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STABILITY OF A CUBIC FUNCTIONAL EQUATION IN p-BANACH SPACES

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Abstract. In this paper, we investigate the stability of a cubic functional equation

$$f(x+ny)+f(x-ny)+f(nx) = n^2f(x+y)+n^2f(x-y)+(n^3-2n^2+2)f(x)$$

in p-Banach spaces and in Banach modules, where $n \geq 2$ is an integer.

1. Introduction

In 1940, S.M. Ulam [12] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by D. H. Hyers [5] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th. M. Rassias [9] gave a generalization of the Hyers' result. In 1994, P. Găvruta [4] also obtained a further generalization of the Rassias' result.

We first recall some basic facts concerning quasi-Banach spaces and some preliminary results.

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DEFINITION 1.1. ([1, 10]) Let X be a linear space. A *quasi-norm* $\|\cdot\|$ is a real-valued function on X satisfying the following:
(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all scalar λ and all $x \in X$.
(iii) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

A quasi-normed space is a linear space together with a specified quasi-norm. A *quasi-Banach space* means a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if the inequality

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

holds for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space*. Clearly, p-norms are continuous, and in fact, if $\|\cdot\|$ is a p-norm on X, then the formula $d(x, y) := \|x - y\|^p$ defines a translation invariant metric for X and $\|\cdot\|^p$ is a p-homogeneous F-norm. The Aoki-Rolewicz theorem [10] (see also [1, 8]) yields that each quasi-norm is equivalent to some p-norm for some $0 < p \leq 1$. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms. In [11], J. Tabor has investigated a version of the Hyers-Rassias-Gajda theorem (see [3]) in quasi-Banach spaces.

The cubic function $f(x) = cx^3 (c \in \mathbb{R})$ satisfies the functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \tag{1.1}$$

In this paper, we will prove the Hyers-Ulam-Rassias stability of the cubic functional equation

$$f(x+ny)+f(x-ny)+f(nx) = n^2f(x+y)+n^2f(x-y)+(n^3-2n^2+2)f(x) \tag{1.2}$$

in p-Banach spaces and in Banach modules, where $n \geq 2$ is an integer.

2. Results in p-Banach spaces

From this section, let X be a linear space and Y a p-Banach space. For a mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x, y) := &\ f(x + ny) + f(x - ny) + f(nx) \\ &-n^2f(x + y) - n^2f(x - y) - (n^3 - 2n^2 + 2)f(x) \end{aligned}$$

for all $x, y \in X$.

Now we have the stability of the equation (1.2).

THEOREM 2.1. *Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X \times X \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{i=1}^{\infty} \frac{1}{n^{3pi}} \varphi(n^{i-1}x, n^{i-1}y)^p < \infty$$

and

$$\|Df(x, y)\| \le \varphi(x, y) \tag{2.1}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying (1.2) *such that*

$$\|f(x) - C(x)\| \le \widetilde{\varphi}(x, 0)^{\frac{1}{p}} \tag{2.2}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (2.1), we gain

$$\left\| f(x) - \frac{1}{n^3} f(nx) \right\|^p \le \frac{1}{n^{3p}} \ \varphi(x, 0)^p$$

for all $x \in X$. Replacing x by nx in the above inequality and then dividing by n^{3p}, we get

$$\left\| \frac{1}{n^3} f(nx) - \frac{1}{n^6} f(n^2x) \right\|^p \le \frac{1}{n^{6p}} \varphi(nx, 0)^p$$

for all $x \in X$. Adding the foregoing two inequalities, we have

$$\left\| f(x) - \frac{1}{n^6} f(n^2x) \right\|^p \le \frac{1}{n^{3p}} \ \varphi(x, 0)^p + \frac{1}{n^{6p}} \varphi(nx, 0)^p$$

for all $x \in X$. Continuing in this way, one can obtain that

$$\left\| f(x) - \frac{1}{n^{3r}} f(n^r x) \right\|^p \le \sum_{i=1}^{r} \frac{1}{n^{3pi}} \varphi(n^{i-1}x, 0)^p \tag{2.3}$$

for all $x \in X$ and all $r \in \mathbb{N}$. For $s = 1, 2, 3, \cdots$, dividing the preceding inequality by n^{3ps} and then substituting x by $n^s x$, we see that

$$\left\| \frac{1}{n^{3s}} f(n^s x) - \frac{1}{n^{3(s+r)}} f(n^{s+r} x) \right\|^p \le \sum_{i=1}^{r} \frac{1}{n^{3pi}} \frac{1}{n^{3ps}} \varphi(n^{i+s-1}x, 0)^p$$

for all $x \in X$ and all $r \in \mathbb{N}$. Taking $s \to \infty$ in the previous inequality, we conclude that $\left\{ \frac{1}{n^{3r}} f(n^r x) \right\}_{r=1}^{\infty}$ is a Cauchy sequence in the p-Banach space Y for all $x \in X$. This implies that the sequence $\left\{ \frac{1}{n^{3r}} f(n^r x) \right\}_{r=1}^{\infty}$

converges in Y for all $x \in X$. Thus we can define a function $C : X \to Y$ by

$$C(x) := \lim_{r\to\infty} \frac{1}{n^{3r}} f(n^r x)$$

for all $x \in X$. Then

$$\begin{aligned}\|DC(x,y)\|^p &= \lim_{r\to\infty} \frac{1}{n^{3pr}} \|Df(n^r x, n^r y)\|^p \\ &\le \lim_{r\to\infty} \frac{1}{n^{3pr}} \varphi(n^r x, n^r y)^p = 0\end{aligned}$$

for all $x, y \in X$. Hence the mapping $C : X \to Y$ satisfies (1.2). By [6], the mapping C also satisfies 1.1). Taking $r \to \infty$ in (2.3), we get the inequality (2.2).

It only remains to show the uniqueness of the cubic mapping $C : X \to Y$. Let $C' : X \to Y$ be another cubic mapping satisfying (1.2) and (2.2). Then

$$\|C(x) - C'(x)\|^p = \frac{1}{n^{3pr}} \|C(n^r x) - C'(n^r x)\|^p \le \frac{2}{n^{3pr}} \widetilde{\varphi}(n^r x, 0)$$

for all $x \in X$. Taking $r \to \infty$ in the above inequality, we conclude that $C(x) = C'(x)$ for all $x \in X$. □

Corollary 2.2. *Let X be a quasi-normed space. Let θ, p, q be real numbers with $\theta \ge 0$, $0 < p < 3$ and $q \ne 0$ and let $f : X \to Y$ be a mapping such that*

$$\|Df(x,y)\| \le \theta(\|x\|^p + \|y\|^q)$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying (1.2) *such that*

$$\|f(x) - C(x)\| \le \frac{\theta}{(n^{3p} - n^{p^2})^{\frac{1}{p}}} \|x\|^p$$

for all $x \in X$.

Theorem 2.3. *Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X \times X \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x,y) := \sum_{i=1}^{\infty} n^{3p(i-1)} \varphi\left(\frac{x}{n^i}, \frac{y}{n^i}\right)^p < \infty$$

and

$$\|Df(x,y)\| \le \varphi(x,y)$$

for all $x, y \in X$. *Then there exists a unique cubic mapping* $C : X \to Y$ *satisfying* (1.2) *such that*

$$\|f(x) - C(x)\| \le \widetilde{\varphi}(x, 0)^{\frac{1}{p}}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let* X *be a quasi-normed space. Let* θ, p, q *be real numbers with* $\theta \ge 0$, $p > 3$ *and* $q \ne 0$ *and let* $f : X \to Y$ *be a mapping such that*

$$\|Df(x, y)\| \le \theta(\|x\|^p + \|y\|^q)$$

for all $x, y \in X$. *Then there exists a unique cubic mapping* $C : X \to Y$ *satisfying* (1.2) *such that*

$$\|f(x) - C(x)\| \le \frac{\theta}{(n^{p^2} - n^{3p})^{\frac{1}{p}}} \|x\|^p$$

for all $x \in X$.

3. Results in Banach modules over a Banach algebra

In this section, let A be a unital Banach algebra with norm $|\cdot|$, $A_1 := \{a \in A : |a| = 1\}$, and let ${}_A\mathcal{M}$ and ${}_A\mathcal{N}$ be left Banach A-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively.

For a mapping $f : {}_A\mathcal{M} \to {}_A\mathcal{N}$, we define

$$D_a f(x, y) := a^3 f(x + ny) + a^3 f(x - ny) + a^3 f(nx)$$
$$-n^2 f(ax + ay) - n^2 f(ax - ay) - (n^3 - 2n^2 + 2)f(ax)$$

for all $a \in A_1$ and all $x, y \in {}_A\mathcal{M}$.

Definition 3.1. A cubic mapping $C : {}_A\mathcal{M} \to {}_A\mathcal{N}$ is called A-cubic if $C(ax) = a^3 C(x)$ for all $a \in A$ and all $x \in {}_A\mathcal{M}$.

From now on, let A be a complex unital Banach $*$-algebra with norm $|\cdot|$.

Theorem 3.2. *Let* $f : {}_A\mathcal{M} \to {}_A\mathcal{N}$ *be a mapping satisfying* $f(0) = 0$ *for which there exists a function* $\phi : ({}_A\mathcal{M})^2 \to [0, \infty)$ *such that*

$$\widetilde{\phi}(x, y) = \sum_{j=0}^{\infty} \frac{1}{n^{3j}} \phi(n^j x, n^j y) < \infty \tag{3.1}$$

and

$$\|D_a f(x,y)\| \le \phi(x,y) \tag{3.2}$$

for all $a \in A_1$ and all $x, y \in {}_A\mathcal{M}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{M}$, then there exists a unique A-cubic mapping $C : {}_A\mathcal{M} \to {}_A\mathcal{N}$ such that

$$\|f(x) - C(x)\| \le \frac{1}{n^3}\widetilde{\phi}(x,0) \tag{3.3}$$

for all $x \in {}_A\mathcal{M}$.

Proof. Similar to the proof of [9, Theorem 3.1], there exists a unique cubic mapping $C : {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying (3.3). In fact, the mapping C is given by

$$C(x) = \lim_{r\to\infty}\left(\frac{1}{n^3}\right)^r f(n^r x)$$

for all $x \in {}_A\mathcal{M}$, where $a = 1$. Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{M}$, by the same reasoning as the proof of [2], the cubic mapping $C : {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfies $C(tx) = t^3C(x)$ for all $t \in \mathbb{R}$ and all $x \in {}_A\mathcal{M}$. That is, C is $\mathbb{R}$-cubic. By letting $x = n^r x$ and $y = 0$ in (3.2), we have

$$\left\| f(n^r ax) - \frac{1}{n^3}a^3 f(n^{r+1}x)\right\| \le \frac{1}{n^3}\phi(n^r x, 0)$$

for all $a \in A_1$ and all $x \in {}_A\mathcal{M}$. Note that for each $a \in A$ and each $z \in {}_A\mathcal{M}$, we have

$$\|az\| \le K|a| \cdot \|z\|$$

for some $K > 0$. Then we get

$$\begin{aligned}
&\left\| f(n^r ax) - a^3 f(n^r x)\right\| \\
&\quad \le \left\| f(n^r ax) - \frac{1}{n^3}a^3 f(n^{r+1}x)\right\| + \left\| \frac{1}{n^3}a^3 f(n^{r+1}x) - a^3 f(n^r x)\right\| \\
&\quad \le \frac{1}{n^3}\phi(n^r x, 0)(1+K)
\end{aligned}$$

for all $a \in A_1$ and all $x \in {}_A\mathcal{M}$. Hence, for all $a \in A_1$ and all $x \in {}_A\mathcal{M}$, $n^{-3r}\|f(n^r ax) - a^3 f(n^r x)\| \to 0$ as $r \to \infty$. We may conclude that

$$C(ax) = \lim_{r\to\infty}\frac{f(n^r ax)}{n^{3r}} = a^3 \lim_{r\to\infty}\frac{f(n^r x)}{n^{3r}}$$

for all $a \in A_1$ and all $x \in {}_A\mathcal{M}$. Note that C is $\mathbb{R}$-cubic and $C(ax) = a^3C(x)$ for all $a \in A_1$. Thus we have

$$C(ax) = C\left(|a| \cdot \frac{a}{|a|}x\right) = |a|^3 C\left(\frac{a}{|a|}x\right) = a^3C(x)$$

for all nonzero $a \in A$ and all $x \in {}_A\mathcal{M}$. Also, $C(0x) = 0^3C(x)$ for all $x \in {}_A\mathcal{M}$. Thus the cubic mapping $C : {}_A\mathcal{M} \to {}_A\mathcal{N}$ is the unique A-cubic mapping as desired. □

4. Results in Banach modules over a C^*-algebra

In this section, we will investigate the stability of the given cubic functional equation (1.2) over a C^*-algebra. Throughout this section, let A be a unital C^*-algebra with a norm $|\cdot|$, and let ${}_A\mathcal{M}$ and ${}_A\mathcal{N}$ be left Banach A-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively. Put $A_1 := \{a \in A|\ |a| = 1\}$, $A_{in} := \{a \in A|\ a \text{ is invertible in } A\}$, $A_{sa} := \{a \in A|\ a^* = a\}$, $\mathcal{U}(A) := \{a \in A|\ aa^* = a^*a = 1\}$, $A^+ := \{a \in A_{sa}|\ Sp(a) \subset [0,\infty)\}$ and $A_1^+ := A_1 \cap A^+$.

For explicitly later use, we state the following lemma.

Lemma 4.1. [7] *Let $a \in A$ and $|a| < 1 - \frac{2}{k}$ for some integer k greater than 2. Then there are unitary elements $u_1, \cdots, u_k \in A$ such that $ka = u_1 + \cdots + u_k$.*

Theorem 4.2. *Let A be of real rank 0 and let $f : {}_A\mathcal{M} \to {}_A\mathcal{N}$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\phi : {}_A\mathcal{M} \times {}_A\mathcal{M} \to [0,\infty)$ satisfying (3.1) and (3.2) for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, y \in {}_A\mathcal{M}$. For each fixed $x \in {}_A\mathcal{M}$, let the sequence $\left\{\frac{f(n^r x)}{n^{3r}}\right\}$ converge uniformly on A_1. If $f(ax)$ is continuous in $a \in A_1 \cup \mathbb{R}$ for each fixed $x \in {}_A\mathcal{M}$ then there exists a unique A-cubic mapping $C : {}_A\mathcal{M} \to {}_A\mathcal{N}$ such that*

$$\|f(x) - C(x)\| \le \frac{1}{n^3}\widetilde{\phi}(x,0) \tag{4.1}$$

for all $x \in {}_A\mathcal{M}$.

Proof. Similar to the proof of Theorem 3.2, there exists a unique cubic mapping $C : {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying (4.1) and

$$C(ax) = a^3C(x) \tag{4.2}$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x \in {}_A\mathcal{M}$. From the continuity and the uniform convergence, one can show that $C(ax)$ is continuous in $a \in A_1$ for each $x \in {}_A\mathcal{M}$.

Let $b \in A_1^+ \setminus A_{in}$. Since $A_{in} \cap A_{sa}$ is dense in A_{sa}, there exists a sequence $\{b_m\}$ in $A_{in} \cap A_{sa}$ such that $b_m \to b$ as $m \to \infty$. Put $d_m := \frac{1}{|b_m|} b_m$. Then $d_m \to b$ as $m \to \infty$. Put $a_m = \sqrt{d_m^* d_m}$. Then $a_m \to b$ as $m \to \infty$ and $a_m \in A_1^+ \cap A_{in}$. By the continuity of C,

$$\lim_{m\to\infty} C(d_m x) = C\left(\lim_{m\to\infty} d_m x\right) = C(bx) \tag{4.3}$$

for all $x \in {}_A\mathcal{M}$. By (4.2),

$$\left\|C(a_m x) - b^3 C(x)\right\| = \left\|a_m^3 C(x) - b^3 C(x)\right\| \to \left\|b^3 C(x) - b^3 C(x)\right\| = 0$$

as $m \to \infty$ for all $x \in {}_A\mathcal{M}$. By (4.3) and the above equation,

$$\begin{aligned} \left\|C(bx) - b^3 C(x)\right\| &\leq \|C(bx) - C(a_m x)\| + \left\|C(a_m x) - b^3 C(x)\right\| \\ &\to 0 \quad \text{as} \quad m \to \infty \end{aligned}$$

for all $x \in {}_A\mathcal{M}$. By (4.2) and the above equation, $C(ax) = a^3 C(x)$ for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{M}$.

The rest of the proof is similar to the proof of Theorem 3.2, which completes the proof. □

THEOREM 4.3. *Let A be of real rank 0, commutative. Let $D := \{a \in A \,|\, Sp(a) \subset \mathbb{C} \setminus [0,\infty)\}$, $E := \{a \in A \,|\, Sp(a) \subset \mathbb{C} \setminus (-\infty, 0]\}$ and $D \cup E$ is dense in A_{in}. And let $f : {}_A\mathcal{M} \to {}_A\mathcal{N}$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\phi : {}_A\mathcal{M} \times {}_A\mathcal{M} \to [0,\infty)$ satisfying* (3.1) *and* (3.2) *for all $a \in \exp(\mathcal{U}(A)) \cup \{1\}$ and all $x, y \in {}_A\mathcal{M} \setminus \{0\}$. For each fixed $x \in {}_A\mathcal{M}$, let the sequence $\left\{\frac{f(n^r x)}{n^{3r}}\right\}$ converge uniformly on A_1. If $f(ax)$ is continuous in $a \in A_1 \cup \mathbb{R}$ for each fixed $x \in {}_A\mathcal{M}$, then there exists a unique A-cubic mapping $C : {}_A\mathcal{M} \to {}_A\mathcal{N}$ such that*

$$\|f(x) - C(x)\| \leq \frac{1}{n^3} \widetilde{\phi}(x, 0) \tag{4.4}$$

for all $x \in {}_A\mathcal{M} \setminus \{0\}$.

Proof. By the same reasoning as in the proof of Theorem 3.2, there exists a unique cubic mapping $C : {}_A\mathcal{M} \to {}_A\mathcal{N}$ satisfying the inequality (4.4) for all $x \in {}_A\mathcal{M}$. By a similar method to the proof of Theorem 3.2, the cubic mapping C satisfies $C(ax) = a^3 C(x)$ for all $a \in \exp(\mathcal{U}(A)) \cup \{1\}$ and all $x \in {}_A\mathcal{M}$.

For every element $a \in A_1 \cap D$, there is a positive integer m such that

$$\left|\frac{1 + \log a}{m}\right| < 1 - \frac{2}{m}.$$

By Lemma 4.1, there are unitary element $\upsilon_1, \cdots, \upsilon_m \in \mathcal{U}(A)$ such that $1 + \log a = \upsilon_1 + \cdots + \upsilon_m$. Then we have

$$\begin{aligned} C(eax) &= C\big(e^{1+\log a}x\big) = C(e^{\upsilon_1+\cdots+\upsilon_m}x) = C(e^{\upsilon_1}\cdots e^{\upsilon_m}x) \\ &= e^{3\upsilon_1}\cdots e^{3\upsilon_m}C(x) = (e^{\upsilon_1+\cdots+\upsilon_m})^3 C(x) \\ &= \big(e^{1+\log a}\big)^3 C(x) = e^3 a^3 C(x) \end{aligned}$$

for all $a \in A_1 \cap D$ and all $x \in {}_A\mathcal{M} \setminus \{0\}$. Hence we obtain $e^3 C(ax) = C(eax) = e^3 a^3 C(x)$ for all $a \in A_1 \cap D$ and all $x \in {}_A\mathcal{M} \setminus \{0\}$. Thus $C(ax) = a^3 C(x)$ for all $a \in A_1 \cap D$ and all $x \in {}_A\mathcal{M}$.

By the same process as the above argument, one can obtain that $C(ax) = a^3 C(x)$ for all $a \in A_1 \cap E$ and all $x \in {}_A\mathcal{M}$. Since $D \cup E$ is dense in A_{in}, $C(ax) = a^3 C(x)$ for all $a \in A_1 \cap A_{in}$ and all $x \in {}_A\mathcal{M}$.

The rest of the proof is the same as in the proof of Theorem 4.2, which completes the proof. □

References

[1] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis, Vol. 1*, Colloq. Publ. 48, Amer. Math. Soc. Providence, 2000.

[2] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg, **62** (1992), 59–64.

[3] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431–434.

[4] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[5] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. **27** (1941), 222–224.

[6] K.-W. Jun, S.-B. Lee and W.-P. Park, *Solution and stability of a cubic functional equation*, Acta Math. Sin., Eng. Ser. **26** (2010), 1255–1262.

[7] R. Kadison and G. Pedersen, *Means and convex combinations of unitary operators*, Math. Scand. **57** (1985), 249–266.

[8] N. Kalton, N. T. Peck, and W. Roberts, *An F-Space Sampler*, London Mathematical Society Lecture Note Series 89, Cambridge University Press, 1984.

[9] Th.M. Rassias, *On the stability of linear mappings in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[10] S. Rolewicz, *Metric Linear Spaces*, PWN-Polish Sci. Publ., Warszawa, Reidel, Dordrecht, 1984.

[11] J. Tabor, *Stability of the Cauchy functional equation in quasi-Banach spaces*, Ann. Polon. Math. **83** (2004), 243–255.

[12] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, 1968, p.63.

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