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THE CHARACTERISTIC CONNECTION ON 6-DIMENSIONAL ALMOST HERMITIAN MANIFOLDS

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Abstract. The characteristic connection is a good substitute for the Levi-Civita connection, especially in studying non-integrable geometries. Unfortunately, not every geometric structure has the characteristic connection. In this paper we consider the space $U(3)/(U(1)\times U(1)\times U(1))$ with an almost Hermitian structure and prove that it has a geometric structure admitting the characteristic connection.

1. Introduction

Recently, the non-integrable geometries are studied by many mathematicians and we refer to the papers ([3], [4], [5], [7]) for more interesting information. A very important tool in studying non-integrable geometries is the characteristic connection ([5]). Given a G-structure, if the holonomy group with respect to the Levi-Civita connection is the whole group $\mathrm{SO}(n)$, the geometric structure is not preserved by the Levi-Civita connection. In some situation it is known that there can exist a unique metric connection with skew symmetric torsion which preserves the geometric structure. We call this the characteristic connection. The characteristic connection and its torsion are very closely related to the string theory in theoretic physics (see [6]).

Many geometric things related to the characteristic connection are also being studied. For example, in paper [2], the Dirac operator with respect to the characteristic connection was studied.

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In [5], studying and defining the non-integrable geometries reasonably, the author found a condition for G-structures admitting the characteristic connection. Specially in almost hermitian 6-dimensional manifolds corresponding to a $U(3)$-structure inside SO(6), the condition can be more comfortable by the consequences of the Gray-Hervella classification for the characteristic connections ([1]).

In this paper, we consider the homogeneous space $U(3)/(U(1) \times U(1) \times U(1))$ which is well known for admitting the nearly kähler structure. Moreover, we take a metric family on the space and Hermitian structures from those metrics. Then we investigate which metrics admit a characteristic connection.

In section 2, we have a look at the definition and the properties of the characteristic connection. We are interested in the situations in which a characteristic connection exists, specially in almost Hermitian manifolds.

In section 3, for the homogeneous space $U(3)/(U(1) \times U(1) \times U(1))$ we calculate the structures which allow a characteristic connection.

In appendix, we summerize some important results on non-integrable geometries related to the characteristic connection.

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2. Characteristic connections

2.1. G-structures on Riemannian manifolds

Let $G \subset \mathrm{SO}(n)$ be a closed subgroup and decompose $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$. Denote the projections onto $\mathfrak{g}$ and $\mathfrak{m}$ by $\mathrm{pr}_{\mathfrak{g}}$ and $\mathrm{pr}_{\mathfrak{m}}$, respectively. For an oriented Riemannian manifold (M^n, g) we denote its frame bundle by $\mathcal{F}(M^n)$. By definition, a G-structure on M^n is a reduction $\mathcal{R} \subset \mathcal{F}(M^n)$ of the frame bundle to the subgroup G.

For a given G-structure a characteristic connection is a metric connection with skew-symmetric torsion preserving the structure. Not every G-structure admits a characteristic connection, but if it exists it will be unique and can be expressed in terms of the geometric data. Hence we call it the characteristic connection. In Appendix, we can see some more details about the characteristic connection concerning non-integrable geometries. For further details we refer to [6].

2.2. The characteristic connections on almost Hermitian manifolds

An almost Hermitian manifold (M^{2n}, g, J) is a manifold with a Riemannian metric g and a g-compatible almost complex structure $J : TM^{2n} \to TM^{2n}$.

By $\Omega(X,Y) := g(JX,Y)$ $X,Y \in TM^{2n}$ we define the Kähler form of (M^{2n}, g, J). And a $(2,1)$-tensor field N called the Nijenhuis tensor is defined by

$$N(X,Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y].$$

By the $(3,0)$-tensor N we mean

$$N(X,Y,Z) = g(N(X,Y), Z), \text{ for } X,Y,Z \in TM^{2n}.$$

We now restrict our attention to the 6-dimensional case which is of special interest among all the even dimensional cases. Let's consider a 6-dimensional almost Hermitian manifold (M^6, g, J) with a $U(3)$-structure in SO(6).

Given a G-structure on a Riemannian manifold, the question about the existence of the characteristic connection is answered in [6] (see also the Appendix of this paper). Specially in the almost Hermitian 6-dimensional case it can be answered more practically studying the Gray-Hervella classifications of almost Hermitian manifolds. We will not go into details about the classification, just refer to [1]. Now we introduce a condition for the existence of the characteristic connection in 6-dimensional case.

Theorem 2.1. ([1] Theorem 4.2) *A 6-dimensional almost Hermitian manifold (M^6, g, J) admits a characteristic connection if and only if its Nijenhuis tensor N is totally skew-symmetric.*

The Nijenhuis tensor N as $(2,1)$-tensor field is already skew-symmetric from the definition (see Lemma 3.2), so the N as $(3,0)$-tensor is totally skew-symmetric if

$$N(X,Y,Z) = -N(X,Z,Y), \text{ for } X,Y,Z \in TM^{2n}.$$

3. The existence of the characteristic connection

3.1. The homogeneous space $U(3)/(U(1) \times U(1) \times U(1))$

We begin with noting a well-known metric family for a homogeneous reductive space without proofs:

PROPOSITION 3.1. *Let $M = G/H$ be the homogeneous space and β a metric of $\mathfrak{g}$, the Lie algebra of G. For $\mathfrak{m} := \mathfrak{h}^\perp$, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ is a reductive decomposition. Furthermore, if $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with relations*

$$[\mathfrak{h}, \mathfrak{m}_1] = \mathfrak{m}_1, \ \ [\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h} \oplus \mathfrak{m}_2,$$

$$[\mathfrak{h}, \mathfrak{m}_2] \subset \mathfrak{m}_2, \ \ [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{h}, \ \ [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1,$$

an $\mathrm{Ad}\,(H)$*-invariant inner product on* $\mathfrak{m}$ *is defined by*

$$\beta_t := \beta|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + 2t\beta|_{\mathfrak{m}_2 \times \mathfrak{m}_2}, \ \ \textit{for each } \ t > 0$$

which induces a left invariant metric g_t on G/H.

We now take $G := U(3)$ and $H := U(1) \times U(1) \times U(1) \subset G$ diagonally embedded. Then $M := G/H$ is a 6-dimensional manifold with

$$\mathfrak{g} = \mathfrak{u}(3) = \{A \in M_3(\mathbb{C}) : A + \bar{A}^t = 0\}, \ \ \mathfrak{h} = \{A \in \mathfrak{u}(3) : \ \ \text{A is diagonal}\}.$$

We define an $\mathrm{Ad}\,(G)$-invariant inner product $\beta := -\mathrm{Re}(\mathrm{tr} AB)/2$ for $A, B \in \mathfrak{u}(3)$ and decompose $\mathfrak{m} = \mathfrak{h}^\perp$ into

$$\mathfrak{m}_1 := \left\{ \begin{bmatrix} 0 & a & b \\ -\bar{a} & 0 & 0 \\ -\bar{b} & 0 & 0 \end{bmatrix} \ : \ a, b \in \mathbb{C} \right\},$$

$$\mathfrak{m}_2 := \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -\bar{c} & 0 \end{bmatrix} \ : \ c \in \mathbb{C} \right\}.$$

Then we can check that this decomposition satisfies the properties of Proposition 3.1 and we have well defined metrics $g_t, t > 0$.

We use the following notations for basis: Let $D_{kl} = (d_{ij})$ be the $n \times n$ matrix with zero entries except that its (k, l)-entry is 1. Furthermore, let $E_{kl} := D_{kl} - D_{lk}$ for $k \neq l$ and $S_{kl} := i(D_{kl} + D_{lk})$. Then

$$\{e_1 := E_{12},\, e_2 := S_{12},\, e_3 := E_{13},\, e_4 := S_{13},$$
$$e_5 := \frac{1}{\sqrt{2t}} E_{23},\, e_6 := \frac{1}{\sqrt{2t}} S_{23}\}$$

is an orthonormal basis of $\mathfrak{m}$ with respect to β_t. As basis for $\mathfrak{h}$ we take $H_k = S_{kk}/2, k = 1, 2, 3$.

LEMMA 3.1. *(i) The isotropy representation* $\mathrm{Ad} : H \to \mathrm{SO}(6)$ *for* $h = \mathrm{diag}(e^{it}, e^{is}, e^{ir})\,(t, s, r \in \mathbb{R})$ *is given by*

$$\mathrm{Ad}\,(h) = \begin{bmatrix} C(t-s) & 0 & 0 \\ 0 & C(t-r) & 0 \\ 0 & 0 & C(s-r) \end{bmatrix},$$

$$\textit{where } C(x) := \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

(ii) The three 2 forms $e_1 \wedge e_2, e_3 \wedge e_4, e_5 \wedge e_6$ are invariant under the isotropy representation given in (i).

Proof. (i) By calculation

$$\begin{aligned}
\mathrm{Ad}\,(h)e_1 &= hE_{12}h^{-1} \\
&= \begin{bmatrix} e^{it} & 0 & 0 \\ 0 & e^{is} & 0 \\ 0 & 0 & e^{ir} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-it} & 0 & 0 \\ 0 & e^{-is} & 0 \\ 0 & 0 & e^{-ir} \end{bmatrix} \\
&= \begin{bmatrix} 0 & e^{i(t-s)} & 0 \\ -e^{i(s-t)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \cos(t-s) \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sin(t-s) \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \cos(t-s)e_1 + \sin(t-s)e_2,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Ad}\,(h)e_2 &= hS_{12}h^{-1} \\
&= \begin{bmatrix} e^{it} & 0 & 0 \\ 0 & e^{is} & 0 \\ 0 & 0 & e^{ir} \end{bmatrix} \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-it} & 0 & 0 \\ 0 & e^{-is} & 0 \\ 0 & 0 & e^{-ir} \end{bmatrix} \\
&= \begin{bmatrix} 0 & ie^{i(t-s)} & 0 \\ ie^{i(s-t)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= -\sin(t-s) \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \cos(t-s) \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= -\sin(t-s)e_1 + \cos(t-s)e_2.
\end{aligned}$$

Analogously

$$\begin{aligned}
\mathrm{Ad}\,(h)e_3 &= \cos(t-r)e_3 + \sin(t-r)e_4, \\
\mathrm{Ad}\,(h)e_4 &= -\sin(t-r)e_3 + \cos(t-r)e_4,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Ad}\,(h)e_5 &= \cos(s-r)e_5 + \sin(s-r)e_6, \\
\mathrm{Ad}\,(h)e_6 &= -\sin(s-r)e_5 + \cos(s-r)e_6.
\end{aligned}$$

(ii) From *(i)*

$$\begin{aligned}\mathrm{Ad}\,(h)(e_1 \wedge e_2) &= \mathrm{Ad}\,(h)e_1 \wedge \mathrm{Ad}\,(h)e_2 \\ &= (\cos(t-s)e_1 + \sin(t-s)e_2 \\ &\quad \wedge(-\sin(t-s)e_1 + \cos(t-s)e_2) \\ &= e_1 \wedge e_2.\end{aligned}$$

The calculations for $e_3 \wedge e_4$ and $e_5 \wedge e_6$ are similar to the above. □

3.2. The existence of the characteristic connection

We first consider the N-tensor, $N(X,Y) = [JX,JY] - J[X,JY] - J[JX,Y] - [X,Y]$, then we have the following lemma concerning the N-tensor.

Lemma 3.2. *The Nijeunhuis tensor N satisfies*

1. $N(X,Y) = -N(Y,X)$,
2. $N(X,JY) = -JN(X,Y) = N(JX,Y)$.

Proof. 1. Since the commutators are skew-symmetric, we have

$$N(X,Y) = [JX,JY] - J[X,JY] - J[JX,Y] - [X,Y] = -N(Y,X).$$

2. We calculate

$$\begin{aligned}N(X,JY) &= [JX,JJY] - J[X,JJY] - J[JX,JY] - [X,JY] \\ &= [JX,-Y] - J[X,-Y] - J[JX,JY] - [X,JY] \\ &= -[JX,Y] + J[X,Y] - J[JX,JY] - [X,JY] \\ &= -J\left(-J[JX,Y] - [X,Y] + [JX,JY] - J[X,JY]\right) \\ &= -JN(X,Y),\end{aligned}$$

and

$$N(JX,Y) = -N(Y,JX) = JN(Y,X) = -JN(X,Y).$$

□

By Lemma 3.1, a 2-form and J on G/H are well defined as follows:

We take a 2-form

$$\Omega(X,Y) := e_{12} - e_{34} + e_{56} =: \beta_t(JX,Y) \text{ with } J^2 = -Id.$$

Then by calculation,

$$\begin{aligned}&J(e_1) = e_2, J(e_2) = -e_1, J(e_3) = -e_4, \\ &J(e_4) = e_3, J(e_5) = e_6, J(e_6) = -e_5.\end{aligned} \tag{3.1}$$

Now we are ready to give the main theorem.

THEOREM 3.1. *On $M = U(3)/U(1) \times U(1) \times U(1)$ we consider a metric family g_t and an almost complex structure J as above. Then the characteristic connection exists only for $t = \frac{1}{2}$.*

Proof. For $X \in TM^{2n}$, Lemma 3.2 implies

$$N(X, JX) = -JN(X, X) = 0,$$

so by (3.1) we have

$$N(e_1, e_2) = N(e_3, e_4) = N(e_5, e_6) = 0.$$

Now we consider

$$\begin{aligned} N(e_1, e_3) &= [Je_1, Je_3] - J[e_1, Je_3] - J[Je_1, e_3] - [e_1, e_3] \\ &= -[e_2, e_4] + J[e_1, e_4] - J[e_2, e_3] - [e_1, e_3], \end{aligned}$$

$$\begin{aligned} N(e_1, e_5) &= [Je_1, Je_5] - J[e_1, Je_5] - J[Je_1, e_5] - [e_1, e_5] \\ &= [e_2, e_6] - J[e_1, e_6] - J[e_2, e_5] - [e_1, e_5], \end{aligned}$$

$$\begin{aligned} N(e_3, e_5) &= [Je_3, Je_5] - J[e_3, Je_5] - J[Je_3, e_5] - [e_3, e_5] \\ &= -[e_4, e_6] - J[e_3, e_6] + J[e_4, e_5] - [e_3, e_5]. \end{aligned}$$

Computing the commutators, for example with the computer, we have:

$$[e_2, e_4] = [e_1, e_3] = -\sqrt{2t}e_5 \text{ and } [e_1, e_4] = -[e_2, e_3] = -\sqrt{2t}e_6,$$

$$[e_1, e_5] = -[e_2, e_6] = \frac{1}{\sqrt{2t}}e_3 \text{ and } [e_1, e_6] = [e_2, e_5] = \frac{1}{\sqrt{2t}}e_4,$$

$$[e_4, e_6] = [e_3, e_5] = -\frac{1}{\sqrt{2t}}e_1 \text{ and } -[e_3, e_6] = [e_4, e_5] = -\frac{1}{\sqrt{2t}}e_2.$$

By (3.1), Lemma 3.2, we obtain

$$\begin{aligned} N(e_1, e_3) &= -JN(e_1, e_4) = N(e_2, e_4) = JN(e_2, e_3) = 4\sqrt{2t}e_5, \\ N(e_1, e_5) &= JN(e_1, e_6) = -N(e_2, e_6) = JN(e_2, e_5) = \frac{-4}{\sqrt{2t}}e_3, \\ N(e_3, e_5) &= JN(e_3, e_6) = N(e_4, e_6) = -JN(e_4, e_5) = \frac{4}{\sqrt{2t}}e_1. \end{aligned}$$

And

$$\begin{aligned} N(e_1, e_4) &= N(e_2, e_3) = -4\sqrt{2t}e_6, \\ N(e_1, e_6) &= N(e_2, e_5) = -\frac{4}{\sqrt{2t}}e_4, \\ N(e_3, e_6) &= N(e_4, e_5) = \frac{4}{\sqrt{2t}}e_2. \end{aligned}$$

We compute

$$N(e_1, e_3, e_5) + N(e_1, e_5, e_3) = 0.$$

After the similar calculations for another indices we observe that N is totally skew-symmetric if and only if

$$4\sqrt{2t} - \frac{4}{\sqrt{2t}} = 0.$$

Hence, the characteristic connection exists only for $t = \frac{1}{2}$. $\square$

4. Appendix

Given G-structure defined by some tensor $\mathcal{T}$, it has been customary that the integrability is defined by $\nabla^g \mathcal{T} = 0$. Here we present another approach to non-integrable geometries from the theory of principal fibre bundles, following the main lines of [5]. We also define the characteristic connection and give a condition for manifolds in which the characteristic connection exists.

Consider a manifold M with a G-structure. Let $\mathcal{R}$ is the restriction of the frame bundle $\mathcal{F}(M)$ to the subgroup $G \subset SO(n)$. The Levi-Civita connection is a 1-form Z on $\mathcal{F}(M)$ with values in the Lie algebra $\mathfrak{so}(n)$. Restricting the Levi-Civita connection to $\mathcal{R}$, we decompose it with respect to $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$, i.e.

$$Z|_{T(\mathcal{R})} := Z^* \oplus \Gamma.$$

So, Z^* is a connection in the principle G-bundle $\mathcal{R}$ and Γ is a 1-form on M with values in the associated bundle $\mathcal{R} \times_G \mathfrak{m}$.

If Γ vanishes, the G-structure $\mathcal{R}$ is called integrable, because this means it is preserved by the Levi-Civita connection and the holonomy group with respect to the Levi-Civita connection is a subgroup of G. All G-structures with $\Gamma \neq 0$ are called non-integrable.

The difference Γ between the Levi-Civita connection and the canonical G-connection induced on the G-structure is a good measure for how much the given G-structure fails to be integrable. By now Γ is widely known as the intrinsic torsion of the G-structure.

We may then ask under which conditions a given G-structure admits a metric connection with skew-symmetric torsion preserving the structure. For any orthonormal basis e_i of $\mathfrak{m}$ consider the G-equivariant map

$$\Theta : \Lambda^3(\mathbb{R}^n) \to \mathbb{R}^n \otimes \mathfrak{m}, \quad \Theta(T) := \sum_i (e_i \lrcorner T) \otimes e_i$$

THEOREM 4.1. *([5]) A G-structure of a Riemannian manifold admits a metric connection with a skew-symmetric torsion if and only if the 1-form Γ belongs to the image of Θ,*

$$2\Gamma = -\Theta(T) \quad \texttt{for some} \quad T \in \Lambda^3(\mathbb{R}^n).$$

DEFINITION 4.2. ([5]) A metric G-connection with the torsion described as in Theorem 4.1 will be called a characteristic connection. By constructiion, the holonomy group with respect to the characteristic connection is a subgroup of G.

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