!n!p!}x^my^nz^p \tag{1.12}$$

$$\left(\left\{r<\frac{1}{4}\wedge t<1\wedge s<(1-t)\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4r}\right]\right\},\right.$$
$$\left.|x|\leq r,\quad |y|\leq s,\quad |z|\leq t\right);$$

$$X_{15}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p}{(c_1)_m(c_2)_{n+p}\,m!n!p!}x^m y^n z^p \tag{1.13}$$
$$\left(\left\{s+2\sqrt{r}<1\wedge t<1\right\},\ |x|\leq r,\quad |y|\leq s,\quad |z|\leq t\right);$$

$$X_{17}\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p\,m!n!p!}x^m y^n z^p \tag{1.14}$$
$$\left(\left\{r<\frac{1}{4}\wedge t<1\wedge s<\left(1-2\sqrt{r}\right)(1-t)\right\},\ |x|\leq r,\quad |y|\leq s,\quad |z|\leq t\right);$$

$$X_{20}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)$$
$$=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_p}{(c_1)_{m+p}(c_2)_n\,m!n!p!}x^m y^n z^p \tag{1.15}$$
$$\left(\left\{s+2\sqrt{r}<1\wedge t<1\right\},\ |x|\leq r,\quad |y|\leq s,\quad |z|\leq t\right).$$

It is *remarked* in passing that the Exton functions have recently been actively investigated (see, e.g., [6, 16, 17, 18, 19]).

2. A set of operator identities

By using Burchnall and Chaundy [2, 3] and Chaundy [4] method together with symbolic operators (1.7) and (1.8), we find the following set of operator identities for hypergeometric Exton functions:

$$X_2\left(a_1,a_2;c_1,c_2,c_3;x,y,z\right)=H_z\left(a_2,c_3\right)$$
$$\cdot(1-z)^{-a_1}F_4\left(\frac{a_1}{2},\frac{a_1}{2}+\frac{1}{2};c_1,c_2;\frac{4x}{(1-z)^2},\frac{4y}{(1-z)^2}\right); \tag{2.1}$$

$$(1-z)^{-a_1}F_4\left(\frac{a_1}{2},\frac{a_1}{2}+\frac{1}{2};c_1,c_2;\frac{4x}{(1-z)^2},\frac{4y}{(1-z)^2}\right)$$
$$=\bar{H}_z\left(a_2,c_3\right)X_2\left(a_1,a_2;c_1,c_2,c_3;x,y,z\right); \tag{2.2}$$

$$X_6\left(a_1, a_2, a_3; c_1, c_2; x, y, z\right) = H_z\left(a_3, c_2\right)(1-z)^{-a_1} H_3\left(a_1, a_2; c_1; \frac{x}{(1-z)^2}, \frac{y}{1-z}\right); \tag{2.3}$$

$$(1-z)^{-a_1} H_3\left(a_1, a_2; c_1; \frac{x}{(1-z)^2}, \frac{y}{1-z}\right) = \bar{H}_z\left(a_3, c_2\right) X_6\left(a_1, a_2, a_3; c_1, c_2; x, y, z\right); \tag{2.4}$$

$$X_8\left(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z\right) = H_z\left(a_3, c_3\right)(1-z)^{-a_1} H_4\left(a_1, a_2; c_1, c_2; \frac{x}{(1-z)^2}, \frac{y}{1-z}\right); \tag{2.5}$$

$$(1-z)^{-a_1} H_4\left(a_1, a_2; c_1, c_2; \frac{x}{(1-z)^2}, \frac{y}{1-z}\right) = \bar{H}_z\left(a_3, c_3\right) X_8\left(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z\right); \tag{2.6}$$

$$X_{14}\left(a_1, a_2, a_3; c_1, c_2; x, y, z\right) = H_z\left(a_3, c_2\right)(1-z)^{-a_2} H_3\left(a_1, a_2; c_1; x, \frac{y}{1-z}\right); \tag{2.7}$$

$$(1-z)^{-a_2} H_3\left(a_1, a_2; c_1; x, \frac{y}{1-z}\right) = \bar{H}_z\left(a_3, c_2\right) X_{14}\left(a_1, a_2, a_3; c_1, c_2; x, y, z\right); \tag{2.8}$$

$$X_{15}\left(a_1, a_2, a_3; c_1, c_2; x, y, z\right) = H_{y,z}\left(a_2, c_2\right)(1-y)^{-a_1}(1-z)^{-a_3} F\left(\frac{a_1}{2}, \frac{a_1}{2}+\frac{1}{2}; c_1; \frac{4x}{(1-y)^2}\right); \tag{2.9}$$

$$(1-y)^{-a_1}(1-z)^{-a_3} F\left(\frac{a_1}{2}, \frac{a_1}{2}+\frac{1}{2}; c_1; \frac{4x}{(1-y)^2}\right) = \bar{H}_{y,z}\left(a_2, c_2\right) X_{15}\left(a_1, a_2, a_3; c_1, c_2; x, y, z\right); \tag{2.10}$$

$$X_{17}\left(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z\right) = H_z\left(a_3, c_3\right)(1-z)^{-a_2} H_4\left(a_1, a_2; c_1, c_2; x, \frac{y}{1-z}\right); \tag{2.11}$$

$$(1-z)^{-a_2} H_4\left(a_1, a_2; c_1, c_2; x, \frac{y}{1-z}\right) = \bar{H}_z\left(a_3, c_3\right) X_{17}\left(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z\right); \tag{2.12}$$

$$
\begin{aligned}
&X_{20}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right)\\
&\quad=H_y\left(a_2,c_2\right)(1-y)^{-a_1}F_3\left(\frac{a_1}{2},a_3,\frac{a_1}{2}+\frac{1}{2},a_4;c_1;\frac{4x}{(1-y)^2},z\right);
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
&(1-y)^{-a_1}F_3\left(\frac{a_1}{2},a_3,\frac{a_1}{2}+\frac{1}{2},a_4;c_1;\frac{4x}{(1-y)^2},z\right)\\
&\quad=\bar{H}_y\left(a_2,c_2\right)X_{20}\left(a_1,a_2,a_3,a_4;c_1,c_2;x,y,z\right);
\end{aligned}
\tag{2.14}
$$

where

$$
\begin{aligned}
&F_3\left(a_1,a_2,b_1,b_2;c;x,y\right)\\
&\quad=\sum_{m,n=0}^{\infty}\frac{(a_1)_m(a_2)_n(b_1)_m(b_2)_n}{(c)_{m+n}\,m!n!}x^m y^n\qquad(|x|<1,\ |y|<1)
\end{aligned}
$$

and

$$
F_4\left(a,b;c_1,c_2;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m(c_2)_n\,m!n!}x^m y^n\qquad\left(\sqrt{|x|}+\sqrt{|y|}<1\right)
$$

are Appell's hypergeometric functions (see [7, p. 224, Eqs. (8) and (9)]) and

$$
H_3\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_{m+n}\,m!n!}x^m y^n\qquad\left(|x|+\left(|y|-\frac{1}{2}\right)^2\le\frac{1}{4}\right)
$$

and

$$
H_4\left(a,b;c_1,c_2;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c_1)_m(c_2)_n\,m!n!}x^m y^n\qquad\left(4|x|\le(|y|-1)^2\right)
$$

are the Horn's hypergeometric functions (see [7, p. 225, Eqs. (15) and (16)]), and F is Gauss hypergeometric function (see [7, p. 56, Eq. (1)]). The proof of the operator identities (2.1) to (2.14) is based on employing of Mellin [20], and Mellin-Barnes integral representation methods (see [7, p. 232]) for hypergeometric functions. The details involved in these derivations are omitted here.

3. Decomposition formulas

In [21, p. 93], it is proved that, for any analytic functions f, the following equalities hold true:

$$
\left(\delta_\xi+\alpha\right)_n\left\{f(\xi)\right\}=\xi^{1-\alpha}\frac{d^n}{d\xi^n}\left\{\xi^{\alpha+n-1}f(\xi)\right\}
\tag{3.1}
$$

$$\left(\delta_\xi := \xi \frac{d}{d\xi};\ \alpha \in \mathbb{C};\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right)$$

and

$$(-\delta_\xi)_n \{f(\xi)\} = (-1)^n \xi^n \frac{d^n}{d\xi^n} \{f(\xi)\} \quad \left(\delta_\xi := \xi \frac{d}{d\xi};\ n \in \mathbb{N}_0\right). \tag{3.2}$$

In view of formulas (3.1) and (3.2), and taking into account the differentiation formula for hypergeometric functions, from operator identities (2.1) to (2.14), we find the following decomposition formulas:

$$\begin{aligned} X_2(a_1, a_2; c_1, c_2, c_3; x, y, z) &= (1-z)^{-a_1} \sum_{i=0}^{\infty} \frac{(a_1)_i (c_3 - a_2)_i}{(c_3)_i\, i!} \left(\frac{z}{z-1}\right)^i \\ &\cdot F_4\left(\frac{a_1+i}{2}, \frac{a_1+i}{2} + \frac{1}{2}; c_1, c_2; \frac{4x}{(1-z)^2}, \frac{4y}{(1-z)^2}\right); \end{aligned} \tag{3.3}$$

$$\begin{aligned} &(1-z)^{-a_1} F_4\left(\frac{a_1}{2}, \frac{a_1}{2} + \frac{1}{2}; c_1, c_2; \frac{4x}{(1-z)^2}, \frac{4y}{(1-z)^2}\right) \\ &= \sum_{i=0}^{\infty} \frac{(a_1)_i (c_3 - a_2)_i}{(c_3)_i\, i!} z^i X_2(a_1 + i, a_2; c_1, c_2, c_3 + i; x, y, z); \end{aligned} \tag{3.4}$$

$$\begin{aligned} X_6(a_1, a_2, a_3; c_1, c_2; x, y, z) &= (1-z)^{-a_1} \sum_{i=0}^{\infty} \frac{(-1)^i (a_1)_i (c_2 - a_3)_i}{(c_2)_i\, i!} \\ &\cdot \left(\frac{z}{1-z}\right)^i H_3\left(a_1 + i, a_2; c_1; \frac{x}{(1-z)^2}, \frac{y}{1-z}\right); \end{aligned} \tag{3.5}$$

$$\begin{aligned} &(1-z)^{-a_1} H_3\left(a_1, a_2; c_1; \frac{x}{(1-z)^2}, \frac{y}{1-z}\right) \\ &= \sum_{i=0}^{\infty} \frac{(a_1)_i (c_2 - a_3)_i (-p)_i}{(c_2)_i\, i!} z^i X_6(a_1 + i, a_2, a_3; c_1, c_2 + i; x, y, z); \end{aligned} \tag{3.6}$$

$$\begin{aligned} X_8(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) &= (1-z)^{-a_1} \sum_{i=0}^{\infty} \frac{(-1)^i (a_1)_i (c_3 - a_3)_i}{(c_3)_i\, i!} \\ &\cdot \left(\frac{z}{1-z}\right)^i H_4\left(a_1 + i, a_2; c_1, c_2; \frac{x}{(1-z)^2}, \frac{y}{1-z}\right); \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
&(1-z)^{-a_1} H_4\left(a_1, a_2; c_1, c_2; \frac{x}{(1-z)^2}, \frac{y}{1-z}\right) \\
&\quad = \sum_{i=0}^{\infty} \frac{(a_1)_i (c_3-a_3)_i}{(c_3)_i\, i!} z^i X_8\left(a_1+i, a_2, a_3; c_1, c_2, c_3+i; x, y, z\right);
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
X_{14}\left(a_1, a_2, a_3; c_1, c_2; x, y, z\right) &= (1-z)^{-a_2} \sum_{i=0}^{\infty} \frac{(-1)^i (a_2)_i (c_2-a_3)_i}{(c_2)_i\, i!} \\
&\quad \cdot \left(\frac{z}{1-z}\right)^i H_3\left(a_1, a_2+i; c_1; x, \frac{y}{1-z}\right);
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&(1-z)^{-a_2} H_3\left(a_1, a_2; c_1; x, \frac{y}{1-z}\right) \\
&\quad = \sum_{i=0}^{\infty} \frac{(a_2)_i (c_2-a_3)_i}{(c_2)_i\, i!} z^i X_{14}\left(a_1, a_2+i, a_3; c_1, c_2+i; x, y, z\right);
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&X_{15}\left(a_1, a_2, a_3; c_1, c_2; x, y, z\right) \\
&= (1-y)^{-a_1} (1-z)^{-a_3} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} (a_1)_i (a_3)_j (c_2-a_2)_{i+j}}{(c_2)_{i+j}\, i! j!} \\
&\quad \cdot \left(\frac{y}{1-y}\right)^i \left(\frac{z}{1-z}\right)^j F\left(\frac{a_1+i}{2}, \frac{a_1+i}{2}+\frac{1}{2}; c_1; \frac{4x}{(1-y)^2}\right);
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&(1-y)^{-a_1} (1-z)^{-a_3} F\left(\frac{a_1}{2}, \frac{a_1}{2}+\frac{1}{2}; c_1; \frac{4x}{(1-y)^2}\right) \\
&\quad = \sum_{i,j=0}^{\infty} \frac{(a_1)_i (a_3)_j (c_2-a_2)_{i+j}}{(c_2)_{i+j}\, i! j!} y^i z^j \\
&\qquad \cdot X_{15}\left(a_1+i, a_2, a_3+j; c_1, c_2+i+j; x, y, z\right);
\end{aligned}
\tag{3.12}
$$

$$\begin{aligned} X_{17}\left(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z\right) \\ &= (1-z)^{-a_2} \sum_{i=0}^{\infty} \frac{(-1)^i \left(a_2\right)_i \left(c_3-a_3\right)_i}{\left(c_3\right)_i i!} \left(\frac{z}{1-z}\right)^i \\ &\quad \cdot H_4\left(a_1, a_2+i; c_1, c_2; x, \frac{y}{1-z}\right); \end{aligned} \tag{3.13}$$

$$\begin{aligned} (1-z)^{-a_2} H_4\left(a_1, a_2; c_1, c_2; x, \frac{y}{1-z}\right) \\ = \sum_{i=0}^{\infty} \frac{\left(a_2\right)_i \left(c_3-a_3\right)_i}{\left(c_3\right)_i i!} z^i X_{17}\left(a_1, a_2+i, a_3; c_1, c_2, c_3+i; x, y, z\right); \end{aligned} \tag{3.14}$$

$$\begin{aligned} X_{20}\left(a_1, a_2, a_3, a_4; c_1, c_2; x, y, z\right) = (1-y)^{-a_1} \sum_{i=0}^{\infty} \frac{(-1)^i \left(a_1\right)_i \left(c_2-a_2\right)_i}{\left(c_2\right)_i i!} \\ \cdot \left(\frac{y}{1-y}\right)^i F_3\left(\frac{a_1+i}{2}, a_3, \frac{a_1+i}{2}+\frac{1}{2}, a_4; c_1; \frac{4x}{(1-y)^2}, z\right); \end{aligned} \tag{3.15}$$

$$\begin{aligned} (1-y)^{-a_1} F_3\left(\frac{a_1}{2}, a_3, \frac{a_1}{2}+\frac{1}{2}, a_4; c_1; \frac{4x}{(1-y)^2}, z\right) \\ = \sum_{i=0}^{\infty} \frac{\left(a_1\right)_i \left(c_2-a_2\right)_i}{\left(c_2\right)_i i!} y^i X_{20}\left(a_1+i, a_2, a_3, a_4; c_1, c_2+i; x, y, z\right). \end{aligned} \tag{3.16}$$

4. Integral representations

We present certain interesting integral representations.

$$\begin{aligned} H_3(a, b; c; x, y) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \xi^{b-1} (1-\xi)^{c-b-1} (1-y\xi)^{-a} \\ \cdot F\left(\frac{a}{2}, \frac{1}{2}+\frac{a}{2}; c-b; \frac{4x(1-\xi)}{(1-y\xi)^2}\right) d\xi \quad (\Re(c) > \Re(b) > 0); \end{aligned} \tag{4.1}$$

$$\begin{aligned}
&H_4\left(a,b;c_1,c_2;x,y\right)\\
&\quad=\frac{\Gamma\left(c_2\right)}{\Gamma\left(b\right)\Gamma\left(c_2-b\right)}\int_0^1\xi^{b-1}\left(1-\xi\right)^{c_2-b-1}\left(1-y\xi\right)^{-a}\\
&\qquad\cdot F\left(\frac{a}{2},\frac{1}{2}+\frac{a}{2};c_1;\frac{4x}{\left(1-y\xi\right)^2}\right)d\xi\quad\left(\Re(c_2)>\Re(b)>0\right);
\end{aligned}\tag{4.2}$$

$$\begin{aligned}
&X_6\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)\\
&=\frac{\Gamma\left(c_1\right)}{\Gamma\left(a_2\right)\Gamma\left(c_1-a_2\right)}\int_0^1\xi^{a_2-1}\left(1-\xi\right)^{c_1-a_2-1}\left(1-z-y\xi\right)^{-a_1}\\
&\quad\cdot H_4\left(a_1,c_2-a_3;c_1-a_2,c_2;\frac{x\left(1-\xi\right)}{\left(1-z-y\xi\right)^2},-\frac{z}{1-z-y\xi}\right)d\xi\\
&\qquad\qquad\left(\Re(c_1)>\Re(a_2)>0\right);
\end{aligned}\tag{4.3}$$

$$\begin{aligned}
&X_8\left(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z\right)\\
&=\frac{\Gamma\left(c_2\right)}{\Gamma\left(a_2\right)\Gamma\left(c_2-a_2\right)}\int_0^1\xi^{a_2-1}\left(1-\xi\right)^{c_2-a_2-1}\left(1-z-y\xi\right)^{-a_1}\\
&\quad\cdot H_4\left(a_1,c_3-a_3;c_1,c_3;\frac{x}{\left(1-z-y\xi\right)^2},-\frac{z}{1-z-y\xi}\right)d\xi\\
&\qquad\qquad\left(\Re(c_2)>\Re(a_2)>0\right);
\end{aligned}\tag{4.4}$$

$$\begin{aligned}
&X_{14}\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)\\
&=\left(1-z\right)^{-a_2}\frac{\Gamma\left(a_1+a_2\right)}{\Gamma\left(a_1\right)\Gamma\left(a_2\right)}\int_0^1\xi^{a_1-1}\left(1-\xi\right)^{a_2-1}\\
&\quad\cdot H_4\left(a_1+a_2,c_2-a_3;c_1,c_2;x\xi^2+\frac{y\xi\left(1-\xi\right)}{1-z},-\frac{z\left(1-\xi\right)}{1-z}\right)d\xi\\
&\qquad\qquad\left(\Re(a_1)>0;\ \Re(a_2)>0\right).
\end{aligned}\tag{4.5}$$

Proof. It is noted that each of the integral representations in Section 4 can be proved mainly by expressing the series definition of the involved special function in each integrand and changing the order of the integral sign and the summation, and finally using the following well-known relationship between the Beta function $B(\alpha,\beta)$ and the Gamma function

Γ:

$$B(\alpha,\,\beta) := \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha)>0;\ \Re(\beta)>0) \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-). \end{cases} \tag{4.6}$$

□

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